Research Article

Exact Peakon, Compacton, Solitary Wave, and Periodic Wave Solutions for a Generalized KdV Equation

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Received 29 August 2013; Accepted 25 September 2013

Academic Editor: Jun-Juh Yan

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We employ the approaches of both dynamical system and numerical simulation to investigate a generalized KdV equation, which is presented by Yin (2012). Some peakon, compacton, solitary wave, smooth periodic wave, and periodic cusp wave solutions are obtained, and the planar graphs of the compactons and the periodic cusp waves are simulated.

1. Introduction

To study the role of nonlinear dispersion in the formation of patterns in the liquid drop, Rosenau and Hyman [1] showed in a particular generalization of the KdV equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \quad (1)$$

which is called $K(2, 2)$ equation. They found some solitary waves with compact support in it, which they called compactons. These compactons had the property that the width was independent of the amplitude. Equation (1) has been studied successfully by some authors [2–8]. However, (1) does not exhibit the usual energy conservation law. Instead of (1), Cooper et al. considered the corresponding generalized KdV equation [9]

$$u_t + uu_x + \alpha(2uu_{xxx} + 4u_x u_{xx}) = 0, \quad (2)$$

which can be derived from a Lagrangian. Equation (2) has the same terms as (1), except the relative weights of the terms. Equation (2) also admits compacton solutions. The stability of the compacton solutions to (2) was studied in [10]. In the presence of a linear dispersion term, (2) turns into a generalized KdV equation with combined dispersion [11]

$$u_t + uu_x + \alpha(2uu_{xxx} + 4u_x u_{xx}) + \beta u_{xxx} = 0. \quad (3)$$

Obviously, for $\beta = 0$, (3) turns into (2). Yin [11] indicated that (3) has two conservative laws and showed that the smooth solitary waves are stable for any speed of wave propagation.

In this paper, we investigated (3) using the approaches of dynamical system and numerical simulation [12–14], we will give some new exact travelling wave solutions and simulate the compactons and the periodic cusp waves.

Next, we always suppose that $\alpha \neq 0$. Using the following independent variable transformation:

$$u(x, t) = \phi(\xi), \quad \xi = x - ct, \quad (4)$$

where c ($c \neq 0$) is the wave speed, and substituting (4) into (3), we obtain

$$-c\phi' + \phi\phi' + \alpha(2\phi\phi''' + 4\phi'\phi'') + \beta\phi''' = 0, \quad (5)$$

where “ $'$ ” is the derivative with respect to ξ .

Integrating (5) once with respect to ξ , we have

$$-c\phi + \frac{1}{2}\phi^2 + \alpha(2\phi\phi'' + (\phi')^2) + \beta\phi'' = g, \quad (6)$$

where g is the integral constant.

Letting $y = d\phi/d\xi$, we get the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g + c\phi - (1/2)\phi^2 - \alpha y^2}{2\alpha\phi + \beta}. \quad (7)$$

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcation sets and phase portraits of system (7), where explicit parametric conditions will be derived. In Section 3, we give some exact travelling wave solutions which include peakon, compacton, solitary wave, smooth periodic wave, and periodic cusp wave solutions of (3). In Section 4, the numerical simulations of the compactons and the periodic cusp waves are given. A short conclusion will be given in Section 5.

2. Bifurcation Sets and Phase Portraits of System (7)

Using the transformation $d\xi = (2\alpha\phi + \beta)d\tau$, it carries (7) into the Hamiltonian system

$$\frac{d\phi}{d\tau} = (2\alpha\phi + \beta)y, \quad \frac{dy}{d\tau} = g + c\phi - \frac{1}{2}\phi^2 - \alpha y^2. \quad (8)$$

System (8) has the following first integral:

$$H(\phi, y) = (2\alpha\phi + \beta)y^2 - \left(2g\phi + c\phi^2 - \frac{1}{3}\phi^3\right) = h. \quad (9)$$

For a fixed h , the level curve $H(\phi, y) = h$ defined by (9) determines a set of invariant curves of system (8) which contains different branches of curves. As h is varied, it defines different families of orbits of (8) with different dynamical behaviors.

Write $\Delta = -2\alpha(\beta^2 + 4\alpha\beta c - 8g\alpha^2)$, $\phi_s = -\beta/2\alpha$. Clearly, when $g > -c^2/2$, system (8) has two equilibrium points at $(\phi_1, 0)$ and $(\phi_2, 0)$ in ϕ -axis, where $\phi_{1,2} = c \pm \sqrt{2g + c^2}$. When $g = -c^2/2$, system (8) has only one equilibrium point at $(c, 0)$ in ϕ -axis. When $\Delta > 0$, there exist two equilibrium points of system (8) in line $\phi = \phi_s$ at $(\phi_s, Y_{\pm})$, where $Y_{\pm} = \pm(1/4\alpha^2)\sqrt{\Delta}$. When $\Delta < 0$, there is no equilibrium point of system (8) in line $\phi = \phi_s$.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (8) at equilibrium point (ϕ_e, y_e) ; then we have $\text{Trace } M(\phi_e, 0) = 0$ and

$$\begin{aligned} J(\phi_1, 0) &= \det M(\phi_1, 0) \\ &= \sqrt{2g + c^2} \left(2\alpha\sqrt{2g + c^2} + (2\alpha c + \beta) \right), \\ J(\phi_2, 0) &= \det M(\phi_2, 0) \\ &= \sqrt{2g + c^2} \left(2\alpha\sqrt{2g + c^2} - (2\alpha c + \beta) \right), \\ J(\phi_s, Y_{\pm}) &= \det M(\phi_s, Y_{\pm}) = -4\alpha^2 Y_{\pm}^2. \end{aligned} \quad (10)$$

For an equilibrium point (ϕ_e, y_e) of a planar integrable system, we know that (ϕ_e, y_e) is a saddle point if $J(\phi_e, y_e) < 0$, a center point if $J(\phi_e, y_e) > 0$ and $\text{Trace } M(\phi_e, y_e) = 0$, and a cusp if $J(\phi_e, y_e) = 0$ and the Poincaré index of (ϕ_e, y_e) is zero.

Since both systems (7) and (8) have the same first integral (9), then two systems above have the same topological phase portraits except the line $\phi = \phi_s$. Therefore we can obtain the

bifurcation sets and phase portraits of system (7) from those of system (8).

