Research Article

Controllability of Second-Order Multiagent Systems with Multiple Leaders and General Dynamics

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This paper proposes a new second-order discrete-time multiagent model and addresses the controllability of second-order multiagent system with multiple leaders and general dynamics. The leaders play an important role in governing the other member agents to achieve any desired configuration. Some sufficient and necessary conditions are given for the controllability of the second-order multiagent system. Moreover, the speed controllability of the second-order multiagent system with general dynamics is discussed. Particularly, it is shown that the controllability of the whole system relies on the number of leaders and the connectivity between the leaders and the members. Numerical examples illustrate the theoretical results.

1. Introduction

Controllability is one of the fundamental issues for coordinated control of multiagent systems which is partly due to the wide applications in communication and computation, as well as cooperative control [1–19].

So far, the issue of controllability shows new features and difficulties, and is still lacking in studies. In [4], the issue of controllability was firstly investigated by the nearest neighbor rules. Tanner had obtained necessary and sufficient conditions of the controllability for first-order multiagent dynamic systems regarding an agent as a leader or the external input. In [5, 7], the controllability for multiagent systems was investigated by the graph theoretic characterization. Moreover, Ji et al. [17] analyzed the multiagent controllability using tree topology. Jafari et al. [18] studied the structural controllability of multiagent systems. In [9, 15,

16], the authors discussed the controllability of discrete-time multiagent systems with a single leader or multiple leaders on fixed networks and switching networks, respectively, and obtained the necessary or sufficient controllable conditions for multiagent systems. References [10, 14] studied the controllability of continuous-time multiagent systems with time-delay and switching topology, respectively.

However, most of the recent research work focuses on the controllability of single integrator or first-order multiagent dynamic systems, such as [4–19]. But the controllability of double integrator or second-order multiagent dynamic systems was seldom studied. Motivated by the works above, in this paper, we focus on discussing the controllability of second-order discrete-time multiagent systems with general dynamic topology. Some sufficient and necessary conditions for controllability are presented. The main contributions of our paper lie in the following. (1) A novel model of

discrete-time multiagent system is a second-order. (2) The influence of leaders on the followers is investigated. (3) The controllability of such second-order discrete-time system with multiple leaders and general dynamics is considered, which cannot be found in the recent literatures. (4) The controllability and the speed controllability of the second-order multiagent system are discussed, respectively. (5) A sufficient and necessary condition for controllability of the second-order system is presented.

This paper is organized as follows. In Section 2, we present some concepts in graph theory. Section 3 gives the model to be studied. In Section 4, main results are presented. In Section 5, numerical examples and simulations are provided to illustrate the theoretical results. A conclusion is made in Section 6.

2. Preliminaries

In this section, some basic definitions and concepts in graph theory [20] are first introduced.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be an undirected graph of order n with the set of nodes $\mathcal{V} = \{1, 2, 3, \dots, n\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $\mathcal{G}$ is denoted by (i, j) , which is an unordered pair of distinct nodes of $\mathcal{V}$. If $i, j \in \mathcal{V}$, and $(i, j) \in \mathcal{E}$, then we say that j is a neighbor of i or i and j are adjacent. The neighborhood set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, where $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$, and a_{ij} is called the coupling weight of edge (i, j) .

Any undirected graph can be represented by its adjacency matrix A , which is a symmetric matrix. A diagonal matrix $D = \text{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^{n \times n}$ is a degree matrix of $\mathcal{G}$ with its diagonal elements $d_i = \sum_{j \in \mathcal{N}_i} a_{ij}$, $i = 1, 2, \dots, n$.

Then, the Laplacian of the graph $\mathcal{G}$ (or matrix A) is defined as

$$L = D - A \in \mathbb{R}^{n \times n}. \quad (1)$$

3. Model

Consider a second-order multiagent system with $N + n_l$ agents, labeled the first N agents from 1 to N as followers and the remainder n_l agents from $N + 1$ to $N + n_l$ as leaders, and each agent moves according to the following dynamics:

$$\begin{aligned} x_i(k+1) &= x_i(k) + v_i(k), \\ v_i(k+1) &= v_i(k) + u_i(k), \end{aligned} \quad (2)$$

with

$$\begin{aligned} u_i(k) &= - \sum_{i \in \mathcal{N}_{ij}} a_{ij} (v_i(k) - v_j(k)) \\ &\quad - \sum_{p \in \mathcal{N}_{ip}} \gamma_{ip} b_{ip} (v_i(k) - v_p(k)), \quad k \in J_k, \end{aligned} \quad (3)$$

where $x_i \in \mathbb{R}^m$ is the state of agent i ($i \in \underline{N} \triangleq \{N+1, \dots, N+n_l\}$) and $x_p \in \mathbb{R}^m$ is the state of agent p ($p \in \underline{N+n_l} \triangleq \{1, 2, \dots, N\}$). $\mathcal{N}_i$ presents the neighbor set of

agent i . $\mathcal{N}_{ij} \cup \mathcal{N}_{ip} = \mathcal{N}_i$, and $\mathcal{N}_{ij} \cap \mathcal{N}_{ip} = \emptyset$. J_k is a discrete-time index set. The coupling matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ with $a_{ij} \geq 0$ and $a_{ii} = 0$ represents the coupling strength among the followers, and $B = [b_{ip}] \in \mathbb{R}^{N \times n_l}$ with $b_{ip} > 0$ represents the coupling strength from the leaders to the followers. $\gamma_{ip} = 1$ if there is information from leader p to follower i ; otherwise $\gamma_{ip} = 0$.

