Research Article

Engineering Design by Geometric Programming

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A geometric program (GP) is a type of mathematical optimization problem characterized by objective and constraint functions, where all functions are of signomial form. The importance of GP comes from two relatively recent developments: (i) new methods can solve even large-scale GP extremely efficiently and reliably; (ii) a number of practical problems have recently been found to be equivalent to or approximated by GP. This study proposes an optimization approach for solving GP. Our approach is first to convert all signomial terms in GP into convex and concave terms. Then the concave terms are further treated with the proposed piecewise linearization method where only $\lceil \log_2(m-1) \rceil$ binary variables are used. It has the following features: (i) it offers more convenient and efficient means of expressing a piecewise linear function; (ii) fewer 0-1 variables are used; (iii) the computational results show that the proposed method is much more efficient and faster than the conventional one, especially when the number of break points becomes large. In addition, the engineering design problems are illustrated to evaluate the usefulness of the proposed methods.

1. Introduction

Consider the following geometric program (GP) [1]:

$$\min f_0(x) \quad (1)$$

$$\text{s.t. } f_k(x) \leq 1, \quad k = 1, 2, \dots, p, \quad (2)$$

$$g_l(x) = 1, \quad l = 1, 2, \dots, q, \quad (3)$$

$$x = (x_1, x_2, \dots, x_n), \quad (4)$$

where $g_l(x)$, $l = 1, 2, \dots, q$, are monomials as in (5), $c_l \geq 0$, and the exponential constants $\alpha_{il} \in \mathbb{R}$, $i = 1, 2, \dots, n$, $l = 1, 2, \dots, q$:

$$g_l(x) = c_l \prod_{i=1}^n x_i^{\alpha_{il}} \quad (5)$$

and $f_k(x)$, $k = 0, 1, \dots, p$, are posynomials as in (6) which are sum of one or more monomials, $c_{hk} \geq 0$, and $\alpha_{ihk} \in \mathbb{R}$, $i = 1, 2, \dots, n$, $h = 1, 2, \dots, H_k$, $k = 0, 1, \dots, p$:

$$f_k(x) = \sum_{h=1}^{H_k} c_{hk} \prod_{i=1}^n x_i^{\alpha_{ihk}}. \quad (6)$$

GP has been applied in many fields of applications including analog/digital circuit design [2–8], chemical engineering [9–11], mechanical engineering [12–19], power control [20], and communication network systems [21–23].

Obtaining the optimal solutions for GP is not straightforward because the signomial terms in the objective function and constraints cannot be solved directly. As a result, many approaches have been developed. Coello and Cortés [24] proposed a genetic algorithm with an artificial immune system to solve a GP in engineering optimization. Nevertheless, this method can only obtain the local optima. Horst and Tuy [25] introduced an analytical approach for solving a problem with Lipschitzian objective and constraints. The restriction of this approach is to find the global optimum only if the range of variables can be reduced by analytical techniques. Serali and Tuncbilek [26] developed a reformulation-linearization technique (RLT) which generates polynomial implied constraints and then linearizes the resulting problem by introducing new variables. Lin and Tsai [14] introduced a generalized method to find multiple optimal solutions of signomial discrete programming problems with free variables. By means of variable substitution and convexification strategies, a signomial discrete programming problem with free variables is

first converted into another convex mixed-integer nonlinear programming problem solvable to obtain an exactly global optimum. Tsai [17] proposed a novel method to solve signomial discrete programming problems. The signomial terms are first convexified following the convexification strategies. The original program is then converted into a convex integer program solvable by commercialized packages to obtain globally optimal solutions. Tsai and Lin [19] developed an efficient method to solve a posynomial geometric program with separable functions. Power transformations and exponential transformations are utilized to convexify and underestimate posynomial terms. The original program therefore can be converted into a convex mixed-integer nonlinear program solvable to obtain a global optimum.

This paper develops an optimization approach for solving GP. Our approach is listed as follows.

- (i) Convert all signomial terms into convex and concave ones.
- (ii) The concave terms are further treated with the proposed piecewise linearization method where only $\lceil \log_2(m-1) \rceil$ binary variables are used.

The rest of this paper is organized as follows. Section 2 introduces the proposed methods. Section 3 provides some numerical examples to illustrate the modeling idea and the usefulness of the proposed method. Section 4 gives our conclusions.