By using the properties of equilibrium points and bifurcation method of dynamical systems, we can show that bifurcation sets and phase portraits of system (7) are drawn in Figures 1 and 2.

3. Exact Travelling Wave Solutions of (3)

Denote that $h_{1,2} = -(2c/3)(3g + c^2) \mp (2/3)(2g + c^2)\sqrt{2g + c^2}$ and $h_s = -\beta(\beta^2 + 6\alpha\beta c - 24g\alpha^2)/24\alpha^3$; we present some exact travelling wave solutions of (3) as follows.

3.1. Peakon Solutions. From Figure 2(d), we see that there are two heteroclinic orbits of system (7) defined by $H(\phi, y) = -\beta(\beta^2 + 12\alpha\beta c + 36\alpha^2 c^2)/96\alpha^3$ connecting with the saddle points $(\phi_s, \pm((2\alpha c + \beta)\sqrt{-6\alpha/8\alpha^2}))$ and $((6\alpha c + \beta)/4\alpha, 0)$ when $\alpha < 0$, $c < -\beta/2\alpha$, and $g = -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$. Their expressions are

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \left(\phi - \frac{6\alpha c + \beta}{4\alpha} \right), \quad \frac{6\alpha c + \beta}{4\alpha} < \phi \leq \phi_s. \quad (11)$$

Substituting (11) into $d\phi/d\xi = y$ and integrating it along the heteroclinic orbits yield equation

$$\int_{\phi}^{\phi_s} \frac{ds}{s - ((6\alpha c + \beta)/4\alpha)} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (12)$$

Completing (12) and using transformation (4), we can get a peakon solution of (3) as follow:

$$u(x, t) = \frac{6\alpha c + \beta}{4\alpha} + \left(\phi_s - \frac{6\alpha c + \beta}{4\alpha} \right) e^{-\omega|x-ct|}, \quad (13)$$

where $\omega = 1/\sqrt{-6\alpha}$. The profiles of (13) are shown in Figures 3(a) and 3(b).

From Figure 2(k), we see that there are two heteroclinic orbits of system (7) defined by $H(\phi, y) = -\beta(\beta^2 + 12\alpha\beta c + 36\alpha^2 c^2)/96\alpha^3$ connecting with the saddle points $(\phi_s, \mp((2\alpha c + \beta)\sqrt{-6\alpha/8\alpha^2}))$ and $((6\alpha c + \beta)/4\alpha, 0)$ when $\alpha < 0$, $c > -\beta/2\alpha$, and $g = -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$. Their expressions are

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \left(\frac{6\alpha c + \beta}{4\alpha} - \phi \right), \quad \phi_s \leq \phi < \frac{6\alpha c + \beta}{4\alpha}. \quad (14)$$

Substituting (14) into $d\phi/d\xi = y$ and integrating it along the heteroclinic orbits yield equation

$$\int_{\phi}^{\phi_s} \frac{ds}{((6\alpha c + \beta)/4\alpha) - s} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (15)$$

Completing (15) and using transformation (4), we can get a peakon solution of (3) which is the same as (13). The profiles are shown in Figures 3(c) and 3(d).

FIGURE 1: Continued.

FIGURE 1: Bifurcation sets and phase portraits of system (7) when $\alpha > 0$.

3.2. Solitary Wave Solutions. From Figure 1(c), we see that there is one homoclinic orbit of system (7) defined by $H(\phi, y) = h_1$ connecting with saddle point $(\phi_1, 0)$ and passing point $(\phi_m, 0)$ when $\alpha > 0$, $c < -\beta/2\alpha$, and $-c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2$, where $\phi_m = c - 2\sqrt{2g + c^2}$. Its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \frac{\phi_1 - \phi}{\phi_s - \phi} \sqrt{(\phi - \phi_m)(\phi_s - \phi)}, \quad \phi_m \leq \phi < \phi_1. \quad (16)$$

Substituting (16) into $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yield equation

$$\int_{\phi_m}^{\phi} \frac{(\phi_s - s) ds}{(\phi_1 - s) \sqrt{(s - \phi_m)(\phi_s - s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (17)$$

Completing (17) and using transformation (4), we can get a solitary wave solution of (3) as follows:

$$u(x, t) = \frac{\phi_1 (\phi_m \cosh^2(\omega\chi) - \phi_s \sinh^2(\omega\chi)) - \phi_m \phi_s}{(\phi_m \sinh^2(\omega\chi) - \phi_s \cosh^2(\omega\chi)) + \phi_1}, \quad (18)$$

$$t = \frac{1}{c} \left(x - \sqrt{6\alpha} (\chi + 2 \tan^{-1}(2\omega \tanh(\omega\chi))) \right),$$

where χ is a new parametric variable and $\omega = (1/2)\sqrt{(\phi_1 - \phi_m)/(\phi_s - \phi_1)}$. The profiles of (18) are shown in Figures 4(a) and 4(b).

From Figure 1(h), we see that there is one homoclinic orbit of system (7) defined by $H(\phi, y) = h_2$ connecting with saddle point $(\phi_2, 0)$ and passing point $(\phi_M, 0)$ when $\alpha > 0$,

FIGURE 2: Continued.

FIGURE 2: Continued.

FIGURE 2: Bifurcation sets and phase portraits of system (7) when $\alpha < 0$. FIGURE 3: Peakons of (3) for $x = 1$.

$c > -\beta/2\alpha$, and $-c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2$, where $\phi_M = c + 2\sqrt{2g + c^2}$. Its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \frac{\phi - \phi_2}{\phi - \phi_s} \sqrt{(\phi_M - \phi)(\phi - \phi_s)}, \quad \phi_2 < \phi \leq \phi_M. \quad (19)$$

Substituting (19) into $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yield equation

$$\int_{\phi}^{\phi_M} \frac{(s - \phi_s) ds}{(s - \phi_2) \sqrt{(\phi_M - s)(s - \phi_s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (20)$$

FIGURE 4: Solitary waves of (3) for $x = 1$.