Throughout this paper, it is assumed that the leader can influence the member followers but cannot be influenced by its neighbors.

Suppose $z = (x_1, x_2, \dots, x_N, v_1, v_2, \dots, v_N)^T$ and $y = (x_{N+1}, x_{N+2}, \dots, x_{N+n_l}, v_{N+1}, v_{N+2}, \dots, v_{N+n_l})^T$ be the state vector of all the followers and all the leaders, respectively. Then, (2) can be rewritten as

$$z(k+1) = Gz(k) + Hy(k), \quad k \in J_k, \quad (4)$$

where

$$G = \begin{pmatrix} I & I \\ 0 & F \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}, \quad (5)$$

$F = I - L - R$, I is the $N \times N$ identity matrix,

$$\begin{aligned} R &= \text{diag} \left\{ \sum_p \gamma_{1p} b_{1p}, \sum_p \gamma_{2p} b_{2p}, \dots, \sum_p \gamma_{Np} b_{Np} \right\} \in \mathbb{R}^{N \times N}, \\ P &= \begin{bmatrix} \gamma_{1(N+1)} b_{1(N+1)} & \gamma_{1(N+2)} b_{1(N+2)} & \cdots & \gamma_{1(N+n_l)} b_{1(N+n_l)} \\ \gamma_{2(N+1)} b_{2(N+1)} & \gamma_{2(N+2)} b_{2(N+2)} & \cdots & \gamma_{2(N+n_l)} b_{2(N+n_l)} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N(N+1)} b_{N(N+1)} & \gamma_{N(N+2)} b_{N(N+2)} & \cdots & \gamma_{N(N+n_l)} b_{N(N+n_l)} \end{bmatrix}. \end{aligned} \quad (6)$$

For simplicity, we denote (6) as

$$P = [p_{N+1}, p_{N+2}, \dots, p_{N+n_l}] \in \mathbb{R}^{N \times n_l}, \quad (7)$$

where

$$\begin{aligned} p_p &= (\gamma_{1p} b_{1p}, \gamma_{2p} b_{2p}, \dots, \gamma_{Np} b_{Np})^T \in \mathbb{R}^{N \times 1}, \\ q &\in \underline{N+n_l} - \underline{N}, \end{aligned} \quad (8)$$

and the matrix $L = [l_{ij}]$ with

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \text{ and } j \in \mathcal{N}_{ij} \\ \sum_{j \in \mathcal{N}_{ij}} a_{ij}, & i = j \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

It can be easily seen that the matrix L satisfies the following:

- (i) the off-diagonal elements are all negative or zero;
- (ii) the row sums are equal to the column sums with the value of zero.

4. Main Results

In the following, we first give the definition of controllability in second-order discrete-time system and the classical criterion of controllability.

Definition 1. A nonzero state z_0 of system (4) is controllable at the initial time $h \in J_k$ if there exists a finite time $l \in J_k, l > h$ and a control input $y(k)$, such that $z(h) = z_0$ and $z(l) = 0$. If any nonzero state z_0 of system (4) is controllable, then system (4) is said to be controllable. If $x(h) = x_0$ and $x(l) = 0$, then system (4) is position controllable, and if $v(h) = v_0$ and $v(l) = 0$, then system (4) is speed controllable.

Definition 2 (controllability matrix). The controllability matrix of system (4) is given by

$$Q = [H \quad GH \quad G^2H \quad \cdots \quad G^{2N-1}H], \quad (10)$$

where matrix $Q \in \mathbb{R}^{2N \times Nn_l}$.

Lemma 3 (Rank test for controllability). *System (4) is controllable if $\text{Rank}(Q) = 2N$.*

Lemma 4 (PBH rank test for discrete-time systems). *System (4) is controllable if (4) satisfies one of the following conditions:*

- (i) $\text{rank}(sI - G, H) = 2N$, for all $s \in \mathbb{C}$;
- (ii) $\text{rank}(\lambda_i I - G, H) = 2N$, where λ_i , for all $i = 1, 2, \dots, 2N$, is the eigenvalue of matrix G .

In general, for second-order multiagent systems, the controllable matrix is too hard to calculate. Therefore, we can use the PBH rank test to justify the controllability of such system. In the following, we will give a more simple and convenient theorem using the PBH rank test.

