Research Article

Some Characteristic Properties of Parallel z -Equidistant Ruled Surfaces

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Some characteristic properties of two ruled surfaces whose principal normal vectors are parallel along their striction curves in E^3 are examined by assuming that the distance between two central planes at suitable points is constant, E^3 . In case of which two ruled surfaces are close, the relationship between the integral invariants of this ruled surfaces is computed.

1. Introduction

The basis notions about ruled surfaces in E^3 are given in [1]. Parallel p -equidistant ruled surfaces are described, and some of their characteristic properties are given in Valeontis's article entitled "Parallel p -Äquidistante Regelflächen" [2]. Integral invariants, shape operators and spherical indicators of Parallel p -equidistant ruled surfaces were computed by Masal and Kuruoğlu in the articles [3–6]. Mannheim curves were described in Liu and Wang's article entitled "Mannheim partner curve in 3-Space" [7]. Some characteristic properties of Mannheim curves and Mannheim offsets of ruled surfaces were studied in [8, 9].

2. Preliminaries

Let $\alpha : I \rightarrow E^3$ be a differentiable curve with arc-length parameter s , and, $\{u_1, u_2, u_3\}$ be the Frenet frame of α at the point $\alpha(s)$, where

$$u_1(s) = \alpha'(s), \quad u_2(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad (1)$$

$$u_3(s) = u_1(s) \wedge u_2(s).$$

The Frenet formulas of α are

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \quad (2)$$

If α is a curve and x is a generator vector, then the ruled surface $X(s, v)$ has the following parameter representation:

$$X(s, v) = \alpha(s) + vx(s). \quad (3)$$

Namely, a ruled surface is a surface generated by the motion of a straight line x along α . Furthermore, if α is a closed curve, then this surfaces is called closed ruled surface. Moreover, the drall P_x , the striction γ , the apex angle λ_x , and the pitch l_x of the closed ruled surface are defined by

$$P_x = \frac{\det(\alpha', x, x')}{\|x'\|^2}, \quad (4)$$

$$\gamma(s) = \alpha(s) - \frac{\langle x'(s), \alpha'(s) \rangle}{\|x'(s)\|^2} x(s),$$

$$\lambda_x = \langle d, x \rangle, \quad l_x = \langle V, x \rangle.$$

The angle of the pitches, pitches, and dralls of the closed ruled surface generated by the Frenet vectors u_1 , u_2 , and u_3 are

$$\begin{aligned} \lambda_{u_1} &= \oint_{(\alpha)} k_2 ds, & l_{u_1} &= \oint_{(\alpha)} ds, \\ \lambda_{u_2} &= 0, & l_{u_2} &= 0, \\ \lambda_{u_3} &= \oint_{(\alpha)} k_1 ds, & l_{u_3} &= 0, \\ p_{u_1} &= 0, \\ p_{u_2} &= \frac{k_2}{k_1^2 + k_2^2}, \\ p_{u_3} &= \frac{1}{k_2}, \end{aligned} \quad (5)$$

see [1].

Definition 1. Let the equation of ruled surface be $X(s, v) = \alpha(s) + v u_1(s)$. The corresponding planes of the spaces $\text{sp}\{u_1, u_2\}$, $\text{sp}\{u_2, u_3\}$, and $\text{sp}\{u_3, u_1\}$ along the striction line of this ruled surfaces are called asymptotic, polar, and central planes, respectively, [9].

Definition 2. If the unit tangent vectors of the anchor curves, $\alpha : I \rightarrow E^3$ and $\bar{\alpha} : I \rightarrow E^3$, are $u_1(s)$ and $v_1(s)$ which are so-called generator vectors, respectively, then parametric equations of the two ruled surfaces are $X(s, v) = \alpha(s) + v u_1(s)$ and $\bar{X}(s, v) = \bar{\alpha}(s) + v v_1(s)$ in E^3 . If the generator vectors are parallel, and the distance p between polar planes in suitable points are constant, then this couple ruled surface are called p -equidistant ruled surfaces [9].