2. Proposed Methods

In this section, the proposed methods will be presented. First, we recall what a signomial function is. A function $f: \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ defined as

$$f(x) = c \prod_{i=1}^n x_i^{\alpha_i}, \quad (7)$$

where $c > 0$ and $\alpha_i \in \mathbb{R}$ for all i , is called a *monomial function* or simply a monomial [1]. Note that the exponents α_i of a monomial can be any real numbers, but the coefficient c must be nonnegative. A sum of monomials, namely, a function of the form

$$f(x) = \sum_{k=1}^N c_k \prod_{i=1}^n x_i^{\alpha_{ik}}, \quad (8)$$

where $c_k > 0$ and $\alpha_{ik} \in \mathbb{R}$, is called a *posynomial function* with N terms or simply a posynomial. A *signomial* is a linear combination of monomials of some positive variables $x_1, \dots, x_n$.

Let f be defined on an open convex set $D \subseteq \mathbb{R}^n$ and be twice differentiable; it is known that (i) f is convex on D if and only if the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite (p.s.d. for short) at each $x \in D$; (ii) if $\nabla^2 f(x)$ is positive definite (p.d. for short) at each $x \in D$, then f is strictly convex. For convenience, we denote by Δ_k the leading principal minors of A .

Lemma 1. Let A be a nonzero $n \times n$ symmetric matrix.

- (a) If A is positive semidefinite, then $\Delta_k \geq 0$, $k = 1, 2, \dots, n$, and not all $\Delta_k = 0$.
- (b) A is positive definite if and only if $\Delta_k > 0$, for all $k = 1, 2, \dots, n$.

The converse of Lemma 1(a) is false. For example, let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$; we have $\langle x, Ax \rangle = x_1^2 - x_3^2$ which is not always nonnegative for all $x \in \mathbb{R}^3$. But $\Delta_1 = 1 \geq 0$, $\Delta_2 = 0 \geq 0$, and $\Delta_3 = 0 \geq 0$. In fact, the converse of Lemma 1(a) is true only for $n = 2$; see [27, page 112].

Lemma 2. Let A be a nonzero symmetric matrix. Then, the following will hold.

- (a) A is p.s.d. if and only if all of its eigenvalues are nonnegative.
- (b) A is p.d. if and only if all of its eigenvalues are positive.

Proposition 3. Let $f_1: \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ be defined as $f_1(x) = c_1 \prod_{i=1}^n x_i^{\alpha_i}$, where $c_1 > 0$ and $\alpha_i \leq 0$ for all $i = 1, 2, \dots, n$. Then f_1 is a convex function.

Proof. Since $c_1 > 0$, it is enough to show that $\tilde{f}_1(x) = \prod_{i=1}^n x_i^{\alpha_i}$ is convex. Let $g(x) = \ln \tilde{f}_1(x) = \sum_{i=1}^n \ln x_i^{\alpha_i} = \sum_{i=1}^n \alpha_i \ln x_i$. Then, we have

$$\begin{aligned} \nabla g(x) &= \left[\frac{\alpha_1}{x_1} \frac{\alpha_2}{x_2} \cdots \frac{\alpha_n}{x_n} \right]^T, \\ \nabla^2 g(x) &= \begin{bmatrix} -\frac{\alpha_1}{x_1^2} & 0 & \cdots & 0 \\ 0 & -\frac{\alpha_2}{x_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{\alpha_n}{x_n^2} \end{bmatrix}. \end{aligned} \quad (9)$$

Because $\alpha_i \leq 0$ for all $i = 1, 2, \dots, n$, we know that all eigenvalues of $\nabla^2 g(x)$ are nonnegative, which implies (by Lemma 1(a)) that $\nabla^2 g(x)$ is positive semidefinite. Thus, $g(x) = \ln f(x)$ is a convex function which yields that $\tilde{f}_1(x)$ is a convex function. $\square$

Proposition 4. Let $f_2: \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ be defined as $f_2(x) = c_2 \prod_{i=1}^n x_i^{\alpha_i}$, where $c_2 < 0$ and $\alpha_i > 0$ for all $i = 1, 2, \dots, n$ with $1 - \sum_{i=1}^n \alpha_i \geq 0$. Then f_2 is a convex function.

Proof. It is not hard to compute that $[\nabla f_2(x)]_i = c_2 \alpha_i x_i^{\alpha_i-1} \prod_{j=1, j \neq i}^n x_j^{\alpha_j}$. In other words,

$$\nabla f_2(x) = \begin{bmatrix} c_2 \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ c_2 \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n} \\ \vdots \\ c_2 \alpha_n x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n-1} \end{bmatrix}. \quad (10)$$

In addition, it can be verified that

$$[\nabla^2 f_2(x)]_{ij} = \frac{\partial^2 f_2(x)}{\partial x_i \partial x_j} = \begin{cases} \frac{\alpha_i \alpha_j}{\partial x_i \partial x_j} f_2(x), & i \neq j, \\ \frac{\alpha_i (\alpha_i - 1)}{\partial x_i^2} f_2(x), & i = j, \end{cases} \quad (11)$$

