# He Chengtian's Inequalities for a Coupled Tangent Nonlinear System Arisen in Packaging System 

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#### Abstract

He Chengtian's inequalities from ancient Chinese algorithm are applied to strong tangent nonlinear packaging system. The approximate solution is obtained and compared with the solution yielded by computer simulation, showing a great high accuracy of this method. The suggested approach provides a novel method to solve some essential problems in packaging engineering.


## 1. Introduction

In order to avoid some restrictions of perturbation method [1], some other methods are developed, such as the homotopy perturbation method (HPM) [2, 3], the variational iteration method (VIM) [4-6], the homotopy analysis method (HAM) [7], and He Chengtian's inequalities which cannot be found in the literature but recently reported in [8]. The max-min approach which is developed from the idea of ancient Chinese mathematics is demonstrated to be of convenient application, less calculation, and high accuracy. Among current researches about He Chengtian's inequalities and their applications [9-11], few involved coupled nonlinear problems. In our previous research [12], He Chengtian's inequalities were introduced to study the nonlinear dropping shock response for coupled cubic nonlinear packaging system, showing the effectiveness of the method. In packaging system, many cushioning materials behave as the tangent nonlinear characteristics $[13,14]$, and the dropping shock response of tangent packaging system with critical component is also studied [15]. In this paper, He Chengtian's inequalities are applied to the coupled nonlinear tangent packaging system with critical component, and the obtained analytical solution is compared with the solution of computer simulation. The aim of this research is to suggest a new and simple mathematical method
for solving the nonlinear dropping shock equations arisen in packaging system.

## 2. Modelling and Equations

The governing equations of tangent nonlinear cushioning packaging system with the critical component can be expressed as [15]

$$
\begin{gather*}
m_{1} x^{\prime \prime}+k_{1}(x-y)=0 \\
m_{2} y^{\prime \prime}+\frac{2 k_{2} d_{b}}{\pi} \tan \frac{\pi}{2 d_{b}} y-k_{1}(x-y)=0 \tag{1}
\end{gather*}
$$

where

$$
\begin{gather*}
x(0)=0, \\
y(0)=0, \\
x^{\prime}(0)=\sqrt{2 g h},  \tag{2}\\
y^{\prime}(0)=\sqrt{2 g h} .
\end{gather*}
$$

Here the coefficients $m_{1}$ and $m_{2}$ denote the mass of the critical component and the main part of the product,
respectively, while $k_{1}$ and $k_{2}$ are the coupling stiffness of the critical component and that of cushioning pad, respectively, $d_{b}$ is the compression limit of the cushioning pad, and $h$ is the dropping height. Equation (1) can be equivalently written in the following forms:

$$
\begin{gather*}
\ddot{X}+\omega_{1}^{2}(X-Y)=0 \\
\ddot{Y}+Y+\frac{1}{3} Y^{3}+\frac{2}{15} Y^{5}+\left(1-\omega_{2}^{2}\right)(X-Y)=0 \tag{3}
\end{gather*}
$$

where

$$
\begin{align*}
& X=\frac{x}{\sqrt{2 d_{b} / \pi}},  \tag{4}\\
& Y=\frac{y}{\sqrt{2 d_{b} / \pi}},  \tag{5}\\
& \tau=\frac{t}{\sqrt{m_{2} / k_{2}}},  \tag{6}\\
& \omega_{1}=\sqrt{\frac{k_{1} m_{2}}{k_{2} m_{1}}},  \tag{7}\\
& \omega_{2}=\sqrt{1+\frac{m_{1} \omega_{1}^{2}}{m_{2}}},  \tag{8}\\
& X(0)=0,  \tag{9}\\
& Y(0)=0,  \tag{10}\\
& \dot{X}(0)=\frac{\sqrt{m_{2} / k_{2}}}{\sqrt{2 d_{b} / \pi}} \sqrt{2 g h},  \tag{11}\\
& \dot{Y}(0)=\frac{\sqrt{m_{2} / k_{2}}}{\sqrt{2 d_{b} / \pi}} \sqrt{2 g h .} \tag{12}
\end{align*}
$$

## 3. Application of He Chengtian's Inequalities

From (3), we can easily obtain

$$
\begin{align*}
Y^{(4)} & +\left(\omega_{1}^{2}+\omega_{2}^{2}+Y^{2}+\frac{2}{3} Y^{4}\right) \ddot{Y}  \tag{13}\\
& +\omega_{1}^{2}\left(Y+\frac{1}{3} Y^{3}+\frac{2}{15} Y^{5}\right)=0
\end{align*}
$$

Rewrite (13) in the form

$$
\begin{align*}
Y^{(4)}=- & {\left[\left(\frac{\omega_{1}^{2}+\omega_{2}^{2}}{Y}+Y^{2}+\frac{2}{3} Y^{3}\right) \ddot{Y}\right.} \\
& \left.+\omega_{1}^{2}\left(1+\frac{1}{3} Y^{2}+\frac{2}{15} Y^{4}\right)\right] Y . \tag{14}
\end{align*}
$$

According to He Chengtian's inequalities, we choose a trialfunction in the form

$$
\begin{equation*}
Y=A \sin (\Omega \tau) \tag{15}
\end{equation*}
$$

which meets the initial conditions as described in (10) and (12).