Theorem 3. The relationships between the angles of the pitch, pitches, and dralls of parallel p -equidistant are given as follows:

$$\begin{aligned} \lambda_{v_1} &= \lambda_{u_1} + a_1, & P_{v_1} &= P_{u_1} = 0, \\ \lambda_{v_2} &= \lambda_{u_2} = 0, & P_{v_2} &= P_{u_2} \frac{d\bar{s}}{ds}, \\ \lambda_{v_3} &= \lambda_{u_3} + a_2, & P_{v_3} &= P_{u_3} \frac{d\bar{s}}{ds}, \\ l_{v_1} &= \frac{k_1}{k_1} (l_{u_1} + a_3), \\ l_{v_2} &= l_{u_2} = l_{v_3} = l_{u_3} = 0, \end{aligned} \quad (6)$$

where

$$\begin{aligned} a_1 &= \oint_{(pu_1+zu_2+qu_3)} k_2 ds, & a_2 &= \oint_{(pu_1+zu_2+qu_3)} k_1 ds, \\ a_3 &= \oint_{(pu_1+zu_2+qu_3)} ds, \end{aligned} \quad (7)$$

see [3].

Definition 4. Consider two space curves $\alpha : I \rightarrow E^3$ and $\beta : I \rightarrow E^3$, where I is a real interval that has at least four continuous derivatives. If there exists a corresponding relationship between the space curves α and β such that the principal normal lines of α coincide with the binormal lines of β at the corresponding points of the curves, then α is called

as a Mannheim curve, and β is called as a Mannheim partner curve of α . The pair of $\{\alpha, \beta\}$ is said to be a Mannheim pair [7].

Definition 5. Let T be the unit tangent vector of the curve, $\alpha : I \rightarrow E^3$. If T makes a constant angle with a fixed line, then the curve α is called the helix curve [1].

Theorem 6. The distance between corresponding points of the Mannheim partner curves in E^3 is constant [8].

Theorem 7. For a curve α in E^3 , there is a curve β , so that $\{\alpha, \beta\}$ is a Mannheim pair [8].

Theorem 8. Let curvature and torsion of α be k_1 and k_2 , respectively, and α is Mannheim curve $\Leftrightarrow \lambda = k_1/(k_1^2 + k_2^2)$, ($\lambda \neq 0 = \text{const}$). See [7].

3. Parallel z -Equidistant Ruled Surfaces

Definition 9. Let α and $\bar{\alpha}$ be two curves, and let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be the Frenet frames of α and $\bar{\alpha}$ at the points $\alpha(s)$ and $\bar{\alpha}(s)$, respectively, in E^3 . If the unit principal normal vectors, $u_2(s)$ and $v_2(s)$, are generator vectors and α and $\bar{\alpha}$ are anchor curve, then parametric equations of the two ruled surfaces are $S(s, v) = \alpha(s) + v u_2(s)$ and $\bar{S}(s, v) = \bar{\alpha}(s) + v v_2(s)$. For this surfaces, if principal normal vectors u_2 and v_2 are parallel, and the distance z between central planes in suitable points are constant, then this couple ruled surface are called parallel z -equidistant ruled surfaces [9].

Let striction curve and curvatures of S ruled surface be $\gamma(s)$, $k_1(s)$, and $k_2(s)$, respectively. Let striction curve and curvatures of $\bar{S}$ ruled surface be $\bar{\gamma}(s)$, $\bar{k}_1(s)$, and $\bar{k}_2(s)$, respectively. In this situation, parametric equations of striction curves are

$$\gamma(s) = \alpha(s) - \frac{\langle u'_2(s), \alpha'(s) \rangle}{\|u'_2(s)\|^2} u_2(s), \quad u'_2(s) \neq 0, \quad (8)$$

$$\bar{\gamma}(s) = \bar{\alpha}(s) - \frac{\langle v'_2(s), \bar{\alpha}'(s) \rangle}{\|v'_2(s)\|^2} v_2(s), \quad v'_2(s) \neq 0.$$