Namely,

$$\nabla^2 f_2(x) = \begin{bmatrix} c_2 \alpha_1 (\alpha_1 - 1) x_1^{-2} \prod_{i=1}^n x_i^{\alpha_i} & c_2 \alpha_1 \alpha_2 x_1^{-1} x_2^{-1} \prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2 \alpha_1 \alpha_n x_1^{-1} x_n^{-1} \prod_{i=1}^n x_i^{\alpha_i} \\ c_2 \alpha_2 \alpha_1 x_1^{-1} x_2^{-1} \prod_{i=1}^n x_i^{\alpha_i} & c_2 \alpha_2 (\alpha_2 - 1) x_2^{-2} \prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2 \alpha_2 \alpha_n x_2^{-1} x_n^{-1} \prod_{i=1}^n x_i^{\alpha_i} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 \alpha_n \alpha_1 x_1^{-1} x_n^{-1} \prod_{i=1}^n x_i^{\alpha_i} & c_2 \alpha_n \alpha_2 x_2^{-1} x_n^{-1} \prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2 \alpha_n (\alpha_n - 1) x_n^{-2} \prod_{i=1}^n x_i^{\alpha_i} \end{bmatrix}. \quad (12)$$

Moreover, the determinant of $\nabla^2 f_2(x)$ can be computed and be shown by induction as

$$\det[\nabla^2 f_2(x)] = (-c_2)^n \left(\prod_{i=1}^n \alpha_i x_i^{\alpha_i - 2} \right) \left(1 - \sum_{i=1}^n \alpha_i \right). \quad (13)$$

We will complete the proof by discussing the following two cases.

Case i. If $1 - \sum_{i=1}^n \alpha_i = 0$, then it is not hard to verify that $\nabla^2 f_2(x) x$ is a zero vector for any $x \in \mathbb{R}^n$. Hence, $x^T \nabla^2 f_2(x) x = 0$ which says that $\nabla^2 f_2(x)$ is a positive semidefinite matrix by definition. Therefore, $f_2(x)$ is convex under this case.

Case ii. If $1 - \sum_{i=1}^n \alpha_i > 0$, then we know from (13) that

$$\Delta_i = (-c_2)^i \left(\prod_{j=1}^i \alpha_j x_j^{\alpha_j - 2} \right) \left(1 - \sum_{j=1}^i \alpha_j \right), \quad (14)$$

where Δ_i denotes the i th principal minor of the Hessian matrix of $f_2(x)$. Note that $c_2 < 0$, $\alpha_i > 0$ for all $i = 1, 2, \dots, n$, and $1 - \sum_{i=1}^n \alpha_i > 0$. Therefore, it can be seen that $\Delta_i > 0$ for all $i = 1, 2, \dots, n$, which implies (by Lemma 1(b)) that $\nabla^2 f_2(x)$ is a positive definite matrix. This says that $f_2(x)$ is strictly convex under this case.

From all the previous, the desired result follows. $\square$

We want to point out that our results also provide an alternative proof for the main result (Theorem 7) of [28]. Indeed, Maranas and Floudas [28] further discuss another condition as follows:

$$\exists j \text{ such that } \alpha_j \geq 1 - \sum_{i \neq j}^n \alpha_i, \quad (15)$$

$$\alpha_i \leq 0, \quad \forall i \neq j, \quad i = 1, 2, \dots, n,$$

to guarantee that f_1 defined as in Proposition 3 is a convex function. Our approach can be also employed to verify this

fact. To see this, we arrange all powers α_i in a decreasing order. In other words, without loss of generality, we assume that

$$\alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n. \quad (16)$$

Notice that condition (15) implies that α_1 is positive and all the other $\alpha_2, \dots, \alpha_n$ are nonpositive with $\alpha_1 \geq 1 - \sum_{i=2}^n \alpha_i$. As mentioned in Proposition 3, we only need to show that the function $\tilde{f}_1(x) = \prod_{i=1}^n x_i^{\alpha_i}$ is convex. By similar arguments as in the proof of Proposition 4, we know that

$$\hat{\Delta}_i = (-1)^i \left(\prod_{j=1}^i \alpha_j x_j^{\alpha_j - 2} \right) \left(1 - \sum_{j=1}^i \alpha_j \right), \quad (17)$$

where $\hat{\Delta}_i$ denotes the i th principal minor of the Hessian matrix of $\tilde{f}_1(x)$. From conditions (15) and (16), it is easily verified that $(1 - \sum_{j=1}^i \alpha_j) < 0$ for each i . It is also not hard to observe that $\prod_{j=1}^i \alpha_j$ is positive if i is odd and is negative if i is even. In summary, there holds

$$(-1)^i \left(\prod_{j=1}^i \alpha_j x_j^{\alpha_j - 2} \right) < 0. \quad (18)$$

The above two inequalities yield that $\hat{\Delta}_i > 0$ for each i . Thus, following the same arguments as in Proposition 4, we can conclude that $\tilde{f}_1$ is also a convex function under condition (15).