Completing (20) and using transformation (4), we can get a solitary wave solution of (3) as follows:

$$u(x, t) = \frac{\phi_2 (\phi_M \cosh^2(\omega\chi) - \phi_s \sinh^2(\omega\chi)) - \phi_M \phi_s}{(\phi_M \sinh^2(\omega\chi) - \phi_s \cosh^2(\omega\chi)) + \phi_2}, \quad (21)$$

$$t = \frac{1}{c} \left(x - \sqrt{6\alpha} \left(\chi - 2 \tanh^{-1}(2\omega \tanh(\omega\chi)) \right) \right),$$

where $\omega = (1/2)\sqrt{(\phi_M - \phi_2)/(\phi_2 - \phi_s)}$. The profiles of (21) are shown in Figures 4(c) and 4(d).

From Figure 2(c), we see that there is one homoclinic orbit of system (7) defined by $H(\phi, y) = h_2$ connecting with saddle point $(\phi_2, 0)$ and passing point $(\phi_M, 0)$ when $\alpha < 0$, $c < -\beta/2\alpha$, and $-c^2/2 < g < -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$, where $\phi_M = c + 2\sqrt{2g + c^2}$. Its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \frac{\phi - \phi_2}{\phi_s - \phi} \sqrt{(\phi_M - \phi)(\phi_s - \phi)}, \quad \phi_2 < \phi \leq \phi_M. \quad (22)$$

Substituting (22) into $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yield equation

$$\int_{\phi}^{\phi_M} \frac{(\phi_s - s) ds}{(s - \phi_2) \sqrt{(\phi_M - s)(\phi_s - s)}} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (23)$$

Completing (23) and using transformation (4), we can get a solitary wave solution of (3) as follows:

$$u(x, t) = \frac{\phi_2 (\phi_M \cosh^2(\omega\chi) - \phi_s \sinh^2(\omega\chi)) - \phi_M \phi_s}{(\phi_M \sinh^2(\omega\chi) - \phi_s \cosh^2(\omega\chi)) + \phi_2}, \quad (24)$$

$$t = \frac{1}{c} \left(x - \sqrt{-6\alpha} \left(\chi + 2 \tanh^{-1}(2\omega \tanh(\omega\chi)) \right) \right),$$

where $\omega = (1/2)\sqrt{(\phi_M - \phi_2)/(\phi_s - \phi_2)}$. The profiles of (24) are shown in Figures 4(e) and 4(f).

From Figure 2(j), we see that there is one homoclinic orbit of system (7) defined by $H(\phi, y) = h_1$ connecting with saddle point $(\phi_1, 0)$ and passing point $(\phi_m, 0)$ when $\alpha < 0$, $c > -\beta/2\alpha$, $-c^2/2 < g < -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$, where $\phi_m = c - 2\sqrt{2g + c^2}$. Its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \frac{\phi_1 - \phi}{\phi - \phi_s} \sqrt{(\phi - \phi_m)(\phi - \phi_s)}, \quad \phi_m \leq \phi < \phi_1. \quad (25)$$

Substituting (25) into $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yield equation

$$\int_{\phi_m}^{\phi} \frac{(s - \phi_s) ds}{(\phi_1 - s) \sqrt{(s - \phi_m)(s - \phi_s)}} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (26)$$

Completing (26) and using transformation (4), we can get a solitary wave solution of (3) as follows:

$$u(x, t) = \frac{\phi_1 (\phi_m \cosh^2(\omega\chi) - \phi_s \sinh^2(\omega\chi)) - \phi_m \phi_s}{(\phi_m \sinh^2(\omega\chi) - \phi_s \cosh^2(\omega\chi)) + \phi_1}, \quad (27)$$

$$t = \frac{1}{c} \left(x - \sqrt{-6\alpha} \left(\chi - 2 \tanh^{-1}(2\omega \tanh(\omega\chi)) \right) \right),$$

where $\omega = (1/2)\sqrt{(\phi_1 - \phi_m)/(\phi_1 - \phi_s)}$. The profiles of (27) are shown in Figures 4(g) and 4(h).

3.3. Smooth Periodic Wave Solutions. From Figure 1(d), we see that there is a periodic orbit of system (7) defined by $H(\phi, y) = \beta^2(3\alpha c + \beta)/12\alpha^3$ enclosing the center point $((4\alpha c + \beta)/2\alpha, 0)$ and passing points $(\phi_s, 0)$ and $((3\alpha c + \beta)/\alpha, 0)$ when $\alpha > 0$, $c < -\beta/2\alpha$, and $g = \beta(4\alpha c + \beta)/8\alpha^2$. Its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\left(\phi - \frac{3\alpha c + \beta}{\alpha} \right) (\phi_s - \phi)}, \quad \frac{3\alpha c + \beta}{\alpha} \leq \phi \leq \phi_s. \quad (28)$$

Substituting (28) into $d\phi/d\xi = y$ and integrating it along the periodic orbit yield equation

$$\int_{(3\alpha c + \beta)/\alpha}^{\phi} \frac{ds}{\sqrt{(s - ((3\alpha c + \beta)/\alpha)) (\phi_s - s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (29)$$

Completing (29) and using transformation (4), we can get a smooth periodic wave solution of (3) as follows:

$$u(x, t) = \phi_s \sin^2(\omega(x - ct)) + \frac{3\alpha c + \beta}{\alpha} \cos^2(\omega(x - ct)), \quad (30)$$

where $\omega = 1/2\sqrt{6\alpha}$. The profiles of (30) are shown in Figures 5(a) and 5(b).

From Figure 1(i), we see that there is a periodic orbit of system (7) defined by $H(\phi, y) = \beta^2(3\alpha c + \beta)/12\alpha^3$ enclosing the center point $((4\alpha c + \beta)/2\alpha, 0)$ and passing points $(\phi_s, 0)$ and $((3\alpha c + \beta)/\alpha, 0)$ when $\alpha > 0$, $c > -\beta/2\alpha$, and $g = \beta(4\alpha c + \beta)/8\alpha^2$. Its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\left(\frac{3\alpha c + \beta}{\alpha} - \phi \right) (\phi - \phi_s)}, \quad \phi_s \leq \phi \leq \frac{3\alpha c + \beta}{\alpha}. \quad (31)$$

Substituting (31) into $d\phi/d\xi = y$ and integrating it along the periodic orbit yield equation

$$\int_{\phi}^{(3\alpha c + \beta)/\alpha} \frac{ds}{\sqrt{(((3\alpha c + \beta)/\alpha) - s) (\phi - s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (32)$$

FIGURE 5: Smooth periodic waves of (3) for $x = 1$.