If the Frenet formulas are written in the last equation, then we have

$$\gamma(s) = \alpha(s) - \frac{k_1(s)}{k_1^2(s) + k_2^2(s)} u_2(s), \quad (9)$$

$$\bar{\gamma}(s) = \bar{\alpha}(s) - \frac{\bar{k}_1(s)}{\bar{k}_1^2(s) + \bar{k}_2^2(s)} v_2(s). \quad (10)$$

If vector $\gamma\bar{\gamma}$ is written related to frame $\{u_1, u_2, u_3\}$, we get

$$\bar{\gamma} - \gamma = p u_1 + z u_2 + q u_3, \quad (11)$$

$$|z| = z, \quad |q| = q, \quad |p| = p,$$

where $p = \langle \gamma\bar{\gamma}, u_1 \rangle$, $z = \langle \gamma\bar{\gamma}, u_2 \rangle$, and $q = \langle \gamma\bar{\gamma}, u_3 \rangle$. Here, $|p|$, $|z|$, and $|q|$ are distance between polar, central, and asymptotic planes, respectively.

Theorem 10. Let striction curves of S and $\bar{S}$ parallel z -equidistant ruled surfaces be γ and $\bar{\gamma}$. Then, the relation between striction curves are given as follows:

$$\begin{aligned} \bar{\gamma} = & \gamma + pu_1 + qu_3 \\ & + \left(\left(k_1 \left(1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right) \right. \\ & \left. - k_2 \left(q' - \left(\frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \right) \times (k_1^2 + k_2^2)^{-1} u_2. \end{aligned} \quad (12)$$

Proof. By substituting (10) into (11), we get

$$\bar{\alpha} - \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} v_2 = \gamma + pu_1 + zu_2 + qu_3. \quad (13)$$

Since vectors v_2 and u_2 are parallel vectors, then we can write

$$\bar{\alpha} = \gamma + pu_1 + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) u_2 + qu_3. \quad (14)$$

By differentiating the last equation, we have

$$\begin{aligned} \bar{\alpha}' = & \left[\left(1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' - \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) k_1 \right] u_1 \\ & + \left[- \left(\frac{k_1}{k_1^2 + k_2^2} \right)' + pk_1 - qk_2 + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right)' \right] \\ & \times u_2 + \left[- \left(\frac{k_1 k_2}{k_1^2 + k_2^2} \right) + q' + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) k_2 \right] \\ & \times u_3. \end{aligned} \quad (15)$$

From the last equation, we have

$$\begin{aligned} \langle \bar{\alpha}', u_2' \rangle = & -k_1 \left[\left(1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right] \\ & + k_2 \left[- \left(\frac{k_1 k_2}{k_1^2 + k_2^2} \right) + q' \right] \\ & + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) (k_1^2 + k_2^2). \end{aligned} \quad (16)$$

Since vectors v_2 and u_2 are parallel vectors, then we can write

$$\bar{\gamma} = \bar{\alpha} - \frac{\langle u_2', \bar{\alpha}' \rangle}{\|u_2'\|^2} u_2. \quad (17)$$

Substituting (14) and (16) into the last equation, then we obtain

$$\begin{aligned} \bar{\gamma} = & \gamma + pu_1 + qu_3 \\ & + \left(\left(k_1 \left(1 + \frac{k_1^2}{k_1^2 + k_2^2} \right) + p' \right) \right. \\ & \left. - k_2 \left(q' - \left(\frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \right) \\ & \times (k_1^2 + k_2^2)^{-1} u_2. \end{aligned} \quad (18)$$

By (11), we have the following results. $\square$

Corollary 11. The distance between central planes of S and $\bar{S}$ parallel z -equidistant ruled surfaces is

$$\begin{aligned} z = & \left(k_1 \left(1 + \left(\frac{k_1^2}{k_1^2 + k_2^2} \right) \right) + p' \right) \\ & - k_2 \left(q' - \left(\frac{k_1 k_2}{k_1^2 + k_2^2} \right) \right) \\ & \times (k_1^2 + k_2^2)^{-1}. \end{aligned} \quad (19)$$

