Research Article

Delay-Dependent Finite-Time and $L_2$-Gain Analysis for Switched Systems with Time-Varying Delay

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For the switched systems, switching behavior always affects the finite-time stability (FTS) property, which was neglected by most previous studies. This paper is mainly concerned with the problem of delay-dependent finite-time and $L_2$-gain analysis for switched systems with time-varying delay. Several less conservative sufficient conditions related to finite-time stability and boundness of switched system with time-varying delays are proposed; the system trajectory stays within a special bound with the information of switching signal. At last, a numerical example is also given to illustrate the efficiency of the developed method.

1. Introduction

During the past decades, the systems with time delays have received much attention since they often fall in various practical systems, for instance, neural networks, networked control systems, engineering systems, biology, economics, and other fields. However, time delay is always frequently the major cause of oscillation and instability; the stability for systems with time delays has been devoted to a lot of effort. Nowadays, the stability of time delay is parted into two classes: delay-independent stability criteria and delay-dependent ones. It should be pointed out that delay-independent criteria turn out to be more conservative, especially for the small-size delays. The problem of delay-dependent stability analysis for time-delay systems has received considerable attention, and lots of important results have also been reported [1–12].

In recent years, there has been an increasing interest in analysis of hybrid and switched systems due to their importance both in theory and its applications. As an important class of hybrid systems, switched linear systems comprise a collection of linear subsystems described by differential equations as well as a switching law to specify the switching during these subsystems. A switched system is a type of hybrid system with a combination of the discrete and continuous dynamical systems. These systems are recognized as models for appearance which cannot be denoted by exclusively continuous or discrete processes. Recently, based on the Lyapunov functions and other analysis tools, the corresponding stability and stabilization for both linear and nonlinear and switched systems have been investigated and many good results have been obtained; for a recent survey on this issue and related matters, one can also refer to [13–26].

Most of the existing results have concentrated on Lyapunov-based asymptotic stability for the switched systems; the behavior of that is above the infinite time interval. However, in practical engineering, there exists the bad transient characteristics; the system is asymptotically stable without unusable. For the systems which work in a short time interval, for instance, control system, missile system, and communication network system, which mainly care about the behavior over a finite time interval. To deal with this problem, in 1961, Dorato presented the idea of finite-time stability in [27]. Over the past few years, many study efforts have been dedicated to the finite-time stability (FTS) of switched
systems due to its wide applications. Comparing with the classical Lyapunov stability, which currently is the focus on a large and growing interdisciplinary area of research. To study the transient behavior of systems, FTS concerns the stability of a system over a finite interval of time and plays an important role. It is important to emphasize the disconnection between classical Lyapunov stability and finite-time stability. The problem about finite-time stabilization has been widely learned in the literature [28–44]. It is worth pointing out that there is a difference between finite-time stability and Lyapunov asymptotic stability, and they are also independent of each other.

Recently, some papers are found to be related to finite-time stability for switched systems. For example, based on the technique of average dwell time, the problem of finite-time stabilization for switched discrete-time systems was developed in [37]. The problem of finite-time stabilization for switched nonlinear discrete-time systems was discussed in [38]. But, to the best of the authors’ knowledge, the finite-time $L_2$-gain problems for switched systems have not been completely studied. The previous research has some conservatism of stability criterion; it is natural to look for an alternative way to reduce the conservatism of corresponding stability criteria. This idea motivates to our study.

The main contribution of this paper is that we present a novel approach for finite-time stability of the given switched system. Moreover, several sufficient conditions ensuring the finite-time stability and boundedness are proposed with different information we know about the switching signal. It shows the less conservative results when more information about the switching signal is available. By selecting the appropriate Lyapunov-Krasovski function, the sufficient conditions are obtained to guarantee finite-time stability of the systems and the closed-loop system trajectory stays in a special bound. The finite-time stability criteria can also be dealt with in the form of linear matrix inequalities and average dwell time. Finally, a numerical example is given to illustrate the effectiveness of the developed techniques.

2. Preliminaries

In this paper, the following switched systems are described as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - \tau (t)) + G_{\sigma(t)}\omega(t),$$

$$z(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}\omega(t), \quad t \geq 0, \quad (1)$$

$$\tilde{x}(t) = \phi(t), \quad t \in [-h, 0],$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system, $z(t) \in \mathbb{R}^m$ is the controlled output, and $\omega(t) \in L_2^2[0, \infty]$ is the external disturbance vector and satisfies the constraint:

$$\int_0^T \omega^T(t) \omega(t) \, dt \leq d, \quad d \geq 0. \quad (2)$$

$A_{\sigma(t)}$, $B_{\sigma(t)}$, $G_{\sigma(t)}$, $C_{\sigma(t)}$, and $D_{\sigma(t)}$ are the constant real matrices with appropriate dimension. $\tau(t)$ represents the mode-dependent time-varying state delay in the switched system and satisfies the following conditions:

$$0 < \tau(t) \leq h < \infty, \quad (3)$$

$$\dot{\tau}(t) \leq \tau, \quad (4)$$

where $h$ is the upper bound of the time-varying delay $\tau(t)$ and $\tau$ is the variation rate of the time-varying delay $\tau(t)$. $\phi(t)$ is the differentiable vector-valued initial function on $[-h, 0]$. $\sigma(t) : [0, \infty) \to \mathcal{N} = \{1, 2, \ldots, N\}$ is the right continuous piecewise constant switching signal to be designed.