Completing (32) and using transformation (4), we can get a smooth periodic wave solution of (3) which is the same as (30). The profiles are shown in Figures 5(c) and 5(d).

3.4. Compacton Solutions. For given $h \in (h_1, h_s)$, $h_s < h_2$ (or $h \in (h_1, h_2)$, $h_s > h_2$) in Figure 1(c), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(a).

For given $h \in (h_s, h_2)$, $h_1 < h_s$ in Figure 1(c), $h \in (h_1, h_2)$ in Figure 1(d), and $h \in (h_s, h_2)$ in Figures 1(e) and 1(j), respectively, the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(b).

For given $h \in (h_s, h_2)$, $h_s > h_1$ (or $h \in (h_1, h_2)$, $h_s < h_1$) in Figure 1(h), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(c).

For given $h \in (h_1, h_s)$, $h_s < h_2$ in Figure 1(h), and $h \in (h_1, h_2)$ in Figure 1(i), $h \in (h_1, h_s)$ in Figures 1(e) and 1(j), respectively, the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(d).

For given $h \in (h_s, h_2)$ in Figures 2(e), 2(f), 2(g), and 2(n), respectively, the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(e).

For given $h \in (h_1, h_s)$ in Figures 2(g), 2(l), 2(m), and 2(n), respectively, the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(f).

From Figure 6(a), we see that there are a periodic orbit and an open curve of system (7) defined by $H(\phi, y) = h$ when $\alpha > 0$, $c < -\beta/2\alpha$, $-c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2$, $h \in (h_1, h_s)$, $h_s < h_2$ (or $h \in (h_1, h_2)$, and $h_s > h_2$). The open curve passes point $(\gamma_1, 0)$ and approaches the line $\phi = \phi_s$; its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\frac{(\phi - \gamma_1)(\phi - \gamma_2)(\phi - \gamma_3)}{\phi_s - \phi}}, \quad \gamma_1 \leq \phi < \phi_s, \quad (33)$$

where γ_1 , γ_2 , and γ_3 ($\gamma_3 < \gamma_2 < \gamma_1 < \phi_s$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx -2.681985997$, $\gamma_2 \approx -2.948887226$, $\gamma_3 \approx -3.369126778$ when $\alpha = 0.2$, $\beta = 1$, $c = -3$, $g = -4.48$, and $h = -8.882$, and $\gamma_1 \approx -0.06941350133$, $\gamma_2 \approx -0.5430641149$, and $\gamma_3 \approx -2.387522384$ when $\alpha = 1$, $\beta = 0$, $c = -1$, $g = -0.25$, and $h = -0.03$.

Substituting (33) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\gamma_1}^{\phi} \sqrt{\frac{\phi_s - s}{(s - \gamma_1)(s - \gamma_2)(s - \gamma_3)}} ds = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (34)$$

FIGURE 6: Continued.

FIGURE 6: The level curves defined by $H(\phi, y) = h$.

Completing (34) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follows:

$$\begin{aligned} & \frac{1}{\alpha_1^2} \left(\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k) \right) \\ &= \frac{\sqrt{(\phi_s - \gamma_2)(\gamma_1 - \gamma_3)}}{2(\phi_s - \gamma_1)\sqrt{6\alpha}} |x - ct|, \end{aligned} \quad (35)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\phi_s - \gamma_2)(u(x, t) - \gamma_1)/(\phi_s - \gamma_1)(u(x, t) - \gamma_2)})$, $\tilde{\phi} = \text{sn}^{-1}(\sqrt{(\phi_s - \gamma_2)(u(x, t) - \gamma_1)/(\phi_s - \gamma_1)(u(x, t) - \gamma_2)})$, $k = \sqrt{(\phi_s - \gamma_1)(\gamma_2 - \gamma_3)/(\phi_s - \gamma_2)(\gamma_1 - \gamma_3)}$, and $\alpha_1^2 = (\phi_s - \gamma_1)/(\phi_s - \gamma_2)$; $\Pi(\cdot, \cdot, \cdot)$ is Legendre's incomplete elliptic integral of the third kind, and $\text{sn}^{-1}(\cdot, \cdot)$ is the inverse function of the Jacobian elliptic function $\text{sn}(\cdot, \cdot)$ [15].

From Figure 6(b), we see that there are a periodic orbit and an open curve of system (7) defined by $H(\phi, y) = h$ if and only if one of the following conditions holds:

- (i) $\alpha > 0$, $c < -\beta/2\alpha$, $-c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2$, $h \in (h_s, h_2)$, and $h_1 < h_s$,
- (ii) $\alpha > 0$, $c < -\beta/2\alpha$, $g = \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_1, h_2)$,
- (iii) $\alpha > 0$, $c \neq -\beta/2\alpha$, $g > \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_s, h_2)$.

The open curve passes point $(\gamma_1, 0)$ and approaches the line $\phi = \phi_s$; its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\frac{(\gamma_1 - \phi)(\phi - \gamma_2)(\phi - \gamma_3)}{\phi - \phi_s}}, \quad \phi_s < \phi \leq \gamma_1, \quad (36)$$

where γ_1, γ_2 , and γ_3 ($\gamma_3 < \gamma_2 < \phi_s < \gamma_1$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx 0.6863083159$,

$\gamma_2 \approx 0.243331752$, and $\gamma_3 \approx -0.1796400679$ when $\alpha = 1, \beta = -1, c = 0.25, g = 0$, and $h = -0.01$ and $\gamma_1 \approx 5.467670461, \gamma_2 \approx -1.060802034, \gamma_3 \approx -1.406868427$ when $\alpha = 1, \beta = 2, c = 1, g = 2$, and $h = 2.72$.