By simple analysis, from (14)-(15), we know that

$$
\begin{align*}
\Omega^{4}= & \left(\omega_{1}^{2}+\omega_{2}^{2}\right) \Omega^{2}-\omega_{1}^{2} \\
& +\left(A^{2} \Omega^{2} \sin ^{2} \Omega \tau+\frac{2}{3} A^{4} \Omega^{2} \sin ^{4} \Omega \tau\right)  \tag{16}\\
& -\left(\frac{1}{3} A^{2} \omega_{1}^{2} \sin ^{2} \Omega \tau+\frac{2}{15} A^{4} \omega_{1}^{2} \sin ^{4} \Omega \tau\right) .
\end{align*}
$$

The maximal and minimal value of $\sin ^{2} \Omega \tau$ are, respectively, 1 and 0 . So, we can immediately obtain

$$
\begin{align*}
f_{\min } & =\left(\omega_{1}^{2}+\omega_{2}^{2}+A^{2}+\frac{2}{3} A^{4}\right) \Omega^{2}-\frac{2 A^{4}+5 A^{2}+15}{15} \omega_{1}^{2} \\
& <\Omega^{4}<\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \Omega^{2}-\omega_{1}^{2}=f_{\max } \tag{17}
\end{align*}
$$

According to He Chengtian's interpolation [8, 12], we obtain

$$
\begin{equation*}
\Omega^{4}=\frac{m f_{\min }+n f_{\max }}{m+n}=\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \Omega^{2}-\omega_{1}^{2}+k M \tag{18}
\end{equation*}
$$

where $m$ and $n$ are weighting factors, $k=m /(m+n)$, and $M=\left(A^{2}+(2 / 3) A^{4}\right) \Omega^{2}-\left(\left(2 A^{4}+5 A^{2}\right) / 15\right) \omega_{1}^{2}$.

Then, the approximate solution of (13) can be written as

$$
\begin{equation*}
Y=A \sin \left[\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \Omega^{2}-\omega_{1}^{2}+k M\right]^{1 / 4} \tau \tag{19}
\end{equation*}
$$

To determine the value of $k$, substituting (19) into (13) results in the following residual [8]:

$$
\begin{equation*}
R(\tau, k)=\left(\Omega^{2}-\frac{1}{3} \omega_{1}^{2}\right) Y^{3}+\left(\frac{2}{3} \Omega^{2}-\frac{2}{15} \omega_{1}^{2}\right) Y^{5}-k M Y \tag{20}
\end{equation*}
$$

And by setting

$$
\begin{equation*}
\int_{0}^{T / 4} R(\tau, k) \sin \Omega \tau d \tau=0 \tag{21}
\end{equation*}
$$

where $T=2 \pi / \Omega$, we obtain the $k$ value as

$$
\begin{equation*}
k=\frac{6 A^{2}\left(\Omega^{2}-(1 / 3) \omega_{1}^{2}\right)+5 A^{4}\left((2 / 3) \Omega^{2}-(2 / 15) \omega_{1}^{2}\right)}{8 M} \tag{22}
\end{equation*}
$$

Substituting (22) into (18) yields

$$
\begin{align*}
8 \Omega^{4}= & 8\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \Omega^{2}-8 \omega_{1}^{2}+6 A^{2}\left(\Omega^{2}-\frac{1}{3} \omega_{1}^{2}\right)  \tag{23}\\
& +5 A^{4}\left(\frac{2}{3} \Omega^{2}-\frac{2}{15} \omega_{1}^{2}\right)
\end{align*}
$$

From (23), we can easily obtain the frequency value $\Omega$. Table 1 gives the values of $\Omega$ with different $\omega_{1}$ and $\omega_{2}$, and Figure 1 shows that the approximate solution, (19), agrees well with the exact solution for various different values of $\omega_{1}$ and $\omega_{2}$, where the initial velocity is assumed as $\dot{Y}(0)=A \Omega=1$.


Figure 1: Comparison of the approximate solution with the exact solution (asterisk: the approximate solution; continuous line: the exact solution).

Table 1: Values of $\Omega$ from (23) with different values of $\omega_{1}$ and $\omega_{2}$.

|  |  | $\omega_{2}$ |  |
| :---: | :---: | :---: | :---: |
| $\omega_{1}$ | 1 | 2 | 3 |
| 3 | 1.08323559 | 0.95968791 | 0.84368664 |
| 4 | 1.10039777 | 1.02416608 | 0.93503514 |
| 5 | 1.10757043 | 1.05683775 | 0.99009216 |

## 4. Conclusion

He Chengtian's inequalities are for the first time applied to study the nonlinear response of coupled tangent packaging system. The results show that this method can be easily used in engineering application with high accuracy without cumbersome calculation.

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