Corresponding to the switching signal $\sigma(t)$, we get the following switching sequence:

$$\sum = \{x_0; (t_0, t_1), (t_1, t_2), \ldots, (i_k, t_k), \ldots, | i_k \in \mathcal{N}, k = 0, 1, \ldots \}, \quad (5)$$

where $t_0$ is the initial time when $t_k \in [t_k, t_{k+1})$, $x(t_0)$ is the initial state and $i_k$th subsystem is active. Therefore, the trajectory $x(t)$ of the switched system (1) is called the trajectory of the $i_k$th subsystem. As assumed before, we get rid of Zeno behavior for all types related to switching signal. Throughout this paper, we assumed that the state of the switched system (1) does not jump at the switching instants; that is, the trajectory $x(t)$ is continuous everywhere.

Firstly, we will give the definitions and lemmas about switched system (1), which plays an major role in the derivation of our results.

**Definition 1.** Switch system (1) is said to be finite-time bounded with respect to $(c_1, c_2, T, S, d)$, if condition (6) holds:

$$\max_{\tau(t) \leq h, 0} \{x^T(t_0) S x(t_0), x^T(t_0) S \dot{x}(t_0)\}$$

$$\leq c_1 \Rightarrow x^T(t) S x(t) < c_2, \quad (6)$$

$$\forall t \in [0, T], \quad \forall \omega(t) : \int_0^T \omega^T(s) \omega(s) \, ds \leq d,$$

where $c_2 > c_1 \geq 0$ and $S > 0$.

**Definition 2.** For $\gamma > 0$, $d > 0$, $T > 0$, $\eta > 0$, $\Lambda > 0$, and $c_2 > c_1 > 0$, system (3) is said to be finite-time stable with a weighted $L_2$ performance $\gamma$ with respect to $(c_1, c_2, T, S, d)$, if the following condition holds:
\[
\int_0^T \left[ \eta s - \ln \frac{\lambda c_2}{\Delta c_1 + d y^2 (1/\eta) (1 - e^{\theta T})} \right] z^T(s) z(s) \, ds \\
\leq y^2 e^{-\eta T} \int_0^T \omega^T(\tau) \omega(\tau) \, d\tau,
\]
and under zero initial condition, it holds for all nonzero \(\omega\): \(\int_0^T \omega^T(\tau) \omega(\tau) \, d\tau \leq d\).

**Definition 3** (see [21]). For any \(T_2 > T_1 \geq 0\), let \(N_\sigma(T_1, T_2)\) denote the switching number of \(\sigma(t)\) during \((T_1, T_2)\). If \(N_\sigma(T_1, T_2) \leq N_0 + (T_2 - T_1)/\tau_a\) holds for \(N_0 \geq 0\) and \(\tau_a > 0\), then \(N_\sigma\) and \(\tau_a\) are called chattering bound and average dwell time, respectively. Here we assume \(N_0 = 0\) for simplicity as supported by reference.

**Lemma 4** (see [9]). Let \(f_i : \mathbb{R}^m \rightarrow \mathbb{R}\ (i = 1, 2, \ldots, N)\) have positive values in an open subset \(D\) of \(\mathbb{R}^m\). Then, the reciprocally convex combination of \(f_i\) over \(D\) satisfies

\[
\min_{\{\beta_i : \beta_i > 0, \sum_i \beta_i = 1\}} \frac{1}{T} \sum_i \beta_i f_i(t) = \sum_i f_i(t) + \max_{g_i, \langle i \neq j \rangle} g_{i,j}(t)
\]
subject to

\[
\left\{ g_{ij} : \mathbb{R}^m \rightarrow \mathbb{R}, \ g_{ij}(t) = g_{i,j}(t), \ \left[ f_i(t), g_{i,j}(t), f_j(t) \right] \geq 0 \right\}.
\]

**Lemma 5** (see [11]). For any constant matrix \(Z \in \mathbb{R}^{n \times n}\), \(Z = Z^T > 0\), scalars \(h > 0\), such that the following integrations are well defined; then

\[
-h \int_{-h}^{t} x^T(s) Z x(s) \, ds \leq -\left[ \int_{-h}^{t} x^T(s) \, ds \right]^T Z \left[ \int_{-h}^{t} x(s) \, ds \right] - \frac{h^2}{2} \int_{-h}^{t} x^T(s) Z x(s) \, ds \, d\theta
\]

\[
\leq -\left[ \int_{-h}^{t} x^T(s) \, ds \right]^T Z \left[ \int_{-h}^{t} x(s) \, ds \right] \leq 0.
\]

**Lemma 6** (Schur’s complement). Given constant matrices \(X, Y, Z\), where \(X = X^T\) and \(0 < Y = Y^T\), then \(X + Z Y^{-1} Z < 0\) if and only if

\[
\begin{bmatrix}
X & Z^T \\
* & -Y
\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}
* & Y \\
- & X
\end{bmatrix} < 0.
\]