Substituting (36) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\phi}^{\gamma_1} \sqrt{\frac{s - \phi_s}{(\gamma_1 - s)(s - \gamma_2)(s - \gamma_3)}} ds = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (37)$$

Completing (37) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follows:

$$\begin{aligned} & \frac{1}{\alpha_1^2} (\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k)) \\ &= \frac{\sqrt{(\gamma_1 - \gamma_2)(\phi_s - \gamma_3)}}{2(\gamma_1 - \phi_s) \sqrt{6\alpha}} |x - ct|, \end{aligned} \quad (38)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\phi_s - \gamma_3)(\gamma_1 - u(x, t))(\gamma_1 - \phi_s)(u(x, t) - \gamma_3)})$, $\tilde{\phi} = \text{sn}^{-1}(\sqrt{(\phi_s - \gamma_3)(\gamma_1 - u(x, t))(\gamma_1 - \phi_s)(u(x, t) - \gamma_3)})$, $k = \sqrt{(\gamma_1 - \phi_s)(\gamma_2 - \gamma_3)/(\gamma_1 - \gamma_2)(\phi_s - \gamma_3)}$, and $\alpha_1^2 = (\phi_s - \gamma_1)/(\phi_s - \gamma_3)$.

From Figure 6(c), we see that there are a periodic orbit and an open curve of system (7) defined by $H(\phi, y) = h$ when $\alpha > 0, c > -\beta/2\alpha, -c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2, h \in (h_s, h_2), h_s > h_1$ (or $h \in (h_1, h_2)$, and $h_s < h_1$). The open curve passes point $(\gamma_3, 0)$ and approaches the line $\phi = \phi_s$; its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\frac{(\gamma_1 - \phi)(\gamma_2 - \phi)(\gamma_3 - \phi)}{\phi - \phi_s}}, \quad \phi_s < \phi \leq \gamma_3, \quad (39)$$

where γ_1, γ_2 , and γ_3 ($\phi_s < \gamma_3 < \gamma_2 < \gamma_1$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx -1.725343175, \gamma_2 \approx -1.954509591$, and $\gamma_3 \approx -2.320147234$ when $\alpha = 0.2, \beta = 1, c = -2, g = -1.985$, and $h = -2.608$ and $\gamma_1 \approx 0.9464101615, \gamma_2 = 0.3$, and $\gamma_3 \approx 0.2535898385$ when $\alpha = 2, \beta = -1, c = 0.5, g = -0.1$, and $h = 0.024$.

Substituting (39) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\phi}^{\gamma_3} \sqrt{\frac{s - \phi_s}{(\gamma_1 - s)(\gamma_2 - s)(\gamma_3 - s)}} ds = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (40)$$

Completing (40) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follow:

$$\begin{aligned} & \frac{1}{\alpha_1^2} (\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k)) \\ &= \frac{\sqrt{(\gamma_1 - \gamma_3)(\gamma_2 - \phi_s)}}{2(\gamma_3 - \phi_s) \sqrt{6\alpha}} |x - ct|, \end{aligned} \quad (41)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\gamma_2 - \phi_s)(\gamma_3 - u(x, t))(\gamma_3 - \phi_s)(\gamma_2 - u(x, t))})$, $\tilde{\phi} = \text{sn}^{-1}(\sqrt{(\gamma_2 - \phi_s)(\gamma_3 - u(x, t))(\gamma_3 - \phi_s)(\gamma_2 - u(x, t))})$,

$k = \sqrt{(\gamma_1 - \gamma_2)(\gamma_3 - \phi_s)/(\gamma_1 - \gamma_3)(\gamma_2 - \phi_s)}$, and $\alpha_1^2 = (\gamma_3 - \phi_s)/(\gamma_2 - \phi_s)$.

From Figure 6(d), we see that there are a periodic orbit and an open curve of system (7) defined by $H(\phi, y) = h$ if and only if one of the following conditions holds:

- (i) $\alpha > 0, c > -\beta/2\alpha, -c^2/2 < g < \beta(4\alpha c + \beta)/8\alpha^2, h \in (h_1, h_s)$, and $h_s < h_2$,
- (ii) $\alpha > 0, c > -\beta/2\alpha, g = \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_1, h_2)$,
- (iii) $\alpha > 0, c \neq -\beta/2\alpha, g > \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_1, h_s)$.

The open curve passes point $(\gamma_3, 0)$ and approaches the line $\phi = \phi_s$, its expression is

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{\frac{(\gamma_1 - \phi)(\gamma_2 - \phi)(\phi - \gamma_3)}{\phi_s - \phi}}, \quad \gamma_3 \leq \phi < \phi_s, \quad (42)$$

where γ_1, γ_2 , and γ_3 ($\gamma_3 < \phi_s < \gamma_2 < \gamma_1$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx 2.557900553, \gamma_2 \approx 0.7351513251$, and $\gamma_3 \approx -2.393051878$ when $\alpha = 1, \beta = -1, c = 0.3, g = 1$, and $h = -1.5$ and $\gamma_1 \approx 5.743824982, \gamma_2 \approx -0.6449004894, \gamma_3 \approx -8.098924493$ when $\alpha = 0.1, \beta = 1, c = -1, g = 7.5$, and $h = 10$.

Substituting (42) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\gamma_3}^{\phi} \sqrt{\frac{\phi_s - s}{(\gamma_1 - s)(\gamma_2 - s)(s - \gamma_3)}} ds = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (43)$$

Completing (43) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follows:

$$\begin{aligned} & \frac{1}{\alpha_1^2} (\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k)) \\ &= \frac{\sqrt{(\gamma_1 - \phi_s)(\gamma_2 - \gamma_3)}}{2(\phi_s - \gamma_3) \sqrt{6\alpha}} |x - ct|, \end{aligned} \quad (44)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\gamma_1 - \phi_s)(u(x, t) - \gamma_3)/(\phi_s - \gamma_3)(\gamma_1 - u(x, t))})$, $\tilde{\phi} = \text{sn}^{-1}(\sqrt{(\gamma_1 - \phi_s)(u(x, t) - \gamma_3)/(\phi_s - \gamma_3)(\gamma_1 - u(x, t))})$, $k = \sqrt{(\gamma_1 - \gamma_2)(\phi_s - \gamma_3)/(\gamma_1 - \phi_s)(\gamma_2 - \gamma_3)}$, and $\alpha_1^2 = (\gamma_3 - \phi_s)/(\gamma_1 - \phi_s)$.

From Figure 6(e), we see that there are three open curves of system (7) defined by $H(\phi, y) = h$ if and only if one of the following conditions holds:

- (i) $\alpha < 0, c < -\beta/2\alpha, g > -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$, and $h \in (h_s, h_2)$,
- (ii) $\alpha < 0, c > -\beta/2\alpha, g > \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_s, h_2)$.