Theorem 5. *System (4) is controllable if $\text{Rank}(P) = m$.*

Proof. By Lemma 4, system (4) is controllable if $\text{rank}(\lambda_i I - G, H) = 2N$, where λ_i , for all $i = 1, 2, \dots, 2N$, is the eigenvalue of matrix G . Then, it is obvious to see that

$$\text{rank}(\lambda_i I - G \quad H) = \text{rank} \begin{pmatrix} \lambda_i I - I & -I & 0 \\ 0 & \lambda_i I - F & P \end{pmatrix} = 2N, \quad (11)$$

if $\text{Rank}(P) = m$. $\square$

Remark 6. From Theorem 5, we can find that the second-order multiagent system (4) is controllable if $n_l \geq N$ (n_l is the number of leaders and N the number of followers); otherwise, the system is always uncontrollable.

Remark 7. Notice that the direct consequence of Theorem 5 is that the controllability of the network (4) of a group of agents relies only on the connectivity between the leaders and members, regardless of the connectivity of the members in the network.

Corollary 8. *If $\text{Rank}(P) = N$, system (4) is speed controllable.*

FIGURE 1: The network topology 1.

FIGURE 2: A straight line configuration with speed.

Under the symmetry condition of the adjacent matrix A , we can have the following result.

Theorem 9. *System (4) is speed controllable if and only if the following conditions hold.*

- (i) *The eigenvalues of F are all distinct.*
- (ii) *All the eigenvectors of F are not orthogonal to at least one column in P simultaneously.*

The proof of Lemma 3 is similar to that of [15, Theorem 1], here omitted.

Remark 10. From Corollary 8 and Theorem 9, it can be easily seen that even though system (4) is speed controllable, system (4) cannot be completely controllable.

5. Numerical Examples and Simulations

This section presents some numerical examples and simulations to illustrate the theoretical results.

Example 1. Consider a seven-agent network with agents 4–7 as the leaders, where the topology of the network is described by Figure 1. From Figure 1, we can see that the number of the leaders is more than that of the followers. For simplicity, let

FIGURE 3: Second-order controllability.

$a_{ij} = b_{ip} = 1$ for $i, j = 1, 2, 3$ and $p = 4, 5, 6, 7$. According to Figure 1, the second-order multiagent system (4) is defined by

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad (12)$$

with

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad (13)$$

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

By computing, $\text{Rank}(P) = 3$ and $\text{Rank}(Q) = \text{Rank}[H \quad GH \quad G^2H \quad \dots \quad G^{2N-1}H] = 6$, then system (4) is indeed controllable.

Figures 2 and 3 show the simulation results of formation control of the second-order network. The follower agents (the black star dots) move from a random initial configuration to desired ones: aligning in a straight line (the black circles) with controllable speed as shown in Figures 2 and 3, respectively.

Example 2. A five-agent network with agents 4 and 5 as the leaders and with fixed topology described by the graph in

FIGURE 4: The network topology 2.

Figure 4. From Figure 4, we can see that the number of the leaders is less than that of the followers.

From Figure 1, the second-order multiagent system (4) is given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (14)$$

with

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (15)$$

FIGURE 5: Speed controllability.

Using Matlab calculation, the eigenvalues of F are

$$\begin{aligned} \text{eig}(F) &= \{-3.7321, -1, -0.2679\}, \\ U^T &= \begin{bmatrix} -0.3251 & 0.8881 & -0.3251 \\ -0.7071 & -0.0000 & 0.7071 \\ 0.6280 & 0.4597 & 0.6280 \end{bmatrix}, \\ U^T P &= \begin{bmatrix} 0.5630 & 0.5630 \\ -0.7071 & 0.7071 \\ 1.0877 & 1.0877 \end{bmatrix}. \end{aligned} \quad (16)$$

It is obvious that the eigenvalues of F are all different, and the elements of $U^T P$ are all nonzero. Therefore, system (4) is speed controllable. However, $\text{Rank}(P) = 2$ and $\text{Rank}(Q) = \text{Rank}[H \quad GH \quad G^2H \quad \dots \quad G^{2N-1}H] = 5$, then system (4) is uncontrollable.

Figure 5 shows the simulation results of the second-order network. It is easily seen that the speeds of the system are controllable, but the positions of the system are uncontrollable. Therefore, the whole second-order system is uncontrollable.

6. Conclusion

This paper has studied the controllability of discrete-time second-order multiagent systems with multiple leaders and general dynamic topology. By applying the PBH rank test technique, some effective sufficient and necessary conditions for the controllability of the multiagent discrete-time systems are obtained. The results in this paper show that the controllability of discrete-time second-order multiagent systems can only depend on the information from the leaders to the

followers, regardless of the connectivity of the members in the network. These studies are helpful in understanding the dynamics of interconnected systems. However, for discrete-time second-order multiagent systems, as in cases often encountered in practice, it is usually difficult to deal with the controllability problem due to complexity of the topology and lack of theoretical tools. Our main result shows an advantage of the second-order topology scheme. We anticipate that our solutions to the above-described problems will offer a theoretical basis and valuable ideas for future applications of networked multiagent systems in the field of coordination control, including formation control and tracking problems.

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