**3. Finite-Time Boundedness Analysis**

**Theorem 7.** System (3) is said to be finite-time stability with respect to \((c_1, c_2, R, d, T)\) if there exist symmetric positive matrices \(P_i, Q_i, s = 1, 2, \ X_{ki} (k = 1, 2), Y_i\) and matrices \(M_i, N_i, V_i\), scalars \(\alpha > 0, \mu \geq 1, \lambda_i > 0\ (i = 1, 2, \ldots, 8)\), \(d > 0, \ h > 0, \ \Lambda > 0, \ \tau > 0, \ r(r);\) such that \(\forall i, j \in \mathcal{N}\), we have the following linear matrix inequalities:

\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & W_i & A_i V_i & h A_i^T M_i + Y_i + Y_i^T & Y_i + Y_i^T & P_i G_i \\
\Xi_{21} & -W_i + X_{2i}/h & B_i V_i^T & h B_i^T M_i^T & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & W_i & A_i V_i^T & Y_i + Y_i^T & h A_i^T N_i^T + Y_i + Y_i^T & P_i G_i \\
\Xi_{21} & -W_i + X_{2i}/h & B_i V_i^T & h B_i^T N_i^T & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & W_i & A_i^T V_i^T & h A_i^T M_i + Y_i + Y_i^T & Y_i + Y_i^T & P_i G_i \\
\Xi_{21} & -W_i + X_{2i}/h & B_i^T V_i & h B_i^T M_i & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

with the average dwell time of the switching signal \(\sigma\) satisfying

\[
\tau_a > \tau_a^* = \frac{T \ln \mu}{\ln (\lambda_1 c_2) - \ln [\Lambda c_1 + d l_8 (1 - e^{-\eta T})] - \eta T}.
\]
where
\[
\Xi_{11i} = \delta P_i + P_i A_i + A_i^T P_i + e^{\delta h} (Q_{11i} + Q_{22i}) + hX_{1i} - Y_i - 2Y_{1i} + 2Y_{2i},
\]
\[
\Xi_{12i} = P_i B_i + \frac{X_{2i}}{h} - W_i,
\]
\[
\Xi_{22i} = r(\tau)Q_{11i} + \frac{X_{2i}}{h} - W_i - W_i^T - \frac{X_{2i}}{h},
\]
\[
\Xi_{4i} = hX_{2i} e^{\delta h} - \frac{\delta h - 1}{\delta^2} Y_i - V_i - V_i^T,
\]
\[
r(\tau) = \begin{cases} 
(1 - \tau) e^{\delta h}, & \text{if } \tau > 1; \\
(1 - \tau), & \text{if } \tau \leq 1.
\end{cases}
\]  

Consider
\[
\Lambda = \lambda_2 + h e^{\delta h} (\lambda_3 + \lambda_4) + h^2 e^{\delta h} (\lambda_5 + \lambda_6) + \frac{1}{2} \lambda_7 h e^{\delta h},
\]
\[
\lambda_1 = \min_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\min}(\mathcal{P}_i) \right\}, \\
\lambda_2 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{P}_i) \right\}, \quad \lambda_3 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{Q}_{1i}) \right\}, \\
\lambda_4 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{Q}_{2i}) \right\}, \\
\lambda_5 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{X}_{1i}) \right\}, \\
\lambda_6 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{X}_{2i}) \right\}, \\
\lambda_7 = \max_{i \in \mathcal{I}, \mathcal{F}} \left\{ \lambda_{\max}(\mathcal{H}_i) \right\}.
\]

**Proof.** Define \(x_i(s) = x(t + s)\). We consider the following Lyapunov-Krasovskii functional:
\[
\begin{align*}
V_{\alpha(t)}(x_i, t) &= \sum_{i=1}^{4} V_{\alpha(t)}(x_i, t), \\
V_{\alpha(t)}(x_i, t) &= x^T(t) e^{\delta t} P_{\alpha(t)} x(t), \\
V_{2\alpha(t)}(x_i, t) &= \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{2\alpha(t)} x(s) ds + \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{2\alpha(t)} x(s) ds, \\
V_{3\alpha(t)}(x_i, t) &= \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{3\alpha(t)} x(s) ds + \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{3\alpha(t)} x(s) ds, \\
V_{4\alpha(t)}(x_i, t) &= \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{4\alpha(t)} x(s) ds + \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) Q_{4\alpha(t)} x(s) ds.
\end{align*}
\]

Taking the time derivative of \(V_{\alpha(t)}(x_i, t)\) along the trajectory of the system (1), one has
\[
\begin{align*}
\dot{V}_{1i}(x_i, t) &= x^T(t) e^{\delta t} (\delta P_i + P_i A_i + A_i^T P_i + e^{\delta h} (Q_{11i} + Q_{22i})) x(t) + 2e^{\delta t} x^T(t) P_i B_i x(t - \tau(t)) \\
&\quad + 2e^{\delta t} x^T(t) P_i G_i \omega(t), \\
\dot{V}_{2i}(x_i, t) &= e^{\delta t} x^T(t) e^{\delta h} (Q_{11i} + Q_{22i}) x(t) + e^{\delta t} x^T(t - h) Q_{1i} x(t - h) \\
&\quad - e^{\delta t} x^T(t - \tau(t)) Q_{2i} x(t - \tau(t)).
\end{align*}
\]