One of them passes point $(\gamma_2, 0)$ and approaches the line $\phi = \phi_s$; its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \sqrt{\frac{(\gamma_1 - \phi)(\phi - \gamma_2)(\phi - \gamma_3)}{\phi_s - \phi}}, \quad \gamma_2 \leq \phi < \phi_s, \quad (45)$$

where γ_1, γ_2 , and γ_3 ($\gamma_3 < \gamma_2 < \phi_s < \gamma_1$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx 2.064177772$, $\gamma_2 \approx -0.3054072893$, and $\gamma_3 \approx -4.758770483$ when $\alpha = -2$, $\beta = 2$, $c = -1$, and $g = 1.5$ and $h = 1$, $\gamma_1 \approx 5.466333036$, $\gamma_2 \approx -1.035930847$, and $\gamma_3 \approx -1.430402188$ when $\alpha = -0.5$, $\beta = -1$, $c = 1$, $g = 2$, and $h = 2.7$.

Substituting (45) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\gamma_2}^{\phi} \sqrt{\frac{\phi_s - s}{(\gamma_1 - s)(s - \gamma_2)(s - \gamma_3)}} ds = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (46)$$

Completing (46) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follows:

$$\begin{aligned} & \frac{1}{\alpha_1^2} (\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k)) \\ &= \frac{\sqrt{(\gamma_1 - \gamma_2)(\phi_s - \gamma_3)}}{2(\phi_s - \gamma_2)\sqrt{-6\alpha}} |x - ct|, \end{aligned} \quad (47)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\phi_s - \gamma_3)(u(x, t) - \gamma_2)/(\phi_s - \gamma_2)(u(x, t) - \gamma_3)})$, $k = \sqrt{(\phi_s - \gamma_2)(\gamma_1 - \gamma_3)/(\gamma_1 - \gamma_2)(\phi_s - \gamma_3)}$, and $\alpha_1^2 = (\phi_s - \gamma_2)/(\phi_s - \gamma_3)$.

From Figure 6(f), we see that there are three open curves of system (7) defined by $H(\phi, y) = h$ if and only if one of the following conditions holds:

- (i) $\alpha < 0, c < -\beta/2\alpha, g > \beta(4\alpha c + \beta)/8\alpha^2$, and $h \in (h_1, h_s)$,
- (ii) $\alpha < 0, c > -\beta/2\alpha, g > -(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2$, and $h \in (h_1, h_s)$.

One of them passes point $(\gamma_2, 0)$ and approaches the line $\phi = \phi_s$; its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \sqrt{\frac{(\gamma_1 - \phi)(\gamma_2 - \phi)(\phi - \gamma_3)}{\phi - \phi_s}}, \quad \phi_s < \phi \leq \gamma_2, \quad (48)$$

where γ_1, γ_2 , and γ_3 ($\gamma_3 < \phi_s < \gamma_2 < \gamma_1$) are three real roots of $\psi^3 - 3c\psi^2 - 6g\psi - 3h = 0$. For example, $\gamma_1 \approx 0.8657007565$, $\gamma_2 \approx -0.091826066674$, and $\gamma_3 \approx -3.77387469$ when $\alpha = -2$, $\beta = -1$, $c = -1$, $g = 0.5$, and $h = 0.1$, and $\gamma_1 \approx 6.154523009$, $\gamma_2 \approx 2.662391044$, and $\gamma_3 \approx 0.1830859473$ when $\alpha = -0.2$, $\beta = 1$, $c = 3$, $g = -3$, and $h = 1$.

Substituting (48) into $d\phi/d\xi = y$ and integrating it along the open curve yield equation

$$\int_{\phi}^{\gamma_2} \sqrt{\frac{s - \phi_s}{(\gamma_1 - s)(\gamma_2 - s)(s - \gamma_3)}} ds = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (49)$$

Completing (49) and using transformation (4), we can get the implicit representation of a compacton solution of (3) as follows:

$$\begin{aligned} & \frac{1}{\alpha_1^2} (\tilde{u} + (\alpha_1^2 - 1) \Pi(\tilde{\phi}, \alpha_1^2, k)) \\ &= \frac{\sqrt{(\gamma_1 - \phi_s)(\gamma_2 - \gamma_3)}}{2(\gamma_2 - \phi_s)\sqrt{-6\alpha}} |x - ct|, \end{aligned} \quad (50)$$

where $\tilde{u} = \text{sn}^{-1}(\sqrt{(\gamma_1 - \phi_s)(\gamma_2 - u(x, t))/(\gamma_2 - \phi_s)(\gamma_1 - u(x, t))})$, $k = \sqrt{(\gamma_1 - \phi_s)(\gamma_2 - u(x, t))/(\gamma_2 - \phi_s)(\gamma_1 - u(x, t))}$, $k = \sqrt{(\gamma_2 - \phi_s)(\gamma_1 - \gamma_3)/(\gamma_1 - \phi_s)(\gamma_2 - \gamma_3)}$, and $\alpha_1^2 = (\gamma_2 - \phi_s)/(\gamma_1 - \phi_s)$.

3.5. Periodic Cusp Wave Solutions. For given $h = h_s$ in Figure 1(e), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(g).

For given $h = h_s$ in Figure 1(j), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(h).

For given $h = h_s$ in Figure 2(e), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(i).

For given $h = h_s$ in Figure 2(l), the level curve defined by $H(\phi, y) = h$ is shown in Figure 6(j).