Remark 5. If $c > 0$, $\alpha_i < 0$, for $i = 1, 2, \dots, m$, and $\alpha_i > 0$, $i = m+1, m+2, \dots, n$, then $f(x)$ can be converted into the following convex function:

$$f'(x) = c \left(\prod_{i=1}^m x_i^{\alpha_i} \right) \left(\prod_{i=m+1}^n z_i^{-\alpha_i} \right), \quad (19)$$

where $z_i = x_i^{-1}$, $i = m+1, m+2, \dots, n$.

Remark 6. If $c < 0$, $\alpha_i > 0$, for $i = 1, 2, \dots, m$, and $\alpha_i < 0$, $i = m+1, m+2, \dots, n$, then $f(x)$ can be converted into the following convex function:

$$f'(x) = c \left(\prod_{i=1}^m z_i^{\alpha_i/t} \right) \left(\prod_{i=m+1}^n z_i^{-\alpha_i/t} \right), \quad (20)$$

where

- (i) t is the smallest positive integer satisfying $1 - (1/t)(\sum_{i=1}^m \alpha_i - \sum_{i=m+1}^n \alpha_i) \geq 0$.
- (ii) $z_i = x_i^t$, for $\alpha_i > 0$, $i = 1, 2, \dots, m$.
- (iii) $z_i = x_i^{-t}$, for $\alpha_i < 0$, $i = m+1, m+2, \dots, n$.

Obviously, all signomial terms are converted into convex and concave terms by the proposed methods. All concave terms in the GP problem can be further represented by an effective piecewise linearization method as follows.

Conventional methods for linearizing a concave function with m break points require $m-1$ binary variables. However,

FIGURE 1: A piecewise linear function of $f(x)$.

when m becomes large, the computation will be very time consuming and may cause a heavy computational burden. An effective piecewise linearization method proposed by Huang [29] is presented in which only $\lceil \log_2(m-1) \rceil$ binary variables are used.

Consider a grid of m points throughout the interval of $[a_1, a_m]$ and denote the break points $a_1, a_2, \dots, a_m$, where $a_1 < a_2 < \dots < a_m$. $f(a_k)$ is evaluated at each a_k , and two adjacent break points $(a_k, f(a_k))$ and $(a_{k+1}, f(a_{k+1}))$ are connected with a straight line as shown in Figure 1.

Let θ be an integer, $1 \leq \theta \leq m-1$, expressed as

$$\theta = 1 + \sum_{j=1}^h 2^{j-1} u_j, \quad (21)$$

where $h = \lceil \log_2(m-1) \rceil$ and $u_j \in \{0, 1\}$. Given the values of u_j , $1 \leq j \leq h$, there is a unique θ .

Let $G(\theta) \subseteq \{1, 2, \dots, h\}$ be a set composed of all indices corresponding to a given value θ , such that

$$\sum_{j \in G(\theta)} 2^{j-1} = \theta - 1. \quad (22)$$

For instance, $G(1) = \emptyset$, $G(2) = \{1\}$, $G(3) = \{2\}$, $G(4) = \{1, 2\}$, $\dots$, $G(10) = \{1, 4\}$, and so forth.

Denote by $\|G(\theta)\|$ the number of elements in $G(\theta)$. For instance, given $1 \leq x \leq 11$, then $h = \lceil \log_2(m-1) \rceil = 4$. $\|G(1)\| = 0$, $\|G(2)\| = 1$, $\|G(4)\| = 2$, and so forth. A list of all θ , u_j , $G(\theta)$, and $\|G(\theta)\|$ is shown in Table 1.

Now we intend to use k as an index to indicate that the k th interval is activated, which results in $a_k \leq x \leq a_{k+1}$ and $\hat{f}(x) = f(a_k) + s_k(x - a_k)$.

Define $m-1$ functions q_θ based on the combination of u_j as follows:

$$q_\theta = \|G(\theta)\| - \sum_{u_j \in D, j \in G(\theta)} u_j + \sum_{u_j \in D, j \notin G(\theta)} u_j, \quad (23)$$

where $\theta = 1, 2, \dots, m-1$, $j = 1, 2, \dots, h$, and $h = \lceil \log_2(m-1) \rceil$.

TABLE 1: Information of interval of x .