Since
\[
0 \leq \tau(t) \leq h,
\]
we define
\[
r(\tau) = \begin{cases} 
(1 - \tau) e^{\delta h}, & \text{if } \tau > 1; \\
(1 - \tau), & \text{if } \tau \leq 1.
\end{cases}
\]  

Then,
\[
\begin{align*}
\dot{V}_{2i}(x_i, t) &= e^{\delta t} x^T(t) e^{\delta h} (Q_{11i} + Q_{22i}) x(t) \\
&\quad - e^{\delta t} x^T(t - h) Q_{1i} x(t - h) \\
&\quad + e^{\delta t} x^T(t - \tau(t)) r(\tau) Q_{2i} x(t - \tau(t)), \\
\dot{V}_{3i}(x_i, t) &= e^{\delta t} x^T(t) hX_{1i} x(t) + e^{\delta t} x^T(t) hX_{2i} x(t) \\
&\quad - \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) X_{1i} x(s) ds \\
&\quad - \int_{t-h}^{t} e^{\delta(t-s)} x^T(s) X_{2i} x(s) ds \\
&\quad \leq e^{\delta t} x^T(t) hX_{1i} x(t) + e^{\delta t} x^T(t) hX_{2i} x(t) \\
&\quad - e^{\delta t} x^T(s) X_{1i} x(s) ds \\
&\quad - e^{\delta t} x^T(s) X_{2i} x(s) ds.
\end{align*}
\]

By Lemma 5, one can obtain
\[
\begin{align*}
- \int_{t-h}^{t} x^T(s) X_{1i} x(s) ds &\leq -\tau(t) U^T_1 X_{1i} U_1, \\
- \int_{t-h}^{t} x^T(s) X_{2i} x(s) ds &\leq -(h - \tau(t)) U^T_2 X_{1i} U_2.
\end{align*}
\]
where

\[ U_1 = \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} x(s) ds, \quad U_2 = \frac{1}{h - \tau(t)} \int_{t-h}^{t-\tau(t)} x(s) ds, \]

\[
\lim_{\tau(t) \to 0} \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} x(s) ds = x(t), \]

\[
\lim_{\tau(t) \to h} \frac{1}{h - \tau(t)} \int_{t-h}^{t-\tau(t)} x(s) ds = x(t-h).
\]

(27)

By using the Leibniz–Newton formula, the following equation is true for any matrices \( M_i, N_i, \) and \( V_i \) with appropriate dimensions:

\[
(2\tau(t) U_1^{\top} M_i + 2(h - \tau(t)) U_2^{\top} N_i + 2\dot{x}(t) V_i) \times [-\dot{x}(t) + A_i x(t) + B_i x(t - \tau(t)) + G_i \omega(t)] = 0.
\]

(28)

From Lemma 4, it yields

\[
-\int_{t-h}^{t} \dot{x}(s) X_2 \dot{x}(s) ds
= -\int_{t-\tau(t)}^{t} \dot{x} X_2 \dot{x}(s) ds
- \int_{t-h}^{t} \dot{x} X_2 \dot{x}(s) ds
\leq - \frac{h}{h - \tau(t)} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix}^T \begin{bmatrix} x(t) - x(t - \tau(t)) \\ x(t - \tau(t)) - x(t - h) \end{bmatrix} \]

\[
\times \begin{bmatrix} 2X_1 \\ h X_2 \end{bmatrix}^T \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \begin{bmatrix} x(t) - x(t - \tau(t)) \\ x(t - \tau(t)) - x(t - h) \end{bmatrix}
\leq - \frac{2}{(h - \tau(t))^2} \begin{bmatrix} x(t) - x(t - \tau(t)) \\ x(t - \tau(t)) - x(t - h) \end{bmatrix}^T \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \begin{bmatrix} x(t) - x(t - \tau(t)) \\ x(t - \tau(t)) - x(t - h) \end{bmatrix}
\]

\[
\dot{V}_{\omega_i}(x_i, t) = e^{\delta t} \dot{x}^3(t) Y_i \dot{x}(t) \int_{0}^{t} e^{-\delta \theta} d\theta
- \int_{t-h}^{t} \dot{x}^3(s) Y_i \dot{x}(s) ds d\theta

= e^{\delta t} \dot{x}^3(t) \left( e^{\delta h} - \frac{\delta h}{\delta t} - 1 \right) Y_i \dot{x}(t)
- e^{\delta t} \int_{t-h}^{t} \dot{x}^3(s) Y_i \dot{x}(s) ds d\theta.
\]

(29)