From Figures 6(g) and 6(h), we see that there are two heteroclinic orbits of system (7) defined by $H(\phi, y) = h_s$ connecting with the saddle points $(\phi_s, Y_{\pm})$ and passing points $(\phi_m, 0)$, and $(\phi_M, 0)$, respectively, when $\alpha > 0, c \neq -\beta/2\alpha$, and $g > \beta(4\alpha c + \beta)/8\alpha^2$, where $\phi_{M,m} = ((\beta + 6\alpha c) \pm \sqrt{\delta})/4\alpha$ and $\delta = -3\beta^2 - 12\alpha\beta c + 36\alpha^2 c^2 + 96\alpha^2 g$. Their expressions are, respectively,

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{(\phi - \phi_m)(\phi_M - \phi)}, \quad \phi_m \leq \phi \leq \phi_s, \quad (51)$$

$$y = \pm \frac{1}{\sqrt{6\alpha}} \sqrt{(\phi - \phi_m)(\phi_M - \phi)}, \quad \phi_s \leq \phi \leq \phi_M. \quad (52)$$

Substituting (51) into the $d\phi/d\xi = y$ and integrating it along the heteroclinic orbit, yields equation

$$\int_{\phi_s}^{\phi} \frac{ds}{\sqrt{(s - \phi_m)(\phi_M - s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (53)$$

Completing (53) and using transformation (4), we can get a periodic cusp wave solution of (3) as follows:

$$\begin{aligned} u(x, t) &= \phi_m \cos^2(\Omega - \omega |x - ct - 2nT|) \\ &+ \phi_M \sin^2(\Omega - \omega |x - ct - 2nT|), \end{aligned} \quad (54)$$

$$x - ct \in [(2n - 1)T, (2n + 1)T],$$

where $n = 0, \pm 1, \pm 2, \dots$, $\omega = 1/2\sqrt{6\alpha}$, $\Omega = \tan^{-1}(\sqrt{(\phi_s - \phi_m)/(\phi_M - \phi_s)})$, and $T = 2|\Omega|$.

Substituting (52) into $d\phi/d\xi = y$ and integrating it along the heteroclinic orbit yield equation

$$\int_{\phi_s}^{\phi} \frac{ds}{\sqrt{(s - \phi_m)(\phi_M - s)}} = \frac{1}{\sqrt{6\alpha}} \int_0^{\xi} ds. \quad (55)$$

Completing (55) and using transformation (4), we can get a periodic cusp wave solution of (3) as follows:

$$\begin{aligned} u(x, t) &= \phi_m \sin^2(\Omega + \omega |x - ct - 2nT|) \\ &\quad + \phi_M \cos^2(\Omega + \omega |x - ct - 2nT|), \quad (56) \\ x - ct &\in [(2n-1)T, (2n+1)T], \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$, $\omega = 1/2\sqrt{6\alpha}$, $\Omega = \tan^{-1}(\sqrt{(\phi_M - \phi_s)/(\phi_s - \phi_m)})$, and $T = 2|\Omega|$.

From Figure 6(i), we see that there are a heteroclinic orbit and an open curve of system (7) defined by $H(\phi, y) = h_s$ when $\alpha < 0$, $c < -\beta/2\alpha$, and $-(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2 < g < \beta(4\alpha c + \beta)/8\alpha^2$. The heteroclinic orbit connecting with the saddle points $(\phi_s, Y_{\pm})$ and passing point $(\phi_M, 0)$; its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \sqrt{(\phi - \phi_M)(\phi - \phi_m)}, \quad \phi_M \leq \phi \leq \phi_s, \quad (57)$$

where $\phi_{m,M} = ((\beta + 6\alpha c) \pm \sqrt{\delta})/4\alpha$ and $\delta = -3\beta^2 - 12\alpha\beta c + 36\alpha^2 c^2 + 96\alpha^2 g$.

Substituting (57) into $d\phi/d\xi = y$ and integrating it along the heteroclinic orbit yield equation

$$\int_{\phi}^{\phi_s} \frac{ds}{\sqrt{(s - \phi_M)(s - \phi_m)}} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (58)$$

Completing (58) and using transformation (4), we can get a periodic cusp wave solution of (3) as follows:

$$\begin{aligned} u(x, t) &= \phi_M \cosh^2(\Omega - \omega |x - ct - 2nT|) \\ &\quad - \phi_m \sinh^2(\Omega - \omega |x - ct - 2nT|), \quad (59) \\ x - ct &\in [(2n-1)T, (2n+1)T], \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$, $\omega = 1/2\sqrt{-6\alpha}$, $\Omega = \tanh^{-1}(\sqrt{(\phi_s - \phi_M)/(\phi_s - \phi_m)})$, and $T = 2|\Omega|$.

From Figure 6(j), we see that there are a heteroclinic orbit and an open curve of system (7) defined by $H(\phi, y) = h_s$ when $\alpha < 0$, $c > -\beta/2\alpha$, and $-(12\alpha^2 c^2 - 4\alpha\beta c - \beta^2)/32\alpha^2 < g < \beta(4\alpha c + \beta)/8\alpha^2$. The heteroclinic orbit connecting with the saddle points $(\phi_s, Y_{\pm})$ and passing point $(\phi_m, 0)$; its expression is

$$y = \pm \frac{1}{\sqrt{-6\alpha}} \sqrt{(\phi_m - \phi)(\phi_M - \phi)}, \quad \phi_s \leq \phi \leq \phi_m, \quad (60)$$

where $\phi_{m,M} = ((\beta + 6\alpha c) \pm \sqrt{\delta})/4\alpha$ and $\delta = -3\beta^2 - 12\alpha\beta c + 36\alpha^2 c^2 + 96\alpha^2 g$.

Substituting (60) into $d\phi/d\xi = y$ and integrating it along the heteroclinic orbit yield equation

$$\int_{\phi_s}^{\phi} \frac{ds}{\sqrt{(\phi_m - s)(\phi_M - s)}} = \frac{1}{\sqrt{-6\alpha}} \int_0^{\xi} ds. \quad (61)$$

Completing (61) and using transformation (4), we can get a periodic cusp wave solution of (3) as follows:

$$\begin{aligned} u(x, t) &= \phi_m \cosh^2(\Omega + \omega |x - ct - 2nT|) \\ &\quad - \phi_M \sinh^2(\Omega + \omega |x - ct - 2nT|), \quad (62) \\ x - ct &\in [(2n-1)T, (2n+1)T], \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$, $\omega = 1/2\sqrt{-6\alpha}$, $\Omega = \tanh^{-1}(\sqrt{(\phi_m - \phi_s)/(\phi_M - \phi_s)})$, and $T = 2|\Omega|$.

4. Numerical Simulations

In this section, we simulate the planar diagrams of the compactons and the periodic cusp waves of (3). From the derivations of (6) and (7), it is seen that if $\phi = \phi(\xi)$ and $y = y(\xi)$ are the parameter expressions of an orbit of system (7) and they satisfy $\phi(0) = \phi_0$ and $y(0) = y_0$, then the diagram of $u = \phi(\xi)$ and the integral curve of (6) with initial conditions $\phi(0) = \phi_0$ and $\phi'(0) = y_0$ are the same. Therefore we can use the simulations of integral curves of (6) to test the validity of exact travelling wave solutions (35), (38), (41), (44), (47), (50), (54), (56), (59), and (62).