Interval of x	θ	u_1	u_2	u_3	u_4	$G(\theta)$	$\ G(\theta)\ $
$1 \leq x \leq 2$	1	0	0	0	0	$\emptyset$	0
$2 \leq x \leq 3$	2	1	0	0	0	1	1
$3 \leq x \leq 4$	3	0	1	0	0	2	1
$4 \leq x \leq 5$	4	1	1	0	0	1, 2	2
$5 \leq x \leq 6$	5	0	0	1	0	3	1
$6 \leq x \leq 7$	6	1	0	1	0	1, 3	2
$7 \leq x \leq 8$	7	0	1	1	0	2, 3	2
$8 \leq x \leq 9$	8	1	1	1	0	1, 2, 3	3
$9 \leq x \leq 10$	9	0	0	0	1	4	1
$10 \leq x \leq 11$	10	1	0	0	1	1, 4	2

The following proposition is deduced.

Proposition 7. Given a set of binary variables $D = \{u_1, u_2, \dots, u_h\}$, where $u_j \in \{0, 1\}$, $j = 1, 2, \dots, h$, and $h = \lceil \log_2(m-1) \rceil$, there is one and only one k , $1 \leq k \leq m-1$, such that

- (i) $q_k = 0$ and $\|G(k)\| = \sum_{u_j \in D, j \in G(k)} u_j - \sum_{u_j \in D, j \notin G(k)} u_j$,
- (ii) $q_\theta \geq 1$ for $1 \leq \theta \leq m-1$, $\theta \neq k$.

Taking Table 1 for instance, given $1 \leq x \leq 11$, $m = 11$. Let $D = \{u_1, u_2, u_3, u_4\} = \{1, 0, 0, 1\}$; then we have

$$q_1 = \|G(1)\| + u_1 + u_2 + u_3 + u_4 = 0 + 1 + 0 + 0 + 1 = 2$$

$$q_2 = \|G(2)\| - u_1 + u_2 + u_3 + u_4 = 1 - 1 + 0 + 0 + 1 = 1$$

$$\vdots$$

$$q_8 = \|G(8)\| - u_1 - u_2 - u_3 + u_4 = 3 - 1 - 0 - 0 + 1 = 3$$

$$q_9 = \|G(9)\| + u_1 + u_2 + u_3 - u_4 = 1 + 1 + 0 + 0 - 1 = 1$$

$$q_{10} = \|G(10)\| - u_1 + u_2 + u_3 - u_4 = 2 - 1 + 0 + 0 - 1 = 0. \quad (24)$$

Since $q_{10} = 0$ and $q_\theta \geq 1$ for $1 \leq \theta \leq 10$, $\theta \neq 10$, then only $a_{10} \leq x \leq a_{11}$ is activated and results in $\hat{f}(x) = f(a_{10}) + s_{10}(x - a_{10})$.

We then have the following theorem.

Theorem 8. Consider a piecewise linear function $\hat{f}(x)$ of a concave function $f(x)$, where x is divided into $m-1$ intervals by m break points, $a_1 < a_2 < \dots < a_m$. Given a set of binary variables $D = \{u_1, u_2, \dots, u_h\}$, where $h = \lceil \log_2(m-1) \rceil$, x and $\hat{f}(x)$ can be expressed as the following linear inequalities:

$$a_\theta - Mq_\theta \leq x \leq a_{\theta+1} + Mq_\theta, \quad (25)$$

$$\begin{aligned} f(a_\theta) + s_\theta(x - a_\theta) - M'q_\theta \\ \leq \hat{f}(x) \leq f(a_\theta) + s_\theta(x - a_\theta) + M'q_\theta, \end{aligned} \quad (26)$$

$$q_\theta = \|G(\theta)\| - \sum_{u_j \in D, j \in G(\theta)} u_j + \sum_{u_j \in D, j \notin G(\theta)} u_j, \quad (27)$$

where $s_\theta = (f(a_{\theta+1}) - f(a_\theta)) / (a_{\theta+1} - a_\theta)$, $\theta = 1, 2, \dots, m-1$. M , and M' are large positive values, $M = \max\{a_m, 1\}$ and $M' = \max\{f(a_1), f(a_2), \dots, f(a_m), 1\}$.

Proof. Given the value of a specific $D = D^0$, $D^0 = \{u_1^0, u_2^0, \dots, u_h^0\}$, consider the following two cases.

Case i. There is one and only one integer θ^0 , $\theta^0 = 1 + \sum_{j=1}^h 2^{j-1} u_j^0$ such that

- (a) $G(\theta^0) = \{j \mid u_j^0 = 1,$
- (b) $\|G(\theta^0)\| = \sum_{j \in G(\theta^0)} u_j^0,$
- (c) $\sum_{u_j^0 \in D, j \notin G(\theta^0)} u_j^0 = 0.$

Expressions (a), (b), and (c) force $q_{\theta^0} = 0$ (from (27)). Then, $a_{\theta^0} \leq x \leq a_{\theta^0+1}$ is activated (from (25)) and results in $\hat{f}(x) = f(a_{\theta^0}) + s_{\theta^0}(x - a_{\theta^0})$ (from (26)).