By using Lemma 5, we have

\[
-\int_{t-h}^{t} \dot{x}^3(s) Y_i \dot{x}(s) ds d\theta
= -\int_{-\tau(t)}^{t} \dot{x}^3(s) Y_i \dot{x}(s) ds d\theta
- \int_{-h}^{t-\tau(t)} \dot{x}^3(s) Y_i \dot{x}(s) ds d\theta
\leq - \frac{2}{(h - \tau(t))^2} \left( \int_{t-h}^{t} \dot{x}^3(s) ds d\theta \right)^T
\times Y_i \left[ \int_{t-\tau(t)}^{t} \dot{x}(s) ds d\theta \right]
- \frac{2}{(h - \tau(t))^2} \left( \int_{-h}^{t-\tau(t)} \dot{x}^3(s) ds d\theta \right)^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
= - \frac{2}{(h - \tau(t))^2} \left[ x(t) - x(t - \tau(t)) \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
- \frac{2}{(h - \tau(t))^2} \left[ x(t - \tau(t)) - x(t - \tau(t)) \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
= - \frac{2}{(h - \tau(t))^2} \left[ x(t) - \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} x(s) ds \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
- \frac{2}{(h - \tau(t))^2} \left[ x(t - \tau(t)) - \frac{1}{h - \tau(t)} \int_{t-h}^{t} x(s) ds \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
= -2 \frac{\tau(t)}{\tau(t)} \left[ x(t) - \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} x(s) ds \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
- 2 \frac{\tau(t)}{h - \tau(t)} \left[ x(t - \tau(t)) - \frac{1}{h - \tau(t)} \int_{t-h}^{t} x(s) ds \right]^T
\times Y_i \left[ \int_{t-h}^{t} \dot{x}(s) ds d\theta \right]
= -2 \left[ x(t) - U_1 \right]^T Y_i \left[ x(t) - U_1 \right]
- 2 \left[ x(t) - U_2 \right]^T Y_i \left[ x(t) - U_2 \right].
\]

(30)

Therefore, for a given \( \eta > 0 \) and from (21)–(30), one can obtain that

\[
\dot{V}_i(x_i, t) - \eta \omega^3 H_\omega(t) \leq e^{\delta t} \xi_i^3(t) \Xi(t),
\]

(31)

where
The LMIs (11) lead to $\tau(t) \to h$ and to $\tau(t) \to 0$, respectively. It is easy to see that $\Xi_{11}$ results from $\Xi_{21}$ | $\tau(t) = h$ and $\Xi_{22}$ results from $\Xi_{12}$ | $\tau(t) = 0$. Thus, we obtain

Thus, from (35)–(37), it yields

$$V_{\sigma(t)}(x_i, t) \leq e^{\eta(t-t_k)} V_{\sigma(t_k)}(x_{i_k}, t_k) + \eta \int_{t_k}^{t} e^{\eta(s-t)} \omega^T(s) H_i \omega(s) \, ds \leq \mu e^{\eta(t-t_k)} V_{\sigma(t_k)}(x_{i_k}, t_k) + \eta \int_{t_k}^{t} e^{\eta(s-t)} \omega^T(s) H_i \omega(s) \, ds$$

Integrating (34) from $t_k$ to $t$, we can get that

$$V_i(x_i, t) < e^{\eta(t-t_k)} V_i(x_{i_k}, t_k) + \eta \int_{t_k}^{t} e^{\eta(s-t)} \omega^T(s) H_i \omega(s) \, ds.$$  

(35)
\[ + \eta \int_0^1 e^{\eta (t-s)} \mu N_\omega(s) \omega(1) H_1 \omega(s) \, ds \leq \mu N_\omega \eta TV_{\sigma(0)} (x_0, 0) \]
\[ + \eta \mu N_\sigma d \lambda_{\text{max}} (H) e^T \int_0^1 e^{-\eta ds} \, ds \leq \mu N_\sigma e^T \]
\[ \times \left\{ V_{\sigma(0)} (x_0, 0) + d \lambda_{\text{max}} (H) \eta \int_0^T e^{-\eta ds} \right\} \leq \mu N_\sigma e^T \left\{ V_{\sigma(0)} (x_0, 0) + d \lambda_{\text{max}} (H) \left( 1 - e^{-\eta T} \right) \right\} \]
\[ \leq \mu N_\sigma e^T \left\{ V_{\sigma(0)} (x_0, 0) + d \lambda_b \left( 1 - e^{-\eta T} \right) \right\} . \]  

(38)

Define $\bar{P}_i = R^{-1/2} P_i R^{-1/2}$, $\bar{Q}_i = R^{-1/2} Q_i R^{-1/2}$ ($i = 1, 2$), $\bar{X}_l = R^{-1/2} X_l R^{-1/2}$ ($l = 1, 2$), $\bar{Y}_l = R^{-1/2} Y_l R^{-1/2}$.