Corollary 12. If $\{\alpha, \gamma\}$ and $\{\bar{\alpha}, \bar{\gamma}\}$ pairs are Mannheim pairs, the distance between central planes of S and $\bar{S}$ parallel z -equidistant ruled surfaces is

$$z = \lambda \left(2 + p' - q' \frac{k_2}{k_1} \right). \quad (20)$$

Corollary 13. Let the unit tangent vectors of the striction curves of surfaces S and $\bar{S}$ be T and $\bar{T}$, respectively. Let δ and $\bar{\delta}$ be the angles between vectors T , $\bar{T}$, and their projection vectors on central plane, respectively. In this case, the distance between central planes of parallel z -equidistant ruled surfaces S and $\bar{S}$ can be obtained by the following equation (Figure 1):

$$\begin{aligned} z = & \left(k_1 \left(\cos \delta \sin \beta \frac{ds_\gamma}{ds} + p' \right) - k_2 \left(\cos \delta \cos \beta \frac{ds_\gamma}{ds} + q' \right) \right) \\ & \times (k_1^2 + k_2^2)^{-1}. \end{aligned} \quad (21)$$

Proof. By some algebraic manipulations, T and $\bar{T}$ can be calculated as follows:

$$T = (\cos \delta \sin \beta u_1 + \sin \delta u_2 + \cos \delta \cos \beta u_3) \frac{ds}{ds_\gamma}, \quad (22)$$

$$\bar{T} = (\cos \bar{\delta} \sin \bar{\beta} v_1 + \sin \bar{\delta} v_2 + \cos \bar{\delta} \cos \bar{\beta} v_3) \frac{d\bar{s}}{ds_{\bar{\gamma}}}. \quad (23)$$

FIGURE 1

By differentiating (14), we have

$$\begin{aligned} \bar{\alpha}' = & \left[\cos \delta \sin \beta \frac{ds_\gamma}{ds} + p' - \left(z + \left(\frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) \right) k_1 \right] u_1 \\ & + \left[\sin \gamma \frac{ds_\gamma}{ds} + pk_1 - qk_2 + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right)' \right] \\ & \times u_2 + \left[\cos \delta \cos \beta \frac{ds_\gamma}{ds} + q' \right. \\ & \left. + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) k_2 \right] u_3. \end{aligned} \quad (24)$$

From the last equation, we can write

$$\begin{aligned} \langle \bar{\alpha}', u_2' \rangle = & (-k_1 \cos \delta \sin \beta - k_1 p' + k_2 \cos \delta \cos \beta) \\ & \times \frac{ds_\gamma}{ds} + k_2 q' + \left(z + \frac{\bar{k}_1}{\bar{k}_1^2 + \bar{k}_2^2} \right) \\ & \times (k_1^2 + k_2^2). \end{aligned} \quad (25)$$

Substituting (14) and (23) into equation $\bar{\gamma} = \bar{\alpha} - (\langle u_2', \bar{\alpha}' \rangle / \|u_2'\|^2) u_2$, we get

$$\begin{aligned} \bar{\gamma} = & \gamma + pu_1 + qu_3 \\ & + \left[\left(k_1 \left(\cos \delta \sin \beta \frac{ds_\gamma}{ds} + p' \right) \right. \right. \\ & \left. \left. - k_2 \left(\cos \delta \cos \beta \frac{ds_\gamma}{ds} + q' \right) \right) \right. \\ & \left. \times (k_1^2 + k_2^2)^{-1} \right] u_2. \end{aligned} \quad (26)$$

From (11), we have

$$\begin{aligned} z = & \left(k_1 \left(\cos \delta \sin \beta \frac{ds_\gamma}{ds} + p' \right) \right. \\ & \left. - k_2 \left(\cos \delta \cos \beta \frac{ds_\gamma}{ds} + q' \right) \right) \\ & \times (k_1^2 + k_2^2)^{-1}. \end{aligned} \quad (27)$$

□

Corollary 14. If $\{\alpha, \gamma\}$ and $\{\bar{\alpha}, \bar{\gamma}\}$ pairs are Mannheim pairs, then the distance between central planes of surfaces S and $\bar{S}$ is

$$z = \lambda \left[\left(\cos \delta \sin \beta - \frac{k_2}{k_1} \cos \delta \cos \beta \right) \frac{ds_\gamma}{ds} + p' - \frac{k_2}{k_1} q' \right]. \quad (28)$$