Case ii. For other $m-1$ integers $1 \leq \theta' \leq m$, $\theta' \neq \theta^0$, since $\|G(\theta')\| > \sum_{j \in G(\theta')} u_j^0 + \sum_{u_j^0 \in D, j \notin G(\theta')} u_j^0$, we then have $q_{\theta'} \geq 1$. In this case, inequalities (25) and (26) are still correct but not being activated. $\square$

Compared with the conventional piecewise linearization methods [30–36], the number of newly added binary variables in the proposed method for a piecewise linear function with m break points is significantly reduced from $m-1$ to $\lceil \log_2(m-1) \rceil$. The algorithm is listed in Algorithm 1 based on Theorem 8.

Algorithm 1 (the proposed piecewise linearization method). *Step 1.* Identify the number of subintervals and choose the break points, a_i , $i = 1, 2, \dots, m$.

Step 2. Calculate the values of $h = \lceil \log_2 m - 1 \rceil$ based on the number of break points selected and obtain q_θ , $\theta = 1, 2, \dots, m-1$, based on the combination of u_h as (27).

Step 3. Generate the set of constraints for x and $\hat{f}(x)$ as (25) and (26), respectively, and solve the resulting problem.

3. Numerical Examples

In this section, we have conducted some engineering design problems to evaluate the usefulness of the proposed methods.

Example 1. Consider the following engineering design problem of a speed reducer which was proposed by Golinski [37] as in Figure 2.

The objective of this problem is to minimize the weight of the speed reducer while satisfying a number of constraints imposed by gear and shaft design practices. There are seven design variables, x_1 (width of the gear face, cm), x_2 (teeth module, cm), x_3 (number of pinion teeth), x_4 (shaft 1 length between bearings, cm), x_5 (shaft 2 length between bearings, cm), x_6 (diameter of shaft 1, cm), and x_7 (diameter of shaft 2, cm). The constraints are characterized by h_1 (upper bound on the bending stress of the gear tooth), h_2 (upper bound on the

FIGURE 2: Schematic of the welded beam problem, Example 2.

contact stress of the gear tooth), h_3, h_4 (upper bounds on the transverse deflection of shafts 1, 2), h_5, h_6 (upper bounds on the stresses in shafts 1, 2), h_7, h_8, h_9 (dimensional restrictions based on space and experience), and h_{10}, h_{11} (dimensional requirements for shafts based on experience).

The problem can be formulated as

$$\begin{aligned} \min \quad & 2.6179x_1x_2^2x_3^2 + 11.7287x_1x_2^2x_3 \\ & - 33.84556x_1x_2^2 - 1.508x_1x_2^6 \\ & - 1.508x_1x_7^2 + 7.4777x_6^3 + 7.4777x_7^3 \\ & + 0.7854x_4x_6^2 + 0.7854x_5x_7^2 \end{aligned} \quad (28)$$

$$\text{s.t.} \quad h_1 = -x_1x_2^2x_3 + 27.0 \leq 0, \quad (29)$$

$$h_2 = -x_1x_2^2x_3^2 + 397.5 \leq 0, \quad (30)$$

$$h_3 = -x_2x_3x_6^4 + 1.93x_4^3 \leq 0, \quad (31)$$

$$h_4 = -x_2x_3x_7^4 + 1.93x_5^3 \leq 0, \quad (32)$$

$$\begin{aligned} h_5 = 555025.0x_4^2 + 1.690 \times 10^7 x_2^2x_3^2 \\ - 12100.0x_2^2x_3^2x_6^6 \leq 0, \end{aligned} \quad (33)$$

$$\begin{aligned} h_6 = 555025.0x_5^2 + 1.575 \times 10^8 x_2^2x_3^2 \\ - 7225.0x_2^2x_3^2x_7^6 \leq 0, \end{aligned} \quad (34)$$

$$h_7 = x_2x_3 - 40.0 \leq 0, \quad (35)$$

$$h_8 = -x_1 + 5x_2 \leq 0, \quad (36)$$

$$h_9 = x_1 - 12x_2 \leq 0, \quad (37)$$

$$h_{10} = -x_4 + 1.5x_6 + 1.9 \leq 0, \quad (38)$$

$$h_{11} = -x_5 + 1.1x_7 + 1.9 \leq 0, \quad (39)$$

$$2.6 \leq x_1 \leq 3.6, \quad 0.7 \leq x_2 \leq 0.8, \quad (40)$$

$$17.0 \leq x_3 \leq 28.0, \quad 7.3 \leq x_4 \leq 8.3,$$

$$7.3 \leq x_5 \leq 8.3, \quad 2.9 \leq x_6 \leq 3.9, \quad 5.0 \leq x_7 \leq 5.5. \quad (41)$$

This problem has been solved by Coello and Cortés [24] using genetic algorithm. The solution is $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (3.5, 0.7, 17.0, 7.300008, 7.715322, 3.350215, 5.286655)$ with the objective value 2994.3419.