Note that
\[ V_{\sigma(0)} (x_0, 0) = \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{P}_i \right) x^T (0) \, R x (0) \]
\[ + \left( \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Q}_1 \right) + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Q}_2 \right) \right) e^T \int_0^T e^{-\eta ds} x^T (s) R x (s) \, ds \]
\[ \times \int_{-h}^0 e^{\delta x} x^T (s) R x (s) \, ds \]
\[ = \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{X}_1 \right) e^{\delta h} \int_{-h}^0 e^{-\delta \lambda x^T (s) R x (s)} \, ds \, d\theta \]
\[ + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{X}_2 \right) e^{\delta h} \int_{-h}^0 e^{-\delta \lambda x^T (s) R x (s)} \, ds \, d\theta \]
\[ + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Y}_1 \right) e^{\delta h} \int_{-h}^0 e^{-\delta \lambda x^T (s) R x (s)} \, ds \, d\theta \]
\[ + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Y}_2 \right) e^{\delta h} \int_{-h}^0 e^{-\delta \lambda x^T (s) R x (s)} \, ds \, d\theta \leq \left\{ \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{P}_i \right) \right\} \]
\[ + \delta e^{\delta h} \left( \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Q}_1 \right) + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Q}_2 \right) \right) \]
\[ + \delta^2 e^{\delta h} \left( \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{X}_1 \right) + \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{X}_2 \right) \right) \]
\[ + \frac{1}{2} \delta^3 e^{\delta h} \max_{i \in \mathcal{F}} \lambda_{\text{max}} \left( \bar{Y}_1 \right) \]
\[ \times \sup_{-h \leq s \leq 0} \{ x^T (s) R x (s), \dot{x}^T (s) R \dot{x} (s) \} \]
\[ \leq \left( \lambda_2 + h e^{\delta h} (\lambda_3 + \lambda_4) \right) \]
\[ + h^2 e^{\delta h} (\lambda_5 + \lambda_6) + \frac{1}{2} \lambda_7 h^2 e^{\delta h} \]
\[ \times \sup_{-h \leq s \leq 0} \{ x^T (s) R x (s), \dot{x}^T (s) R \dot{x} (s) \} \]
\[ \leq \left( \lambda_2 + h e^{\delta h} (\lambda_3 + \lambda_4) + h^2 e^{\delta h} (\lambda_5 + \lambda_6) \right) \]
\[ + \frac{1}{2} \lambda_7 h^3 e^{\delta h} \]
\[ \times \sup_{-h \leq s \leq 0} \{ x^T (s) R x (s), \dot{x}^T (s) R \dot{x} (s) \} \]
\[ \leq \left( \lambda_2 + h e^{\delta h} (\lambda_3 + \lambda_4) + h^2 e^{\delta h} (\lambda_5 + \lambda_6) \right) \]
\[ + \frac{1}{2} \lambda_7 h^3 e^{\delta h} \right \} c_1 = \Lambda c_1. \]  

Thus,
\[ V_{\sigma(t)} (x_t, t) \leq \mu T e^{\eta T} \left\{ \Lambda c_1 + \lambda_b \left( 1 - e^{-\eta T} \right) \right\} \]
\[ = \delta^{(\alpha + \ln \mu / \tau \gamma + \eta T)} \left\{ \Lambda c_1 + \lambda_b \left( 1 - e^{-\eta T} \right) \right\} . \]  

(40)

On the other hand,
\[ V_{\sigma(t)} (x_t, t) \geq \min_{i \in \mathcal{F}} \{ \bar{P}_i \} x^T (t) R x (t) = \lambda_1 x^T (t) R x (t) . \]  

(41)

From (40) and (41), one obtains
\[ x^T (t) R x (t) \leq \frac{\Lambda c_1 + \lambda_b \left( 1 - e^{-\eta T} \right)}{\lambda_1} \delta^{(\alpha + \ln \mu / \tau \gamma + \eta T)} . \]  

(42)

When $\mu = 1$, which is the trivial case, from (13), $x^T (t) R x (t) < c_2 e^{-\eta T} \delta^{(\alpha + \eta T)} = c_2$. When $\mu \geq 1$, from (13), $\ln (\lambda_1 c_2) - \ln [\Lambda c_1 + \lambda_b (1 - e^{-\eta T})] - \alpha T > 0$, we have
\[ \frac{T}{\tau_a} < \frac{\ln (\lambda_1 c_2) - \ln \left[ \Lambda c_1 + \lambda_b \left( 1 - e^{-\eta T} \right) \right] - \eta T}{\ln \mu} \]
\[ = \frac{\ln (\lambda_1 c_2 e^{-\alpha T} / \left( \Lambda c_1 + \lambda_b \left( 1 - e^{-\eta T} \right) \right) )}{\ln \mu} . \]  

(43)

Substituting (43) into (42) yields
\[ x^T (t) R x (t) < c_2. \]  

(44)

The proof is completed. \(\square\)

**Remark 8.** It should be mentioned that to reduce the conservatism, $-\int_{-\tau(t)}^0 \int_{-\theta}^0 \dot{x}^T (s) Z_x x dsd\theta$ and $-\int_{-\tau(t)}^0 \int_{-\theta}^0 \dot{x}^T (s) Z_x x dsd\theta$ are bounded with $-2[x(t) - U_1] R^T Z_x [x(t) - U_1]$ and $-2[x(t) - U_2] R^T Z_x [x(t) - U_2]$, respectively.

**Remark 9.** One can clearly see that criteria given in Theorem 7 are delay-dependent. It is well known that the delay-dependent criteria are less conservative than delay-independent criteria when the delay is small or belongs to a given interval. Interval time-varying delay is a time delay that varies in an interval in which the lower bound is not restricted to zero.