Theorem 15. Let S and $\bar{S}$ be parallel z -equidistant ruled surfaces. Then, the relation between Frenet frame of $\alpha, \{u_1, u_2, u_3\}$ and of $\bar{\alpha}, \{v_1, v_2, v_3\}$ is given as follows:

$$\begin{aligned} v_1 &= \cos \varphi u_1 - \sin \varphi u_3, \\ v_2 &= u_2, \\ v_3 &= \sin \varphi u_1 + \cos \varphi u_3, \end{aligned} \quad (29)$$

where φ is the angle between the vector v_1 and the vector u_1 .

Proof. Let φ be the angle between the vector v_1 and the vector u_1 . In this case, we can write

$$v_1 = \cos \varphi u_1 + \sin \varphi u_3. \quad (30)$$

Since the vector v_2 is parallel to the vector u_2 , we have

$$\begin{aligned} v_2 &= u_2, \\ v_3 &= v_1 \wedge v_2, \\ v_3 &= -\sin \varphi u_1 + \cos \varphi u_3. \end{aligned} \quad (31)$$

This completes the proof of the theorem. □

Theorem 16. Let S and $\bar{S}$ be parallel z -equidistant ruled surfaces. Let s and $\bar{s}$ be arc parameters of anchor curves of S

and $\bar{S}$, respectively. If k_1, k_2 and $\bar{k}_1, \bar{k}_2$ are curvatures of anchor curves of S and $\bar{S}$, respectively, there are following equations between these curvatures:

$$\bar{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\bar{s}}, \quad (32)$$

$$\bar{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\bar{s}}. \quad (33)$$

Proof. Since S and $\bar{S}$ parallel z -equidistant ruled surfaces, $u_2(s) = v_2(\bar{s})$. Differentiating this equation related to s , we have

$$-k_1 u_1 + k_2 u_3 = (-\bar{k}_1 v_1 + \bar{k}_2 v_3) \frac{d\bar{s}}{ds}. \quad (34)$$

Multiplying the last equation with v_1 and v_3 , we have

$$\bar{k}_1 = (\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\bar{s}}, \quad (35)$$

$$\bar{k}_2 = (\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\bar{s}}. \quad \square$$

Theorem 17. *The relations between apex angles of closed parallel z -equidistant ruled surfaces S and $\bar{S}$*

$$\begin{aligned} (1) \lambda_{v_1} &= \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1, \quad b_1 = \\ &\oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s} + \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_2 d\bar{s} \\ (2) \lambda_{v_2} &= \lambda_{u_2} = 0 \\ (3) \lambda_{v_3} &= -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2, \quad b_2 = \\ &\oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s} + \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_1 d\bar{s}. \end{aligned}$$

Proof. The apex angle of closed ruled surface which is generated by the unit tangent vector v_1 is $\lambda_{v_1} = \oint_{(\bar{\alpha})} \bar{k}_2 d\bar{s}$. Substituting (14) into the last equation, we get

$$\begin{aligned} \lambda_{v_1} &= \oint_{(\gamma + pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s}, \\ \lambda_{v_1} &= \oint_{(\gamma)} \bar{k}_2 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s}. \end{aligned} \quad (36)$$

Substituting (9) into the last equation, we get

$$\begin{aligned} \lambda_{v_1} &= \oint_{(\alpha - (k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_2 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s}, \\ \lambda_{v_1} &= \oint_{(\alpha)} \bar{k}_2 d\bar{s} + \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_2 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s}. \end{aligned} \quad (37)$$

Impending,

$$\begin{aligned} b_1 &= \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_2 d\bar{s} \\ &+ \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_2 d\bar{s}, \\ \lambda_{v_1} &= \oint_{(\alpha)} \bar{k}_2 d\bar{s} + b_1. \end{aligned} \quad (38)$$