For instance, the nonlinear terms $x_1 x_2^2 x_3^2$, $x_1 x_2^2 x_3$, and $x_4 x_6^2$ can be convexified as $z_{1,1}^{-1} z_{2,1}^{-2} z_{3,1}^{-2}$, $z_{1,1}^{-1} z_{2,1}^{-2} z_{3,1}^{-1}$, and $z_{4,1}^{-1} z_{6,1}^{-2}$ by Remark 5, where $z_{k,1} = x_k^{-1}$, $k = 1, 2, \dots, 7$. The nonlinear terms $-x_1 x_2^2$, $-x_1 x_3^2$, and $-x_1 x_7^2$ can be convexified as $-z_{1,2}^{1/3} z_{2,2}^{2/3}$, $-z_{1,2}^{1/3} z_{6,2}^{2/3}$, and $-z_{1,2}^{1/3} z_{7,2}^{2/3}$ by Remark 6, where $z_{1,2} = x_1^3$, $z_{2,2} = x_2^3$, $z_{6,2} = x_6^3$, and $z_{7,2} = x_7^3$. The concave terms can be further represented by the proposed piecewise linearization method. Take $z_{1,1} = x_1^{-1}$ for instance; suppose 41 equal-distance break points are used to linearize $z_{1,1} = x_1^{-1}$, as given by $a_{1,1} = 2.600, a_{1,2} = 2.625, a_{1,3} = 2.650, \dots$, and $a_{1,41} = 3.600$. Then $z_{1,1} = x_1^{-1}$ can be represented by Algorithm 2 as follows.

Algorithm 2. Step 1. The proposed method requires the use of $\lceil \log_2(41 - 1) \rceil = 6$ binary variables to linearize $z_{1,1} = x_1^{-1}$ in a piecewise manner. Compared with the conventional piecewise linearization methods, the number of newly added binary variables for a piecewise linear function with 41 break points is significantly reduced from 40 to 6.

Step 2. $h_{1,1} = \lceil \log_2(41 - 1) \rceil = 6$. A set of $q_{1,\theta}$, $\theta = 1, 2, \dots, 40$, based on the combination of $u_{1,h_{1,1}}$ is expressed as (42)

$$\begin{aligned} q_{1,1} &= 0 + u_{1,1} + u_{1,2} + u_{1,3} + u_{1,4} + u_{1,5} + u_{1,6} \\ q_{1,2} &= 1 - u_{1,1} + u_{1,2} + u_{1,3} + u_{1,4} + u_{1,5} + u_{1,6} \\ &\vdots \\ q_{1,40} &= 4 - u_{1,1} - u_{1,2} - u_{1,3} + u_{1,4} + u_{1,5} - u_{1,6}. \end{aligned} \quad (42)$$

Step 3. Based on Theorem 8, x_1 and $z_{1,1}$ can be expressed as the following linear inequalities:

$$\begin{aligned} 2.600 - Mq_{1,1} &\leq x_1 \leq 2.625 + Mq_{1,1} \\ 2.625 - Mq_{1,2} &\leq x_1 \leq 2.650 + Mq_{1,2} \\ &\vdots \\ 3.575 - Mq_{1,40} &\leq x_1 \leq 3.600 + Mq_{1,40}, \end{aligned} \quad (43)$$

TABLE 2: Comparison of Example 1.

Item	Coello and Cortés [24]	Proposed method
x_1	3.50	3.4956520
x_2	0.70	0.7000002
x_3	17.0	17.0
x_4	7.300008	7.30000007
x_5	7.715322	7.7120386
x_6	3.350215	3.3433720
x_7	5.286655	5.2853520
Objective	2994.3419	2989.5140

FIGURE 3: Golinski's speed reducer problem, Example 1.

$$\begin{aligned} 2.600^{-1} + s_{1,1} (x_1 - 2.600) - M' q_{1,1} \\ \leq z_{1,1} \leq 2.600^{-1} + s_{1,1} (x_1 - 2.600) + M' q_{1,1} \\ 2.625^{-1} + s_{1,2} (x_1 - 2.625) - M' q_{1,2} \\ \leq z_{1,1} \leq 2.625^{-1} + s_{1,2} (x_1 - 2.625) + M' q_{1,2} \quad (44) \\ \vdots \\ 3.575^{-1} + s_{1,40} (x_1 - 3.575) - M' q_{1,40} \\ \leq z_{1,1} \leq 3.575^{-1} + s_{1,40} (x_1 - 3.575) + M' q_{1,40}, \end{aligned}$$

where $s_{1,\theta} = (a_{1,\theta+1}^{-1} - a_{1,\theta}^{-1}) / (a_{1,\theta+1} - a_{1,\theta})$, $\theta = 1, 2, \dots, 40$. M and M' are large positive values.