Example 1. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(a); that is $\alpha = 0.2$, $\beta = 1$, $c = -3$, $g = -4.48$, and $h = -8.88$, and let $\phi(0) = -2.6$ in (33), then it follows that $\phi'(0) = y(0) \approx 0.3651483603$ or $\phi'(0) = y(0) \approx -0.3651483603$. Taking $\phi(0) = -2.6$, and $y(0) = 0.3651483603$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (33) as Figure 7(a).

Example 2. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(b); that is $\alpha = 0.2$, $\beta = 0$, $c = -1$, $g = -0.25$, and $h = 0.2$, and let $\phi(0) = 0.001$ in (36); then it follows that $\phi'(0) = y(0) \approx 22.33265544$ or $\phi'(0) = y(0) \approx -22.33265544$. Taking $\phi(0) = 0.001$, and $y(0) = 22.33265544$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (36) as Figure 7(b).

Example 3. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(c); that is $\alpha = 0.2$, $\beta = 1$, $c = -2$, $g = -1.985$, $h = -2.608$ and let $\phi(0) = -2.4$ in (39); then it follows that $\phi'(0) = y(0) \approx 0.4472135955$ or $\phi'(0) = y(0) \approx -0.4472135955$. Taking $\phi(0) = -2.4$ and $y(0) = -0.4472135955$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (39) as Figure 7(c).

Example 4. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(d); that is $\alpha = 0.2$, $\beta = 0$, $c = 2$, $g = 0$, and $h = -5$, and let $\phi(0) = -1.2$ in (42); then it follows that $\phi'(0) = y(0) \approx 1.793506807$ or $\phi'(0) = y(0) \approx -1.793506807$. Taking $\phi(0) = -1.2$ and $y(0) = 1.793506807$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (42) as Figure 7(d).

FIGURE 7: Continued.

FIGURE 7: The numerical simulations of integral curves of (3).

Example 5. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(e); that is $\alpha = -0.2, \beta = -1, c = -3, g = -4.43, h = -8.58$, and let $\phi(0) = -2.9$ in (45); then it follows that $\phi'(0) = y(0) \approx 0.2922613022$ or $\phi'(0) = y(0) \approx -0.2922613022$. Taking $\phi(0) = -2.9$ and $y(0) = 0.2922613022$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (45) as Figure 7(e).

Example 6. Choose α, β, c, g , and h satisfying the parametric conditions of Figure 6(f); that is $\alpha = -0.2, \beta = -1, c = -2, g = -1.92$, and $h = -2.36$, and let $\phi(0) = -2.1$ in (48); then it follows that $\phi'(0) = y(0) \approx 0.4257346591$ or $\phi'(0) = y(0) \approx -0.4257346591$. Taking $\phi(0) = -2.1$ and $y(0) = 0.4257346591$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the open curve (48) as Figure 7(f).

Example 7. Choose α, β, c , and g satisfying the parametric conditions of Figure 6(g); that is $\alpha = 0.2, \beta = 1, c = -3$, and $g = -4.1$. Let $\phi(0) = -4$ in (51); then it follows that $\phi'(0) = y(0) \approx 1.172603942$ or $\phi'(0) = y(0) \approx -1.172603942$. Taking $\phi(0) = -4$ and $y(0) = 1.172603942$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (51) as Figure 7(g). Let $\phi(0) = -2$ in (52), then it follows that $\phi'(0) = y(0) \approx 0.7337120246$ or $\phi'(0) = y(0) \approx -0.7337120246$. Taking $\phi(0) = -2, y(0) = 0.7337120246$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (52) as Figure 7(h).

Example 8. Choose α, β, c , and g satisfying the parametric conditions of Figure 6(h); that is $\alpha = 0.2, \beta = -1, c = 3$, and $g = -3$. Let $\phi(0) = 2$ in (51), then it follows that $\phi'(0) = y(0) \approx 2.457912394$ or $\phi'(0) = y(0) \approx -2.457912394$. Taking $\phi(0) = 2$ and $y(0) = 2.457912394$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (51) as Figure 7(i). Let $\phi(0) = 5$ in (52); then it follows that $\phi'(0) = y(0) \approx 2.188957133$ or $\phi'(0) = y(0) \approx -2.188957133$. Taking $\phi(0) = 5, y(0) = 2.188957133$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (52) as Figure 7(j).

Example 9. Choose α, β, c , and g satisfying the parametric conditions of Figure 6(i); that is $\alpha = -0.2, \beta = -1, c = -3$, and $g = -4.43$ and let $\phi(0) = -2.7$ in (57), then it follows that $\phi'(0) = y(0) \approx 0.2397915762$ or $\phi'(0) = y(0) \approx -0.2397915762$. Taking $\phi(0) = -2.7$ and $y(0) = 0.2397915762$ as initial values of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (57) as Figure 7(k).

Example 10. Choose α, β, c , and g satisfying the parametric conditions of Figure 6(j); that is $\alpha = -1, \beta = 2, c = 2$, and $g = -1.6$ and let $\phi(0) = -1.1$ in (60); then it follows that $\phi'(0) = y(0) \approx 0.2272296966$ or $\phi'(0) = y(0) \approx -0.2272296966$. Taking $\phi(0) = -1.1$ and $y(0) = 0.2272296966$ as initial values

of (6), using the Maple, we get the numerical simulation of the integral curve which corresponds to the heteroclinic orbit (60) as Figure 7(l).

Figure 7 implies that our theoretic results of the compacton and the periodic cusp wave solutions are identical with the numerical simulations.

5. Conclusion

In this paper, we studied the bifurcations of travelling wave solutions of (3) using the approach of dynamical system and obtained some exact peakon, compacton, solitary wave, smooth periodic wave, and periodic cusp wave solutions. Also, the compactons and the periodic cusp waves are simulated by approach of numerical simulation. The results of this paper have enriched results of references [9–11].

Acknowledgments

This work is supported by the Natural Science Foundation of Yunnan Province, China (no. 2013FZ117), and the National Natural Science Foundation of China (no. 11161020).

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