**Remark 10.** In Theorem 7, the free weight matrix technology is employed, which reduces the conservativeness of the stability condition.
### 4. Finite-Time Weighted $L_2$-Gain Analysis

**Theorem 11.** System (3) is finite-time bounded with respect to $(c_1, c_2, R, d, T)$ if there exist symmetric positive matrices $P_i$, $Q_{ij}$, $X_{ij}$, $Y_i$, and matrices $M_i$, $N_i$, $V_i$, scalars $\alpha \geq 0$, $\mu \geq 1$, $\Lambda_i > 0$ ($i = 1, 2, \ldots, 7$), $d > 0$, $h > 0$, $\Lambda > 0$, $\gamma > 0$, $\tau > 0$, $r(\tau)$, such that for all $\mathcal{V} \in \mathcal{N}$; we have the following linear matrix inequalities:

$$
\Sigma_1 = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & W_i & A_i^T V_i^T & hA_i^T M_i^T + Y_i + Y_i^T & Y_i + Y_i^T & P_i G_i & C_i^T \\
* & \Xi_{21} & -W_i + \frac{X_{2i}}{h} B_i^T V_i^T & hB_i M_i^T & 0 & 0 & 0 & 0 \\
* & * & -Q_{ij} - \frac{X_{2j}}{h} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44} & -hM_i^T & 0 & V_i G_i & 0 \\
* & * & * & * & -hX_{1i} - Y_i - Y_i^T & 0 & hM_i G_i & 0 \\
* & * & * & * & * & -Y_i - Y_i^T & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & D_i^T \\
* & * & * & * & * & * & * & -I \\
\end{bmatrix} < 0,
$$

(45)

$$
\Sigma_2 = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & W_i & A_i^T V_i^T & hA_i^T N_i^T + Y_i + Y_i^T & Y_i + Y_i^T & P_i G_i & C_i^T \\
* & \Xi_{21} & -W_i + \frac{X_{2i}}{h} B_i^T V_i^T & hB_i N_i^T & 0 & 0 & 0 & 0 \\
* & * & -Q_{ij} - \frac{X_{2j}}{h} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44} & -hN_i^T & 0 & V_i G_i & 0 \\
* & * & * & * & -hX_{1i} - Y_i - Y_i^T & 0 & hN_i G_i & 0 \\
* & * & * & * & * & -Y_i - Y_i^T & 0 & 0 \\
* & * & * & * & * & * & -\gamma^2 I & D_i^T \\
* & * & * & * & * & * & * & -I \\
\end{bmatrix} < 0,
$$

When $t \in [t_k, t_{k+1}]$, where $t_k$ is the $k$th switching instant,

$$
V_{\sigma(t)}(x_i, t) < e^{\gamma(t-t_k)} V_{\sigma(t_k)}(x_i, t_k) - \int_{t_k}^{t} e^{\gamma(t-s)} J(s) ds. 
$$

(51)

Notice that $x(t_k) = x(t'_k)$; then one obtains

$$
V_{\sigma(t_k)}(x(t_k), t_k) \leq \mu V_{\sigma(t'_k)}(x(t'_k), t_k).
$$

(52)

For any $t \in [0, T]$, one has

$$
V_{\sigma(t)}(x_i, t) \leq e^{\gamma(t-t_{k+1})} V_{\sigma(t_{k+1})}(x_i, t_{k+1}) + \eta \int_{t_k}^{t} e^{\gamma(t-s)} J(s) ds \\
\leq \mu e^{\gamma(t-t_{k+1})} V_{\sigma(t_{k+1})}(x_i, t_{k+1}) \\
+ \eta \mu \int_{t_{k-1}}^{t} e^{\gamma(t-s)} J(s) ds + \eta \int_{t_k}^{t} e^{\gamma(t-s)} J(s) ds \\
+ \eta \mu \int_{t_{k-1}}^{t} e^{\gamma(t-s)} J(s) ds \\
+ \eta \int_{t_k}^{t} e^{\gamma(t-s)} J(s) ds
$$

(53)

### Proof.
Choose the same Lyapunov-Krasovskii functional as in Theorem 7; after some mathematical manipulation and Schur complement, we can get

$$
V_{\sigma(t)}(x_i, t) + e^T(t) e(t) - \gamma^2 \omega^T(t) \omega(t) = \xi^T(t) \Sigma_\alpha \xi(t),
$$

(48)

with the average dwell time of the switching signal $\sigma$ satisfying

$$
\tau_a > \tau_a^* = \frac{T \ln \mu}{\ln (\lambda_1 c_2) - \ln (\Lambda c_1 + d y^2 (1/\eta) (1 - e^{-\eta T})) - \eta T},
$$

(47)

Define

$$
J(t) = e^T(t) e(t) - \gamma^2 \omega^T(t) \omega(t).
$$

(49)

From (45), we obtain

$$
\dot{V}_{\sigma(t)}(x_i, t) - \eta V_{\sigma(t)}(x_i, t) + J(t) < 0.
$$

(50)
\[ \leq \cdots \leq \mu^{N_\sigma(0,t)} e^{\Psi_V(0,t)} (x_0,0) + \eta \mu^{N_\sigma(0,t)} \int_0^{t_1} e^{\eta(t-s)} f(s) ds \]
\[ + \cdots + \eta \int_0^t e^{\eta(t-s)} f(s) ds \]
\[ \leq \mu^{N_\sigma(0,T)} e^{\Psi_V(0,T)} (x_0,0) + \eta \mu^{N_\sigma(0,T)} \int_0^T e^{\eta(t-s)} \mu^{N_\sigma(s,T)} f(s) ds. \]