Solving the transformed program with MATLAB R13, Table 2 shows the results computed by Coello and Cortés [24] and the proposed method.

Example 2 (see [38]). Consider the design of a welded beam which is to be rigidly attached to a fixed support and is subjected to a load as illustrated in Figure 3. The welded beam is to consist of 1010 steel and is to support a force F of 6000 lb. The length L is assumed to be specified at 14 in.

Three methods (brazed, bolted, and welded connections) of attachment and three materials (steel, brass, and aluminum) are considered. The choices of attachment and materials are to be made, which minimizes cost while satisfying a limit on the allowable bending stress, the buckling load, and the allowable deflection at the load application point.

The various materials have different elastic, shear moduli, and different costs.

The design variables are defined as x_1 (a continuous variable for thickness of weld), x_2 (a continuous variable for width of beam), x_3 (a continuous variable for length of connection), x_4 (a continuous variable for thickness of beam), x_5 (a continuous variable for diameter of bolts), x_6 (a 0-1 variable for brazed connections), x_7 (a 0-1 variable for bolted connections), x_8 (a 0-1 variable for welded connections), x_9 (a 0-1 variable for steel material), x_{10} (a 0-1 variable for brass material), and x_{11} (a 0-1 variable for aluminum material).

The problem can be formulated as (45)–(50)

$$\begin{aligned} \min \quad & 1.2x_2x_3x_6 + x_5x_7 + x_1^2x_3x_8 \\ & + 0.04811Lx_2x_4x_9 + 0.04811x_2x_3x_4x_9 \\ & + 0.06Lx_2x_4x_{10} + 0.06x_2x_3x_4x_{10} \\ & + 0.015Lx_2x_4x_{11} + 0.015x_2x_3x_4x_{11} \end{aligned} \quad (45)$$

$$\text{s.t} \quad h_1 = -\sigma_{\text{allowable}} + \frac{6FL}{x_2^2x_4} \leq 0, \quad (46)$$

$$h_2 = -P_c + 6000 \leq 0, \quad (47)$$

$$h_3 = -\delta_{\text{max}} + \frac{4FL^3}{Ex_2^3x_4} \leq 0, \quad (48)$$

$$x_6 + x_7 + x_8 = 1, \quad (49)$$

$$x_9 + x_{10} + x_{11} = 1, \quad (50)$$

where $\sigma_{\text{allowable}}$ is the maximum bar bending stress. P_c is the bar buckling load which is a function of the beam dimensions and material properties. δ_{max} is the maximum bar deflection. E is the Young's modulus, 30×10^6 psi. More detailed description including the derivation of P_c can be found in Reklaitis et al. [38].

Following the same procedure as in Example 1, the nonlinear terms $x_2x_3x_6$, x_5x_7 , $x_1^2x_3x_8$, $x_2x_4x_9$, $x_2x_3x_4x_9$, $x_2x_4x_{10}$, $x_2x_3x_4x_{10}$, $x_2x_4x_{11}$, and $x_2x_3x_4x_{11}$ can be convexified by Remark 5. The concave terms are further represented by the proposed piecewise linearization method. Solving the transformed program with MATLAB R13, we can obtain the global optimum with $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}) = (0.320, 14.109, 5.720, 0.321, 4.992, 0, 0, 1, 0, 0, 1)$ and the objective value is 1.969 within the tolerance error 0.0001.

4. Conclusions

This paper proposes an optimization approach for solving geometric programming problems. Our approach is first to convert all signomial terms in GP into convex and concave terms by the proposed methods. Then the concave terms are further treated with the proposed piecewise linearization method where only $\lceil \log_2(m-1) \rceil$ binary variables are used. It has the following features.

- (i) It offers more convenient and efficient means of expressing a piecewise linear function.

- (ii) Fewer 0-1 variables are used.

- (iii) The computational results show that the proposed method is much more efficient and faster than the conventional one, especially when the number of break points becomes large.

Numerical examples in real applications are illustrated to demonstrate the usefulness of the proposed methods.

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