Substituting (33) into the last equation, we have

$$\lambda_{v_1} = \sin \varphi \oint_{(\alpha)} k_1 ds + \cos \varphi \oint_{(\alpha)} k_2 ds + b_1. \quad (39)$$

From (5), we can write

$$\lambda_{v_1} = \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1. \quad (40)$$

From (5), the apex angles of closed ruled surfaces which are generated with vector v_2 and u_2 are

$$\lambda_{v_2} = \lambda_{u_2} = 0. \quad (41)$$

The apex angle of closed ruled surface which is generated with vector v_3 is $\lambda_{v_3} = \oint_{(\bar{\alpha})} \bar{k}_1 d\bar{s}$. Substituting (14) into the last equation, we get

$$\begin{aligned} \lambda_{v_3} &= \oint_{(\gamma + pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s}, \\ \lambda_{v_3} &= \oint_{(\gamma)} \bar{k}_1 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s}. \end{aligned} \quad (42)$$

Substituting (9) into the last equation, we have

$$\begin{aligned} \lambda_{v_3} &= \oint_{(\alpha - (k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_1 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s}, \\ \lambda_{v_3} &= \oint_{(\alpha)} \bar{k}_1 d\bar{s} + \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_1 d\bar{s} \\ &+ \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s}. \end{aligned} \quad (43)$$

Impending,

$$\begin{aligned} b_2 &= \oint_{(pu_1 + (z + (\bar{k}_1/(\bar{k}_1^2 + \bar{k}_2^2)))u_2 + qu_3)} \bar{k}_1 d\bar{s} \\ &+ \oint_{((-k_1/(k_1^2 + k_2^2)) \cdot u_2)} \bar{k}_1 d\bar{s}, \\ \lambda_{v_3} &= \oint_{(\alpha)} \bar{k}_1 d\bar{s} + b_2. \end{aligned} \quad (44)$$

Substituting (32) into the last equation, we have

$$\lambda_{v_3} = -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2. \quad (45)$$

Corollary 18. If $\{\alpha, \gamma\}$ and $\{\bar{\alpha}, \bar{\gamma}\}$ pairs are Mannheim pairs, then the relations between apex angles of closed parallel z-equidistant ruled surfaces S and $\bar{S}$ are given as follows:

$$\begin{aligned} (1) \lambda_{v_1} &= \cos \varphi \lambda_{u_1} + \sin \varphi \lambda_{u_3} + b_1, \quad b_1 = \\ &\oint_{(pu_1+(z+\bar{\lambda})u_2+qu_3)} \bar{k}_2 d\bar{s} + \oint_{(-\lambda \cdot u_2)} \bar{k}_2 d\bar{s}, \\ (2) \lambda_{v_2} &= \lambda_{u_2} = 0, \\ (3) \lambda_{v_3} &= -\sin \varphi \lambda_{u_1} + \cos \varphi \lambda_{u_3} + b_2, \quad b_2 = \\ &\oint_{(pu_1+(z+\bar{\lambda})u_2+qu_3)} \bar{k}_1 d\bar{s} + \oint_{(-\lambda \cdot u_2)} \bar{k}_1 d\bar{s}. \end{aligned}$$

Theorem 19. If we specially take helix curve instead of anchor curve of closed parallel z-equidistant ruled surface, there is relation between l_{u_1} and l_{v_1} as follows:

$$\begin{aligned} l_{v_1} &= \left(\cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_1} \right) l_{u_1} + b_3, \\ b_3 &= \oint_{(pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s} + \oint_{((-k_1/(k_1^2+k_2^2)) \cdot u_2)} d\bar{s}. \end{aligned} \quad (46)$$

Proof. From (5), the pitch of closed ruled surface which is generated vector v_1 is $l_{v_1} = \oint_{(\bar{\alpha})} d\bar{s}$. Substituting (14) into the last equation, we get

$$\begin{aligned} l_{v_1} &= \oint_{(\gamma+pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s}, \\ l_{v_1} &= \oint_{(\gamma)} d\bar{s} + \oint_{(pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s}. \end{aligned} \quad (47)$$