Under zero initial condition, we have
\[ \int_0^T e^{-\eta s} \mu^{N_\sigma(s,T)} f(s) ds < 0, \]
which implies that
\[ \int_0^T e^{-\alpha s} \mu^{N_\sigma(s,T)} z^T(s) z(s) ds \]
\[ < \int_0^T \mu^{N_\sigma(s,T)} \gamma^2 \omega^T(s) \omega(s) ds. \]

Multiplying both sides of (55) by \( e^{-N_\sigma(0,T)} \) yields
\[ \int_0^T e^{-\alpha s} \mu^{N_\sigma(0,s)} z^T(s) z(s) ds \]
\[ < \int_0^T \mu^{-N_\sigma(0,s)} \gamma^2 \omega^T(s) \omega(s) ds. \]

It is easy to deduce from (47) that
\[ N_\sigma(0,s) \leq \frac{2}{\tau_a} \left( \ln \left( \lambda_{1,2} / \left( \Lambda c_1 + d \gamma^2 (1/\eta)(1-e^{\eta T}) \right) \right) - \eta s \right) / \ln \mu. \]

Since \( \mu \geq 1 \), we have
\[ \int_0^T \mu^{\ln(\eta s - \ln(\lambda_{1,2} / (\Lambda c_1 + d \gamma^2 (1/\eta)(1-e^{\eta T}))) / \ln \mu)} \]
\[ \times z^T(s) z(s) ds \]
\[ \leq \int_0^T \mu^{N_\sigma(0,s)} e^{-\eta s} z^T(s) z(s) ds \]
\[ \leq \int_0^T \mu^{N_\sigma(0,s)} e^{-\eta s} \gamma^2 \omega^T(s) \omega(s) ds \]
\[ \leq e^{-\eta T} \int_0^T \gamma^2 \omega^T(s) \omega(s) ds. \]

Therefore, we can obtain
\[ \int_0^T \left[ \eta s - \ln \left( \Lambda c_1 + d \gamma^2 (1/\eta)(1-e^{\eta T}) \right) \right] z^T(s) z(s) ds \]
\[ \leq \gamma^2 e^{-\eta T} \int_0^T \omega^T(s) \omega(s) ds. \]

According to Definition 2, we know that Theorem 11 holds. This completes the proof. \( \square \)

Remark 12. It can be seen that Theorem 11 presents a more general result than [33–42] which contain the decay \( \eta \); such a condition is derived by advantage of the simultaneous usage of the Lyapunov functional and the state variable transformation.

Remark 13. In many real applications, the minimum value of \( \gamma^2_{\min} \) is of interest. In Theorem 11, with a fixed \( \mu \) and \( \eta \), \( \gamma^2_{\min} \) can be obtained through following procedure:
\[ \min \gamma^2 \]
\[ \text{s.t. LMIs (37)–(39)).} \]

5. Illustrative Example

Example 1. Consider the system (1) with the following data:
\[ A_1 = \begin{bmatrix} -1.7 & 1.7 & 0 \\ 1.3 & -1 & 0.7 \\ 0.7 & 1 & -0.6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0.7 & 1 & -0.6 \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} 1.5 & -1.7 & 0.1 \\ -1.3 & 1 & -0.3 \\ -0.7 & 1 & 0.6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.3 & -1 & 0.1 \\ 1.5 & 1 & 0.6 \end{bmatrix}, \]
\[ G_1 = G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad x(t) = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}, \]
\[ \omega(t) = \begin{bmatrix} 0.03 \sin(t) \\ 0.02 \cos(2t) \\ 0.015 \sin(t + 1) + \cos(t - 2) \end{bmatrix}, \quad t \in [-\tau, 0]. \]

The values of \( c_1, c_2, T, d \) and matrix \( R \) are given as follows:
\[ c_1 = 0.5, \quad T = 10, \quad R = I, \quad d = 0.01, \quad \delta = 0.042, \quad \eta = 0.075. \]

Through Theorem 7, when \( c_2 = 50 \), we can see that the admissible maximum \( h \) computed by [36] is 0.2, and the maximal value of \( \tau \) in this paper is 0.262. It is obvious that our method shows less conservatism result than that in [36]. Moreover, when \( \tau = 0.2 \), the optimal bound with minimum value of \( c_{2 \min} \) relies on the parameter \( \eta \). One can obtain clearly that Theorem 7 in our paper can indeed provide much smaller admissible \( c_{2 \min} \) than the stability criteria in [36] which shows the less conservative result in this paper.
6. Conclusions

In this paper, we have examined the problems of delay-dependent finite-time and $L_2$-gain analysis for switched systems with time-varying delay by allowing new Lyapunov-Krasovskii functional and average dwell time. A numerical example has also been given to demonstrate the effectiveness of the proposed approach. It is possible to extend the main results obtained in this paper to real systems which contain some practical constraints, for example, the failures of controllers. Details will be reported in our follow-up work in the future.

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