Substituting (9) into the last equation, we get

$$\begin{aligned} l_{v_1} &= \oint_{(\alpha-(k_1/(k_1^2+k_2^2)) \cdot u_2)} d\bar{s} \\ &+ \oint_{(pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s}, \\ l_{v_1} &= \oint_{(\alpha)} d\bar{s} + \oint_{((-k_1/(k_1^2+k_2^2)) \cdot u_2)} d\bar{s} \\ &+ \oint_{(pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s}. \end{aligned} \quad (48)$$

Impending,

$$\begin{aligned} b_3 &= \oint_{(pu_1+(z+(\bar{k}_1/(\bar{k}_1^2+\bar{k}_2^2)))u_2+qu_3)} d\bar{s} + \oint_{((-k_1/(k_1^2+k_2^2)) \cdot u_2)} d\bar{s}, \\ l_{v_1} &= \oint_{(\alpha)} d\bar{s} + b_3. \end{aligned} \quad (49)$$

Substituting (32) into this last equation, we have

$$l_{v_1} = \oint_{(\alpha)} \left(\cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_1} \right) ds + b_3. \quad (50)$$

Since anchor curve is helix curve, then curvatures are constant. In this situation, we have

$$l_{v_1} = \left(\cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_1} \right) \oint_{(\alpha)} ds + b_3. \quad (51)$$

Substituting (5) into the last equation, we obtain

$$l_{v_1} = \left(\cos \varphi \frac{k_1}{k_1} - \sin \varphi \frac{k_2}{k_1} \right) l_{u_1} + b_3. \quad (52)$$

Theorem 20. Let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be Frenet frame of anchor curves of S and $\bar{S}$ closed parallel z-equidistant ruled surfaces. Then, the relations between dralls of ruled surfaces S and $\bar{S}$ are given as follows:

$$\begin{aligned} P_{v_1} &= P_{u_1} = 0, \\ P_{v_2} &= P_{u_2} \left(\cos \varphi + \sin \varphi \frac{k_1}{k_2} \right) \frac{d\bar{s}}{ds}, \\ P_{v_3} &= P_{u_3} \frac{k_2}{(\sin \varphi k_1 + \cos \varphi k_2)} \frac{d\bar{s}}{ds}. \end{aligned} \quad (53)$$

Proof. From (5), the drall of closed ruled surface which is generated vector v_1 is

$$P_{v_1} = 0. \quad (54)$$

From (5), the drall of closed ruled surface which is generated vector v_2 is

$$P_{v_2} = \frac{\bar{k}_2}{\bar{k}_1^2 + \bar{k}_2^2}. \quad (55)$$

Substituting (32) and (33) into the last equation, we have

$$\begin{aligned} P_{v_2} &= \left((\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\bar{s}} \right) \\ &\times \left(\left[(\cos \varphi k_1 - \sin \varphi k_2) \frac{ds}{d\bar{s}} \right]^2 \right. \\ &\left. + \left[(\sin \varphi k_1 + \cos \varphi k_2) \frac{ds}{d\bar{s}} \right]^2 \right)^{-1}, \end{aligned} \quad (56)$$

$$P_{v_2} = \cos \varphi \frac{k_2}{(k_1^2 + k_2^2)} \frac{d\bar{s}}{ds} + \sin \varphi \frac{k_2}{(k_1^2 + k_2^2)} \cdot \frac{k_1}{k_2} \frac{d\bar{s}}{ds}.$$

From (5),

$$P_{v_2} = P_{u_2} \left(\cos \varphi + \sin \varphi \frac{k_1}{k_2} \right) \frac{d\bar{s}}{ds}. \quad (57)$$

The drall of closed ruled surface which is generated vector v_3 is $P_{v_3} = 1/\bar{k}_2$,

$$P_{v_3} = \frac{1}{k_2} \frac{k_2}{\bar{k}_2}. \quad (58)$$

Substituting (5) and (33) into this last equation, we have

$$P_{v_3} = P_{u_3} \frac{k_2}{(\sin \varphi k_1 + \cos \varphi k_2)} \frac{d\bar{s}}{ds}. \quad (59)$$

□

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