Research Article

On Structural Patterns of Mechanical Systems with Impacts with One and Two Degrees-of-Freedom

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Structural patterns of mechanical systems with impacts with one degree-of-freedom and two degrees-of-freedom, with elastic connections, have been identified and described. For their identification, a general method proposed by the author has been applied. This method uses (i) a matrix representation of the system with impacts, (ii) procedures that enable generations of all combinations of such systems as well as their identification and elimination of redundant equivalent combinations, and (iii) a procedure for elimination of disconnected systems.

1. Introduction

Systems with one degree-of-freedom belong to the simplest mechanical systems. These systems have been analyzed by numerous authors (e.g., [1–44]), and their dynamics have been investigated in a wide range. The following issues have been considered in the above-mentioned publications: steady periodic motions, subharmonic motions, stability of vibrations, elastic and plastic impacts, time and phase characteristics, behavior of the system under an influence of variations in the rigidity coefficient, the viscous damping coefficient, or the dry friction coefficient.

Special attention should be here drawn to works by Peterka and his coworkers, Nordmark, Chin et al., Ivanov, Shaw, and by Shaw and Holmes. Peterka et al. [32–34, 75] determine the regions of stable periodic motions and show a transition to chaotic motions both for the case of a linear, elastic-damping supporting structure, and for the case where dry friction occurs in the system. They present the results of numerical investigations in the form of time functions, phase trajectories, and Poincare maps. Nordmark [29] conducted the investigations on a motion of the one-degree-of-freedom oscillator in which impacts occur under the influence of external excitation. He undertook a trial to

describe the moment of impact and showed that a change in external parameters could lead to qualitative changes in the motion character, from a periodic motion, through bifurcation cycles, up to chaotic motions. Moreover, using analytical methods, he presented a new impact phenomenon of the grazing type and a new type of bifurcation that follows from it. Various types of grazing bifurcations were discussed by Chin et al. [7]. The grazing type impacts were dealt with by Ivanov ([17, 18]) as well. Shaw and Holmes [37–39] discuss a simple conservative system with one limiting stop. On the assumption that the coefficient of restitution equals zero, the authors reduce the considerations to a one-dimensional map whose analysis shows that stable orbits exist for nearly all frequencies of the excitation force. Shaw [36] observed harmonic, subharmonic, and chaotic motions while analyzing the linear model of an oscillator with two limiting stops subjected to harmonic excitation, in which the processes occurring in the system at the instant of impact are modeled with Newton's hypothesis (Goldsmith [79]).

In many studies (e.g., [4, 19, 26, 49, 80–85]), the impact is modeled as an abrupt change in rigidity. It allows one to use various analytical methods in the dynamical analysis of the system. For instance, Blankenship and Kahraman [4] as well as Kahraman and Singh [19] analyzed a vibroimpact

system with clearances subjected to parametric and periodic excitation. The results of the numerical analyses conducted on the basis of the methods of the harmonics balance were confirmed by the experimental results. Kim and Noah [80] employed the harmonics balance method connected with a Fourier transform procedure to analyze the system dynamics with linear characteristics in segments.

The literature devoted to mechanical systems with impacts on two degrees-of-freedom is much less extensive. Although, in this case, the first studies were published slightly later, interesting applications were already mentioned in them. Special attention should be paid to Sadek [71], Masri [86], Masri and Caughey [70], Masri and Ibrahim [73], Araki et al. [87], Bapat and Sankar [88], Cempel [5], Peterka [32, 33], Chatterjee et al. [72], Bazhenov et al. [55], Peterka and Blazejczyk-Okolewska [75], including theoretical, experimental, and numerical analyses of impact vibration dampers, Karpenko et al. ([89–91]), providing extensive analytical, numerical, and experimental studies of a Jeffcott rotor with a snubber ring, and Koizumi [92], Park [93], and Luo et al. [60], describing the applications of impact oscillators as models of moulding machines. Numerous examples of other applications of systems with impacts can be found in A. E. Kobrinskii and A. Kobrinskii [20], Babicki [47], and Jerrelind and Stensson [94].

Lately the efforts of researchers investigating the dynamics of vibroimpact systems have been focused on the theory of stability, bifurcations, the reasons for occurrence of chaos (e.g., Aidanpää and Gupta [51], Brach [95], Chen [96], Chin et al. [7], Blazejczyk-Okolewska and Kapitaniak [76, 77], Czolczynski [57], Czolczynski and Kapitaniak [58], Di Bernardo et al. [9], Han et al. [67], Ivanov [18], Awrejcewicz and Lamarque [97], Bishop [98], Blazejczyk-Okolewska et al. [99], Brogliato [100], Guckenheimer and Holmes [101], Ibrahim [102], Rand and Moon [103], and Thompson and Bishop [104]), and control of such systems (e.g., Awrejcewicz et al. [2], Gutiérrez and Arrowsmith [105], de Souza and Caldas [66], de Souza et al. [106], Lee and Yan [23], Luo [107], Luo and Lv [62], Zhao and Wang [108], and Zinjade and Mallik [109]).

In the literature survey presented here, first of all the publications devoted to new dynamic behaviors have been mentioned. The occurrence of specific behaviors of the system is strictly related to the physical model assumed for its analysis. In Blazejczyk-Okolewska et al. [110], numerous schemes of models of mechanical systems with impacts were presented. For the initial analysis of the above-mentioned systems, physical models in which all linear and nonlinear supporting structures are described with the most elementary rheological model, a spring with linear characteristics, can be used.

The comparative studies of physical models of vibroimpact systems used in scientific considerations have led to the determination of assumptions and principles of classification of mechanical systems with impacts in which models are rigid bodies that can move along a straight line without a possibility to rotate (Blazejczyk-Okolewska et al. [110]) and to the development of the method that allows one to determine

structural patterns of all such systems with an arbitrary number of degrees-of-freedom (Blazejczyk-Okolewska [111]). By a structural pattern of the mechanical system with impacts is understood a certain series of systems characterized by a specified structure of component elements (a definite configuration of fenders and connections). All structural patterns of mechanical systems constitute a set in which a kind of the connection (a spring or a damper), its character (linearity or nonlinearity), and the way the impact phenomenon is modeled are not differentiating parameters.

In the present study, the above-mentioned method has been illustrated on the example of one- and two-degrees-of-freedom systems. All structural patterns of mechanical one- and two-degrees-of-freedom systems with impacts, with arbitrary connections, have been identified and described with this method. Next, structural patterns have been assigned to vibroimpact systems selected from the literature. The knowledge of all systems with impacts of a given number of degrees-of-freedom allows us to state which types have been already analyzed and which have not been investigated yet. The proposed classification of mechanical systems with impacts according to their structure allows us to rearrange the knowledge on systems with impacts and is the basis for understanding the sources of their diversity. Providing a full set of objects to be analyzed, it gives hints for new ideas and directions in designing technical devices.

In the considerations below, the terminology, concepts, and notations introduced by Blazejczyk-Okolewska [111] have been used.

2. Systems with Two Degrees-of-Freedom

Systems with two degrees-of-freedom ($n = 2$) have one connectedness zone. Numerous equivalent combinations and many disconnected systems can be identified among them.

The maximal number of springs (spring connections) $s = 3$ (k_1 is the spring connecting the subsystem of a mass m_1 with the frame, k_{12} the spring connecting the subsystem of a mass m_1 with the subsystem of the mass m_2 , and k_2 the spring connecting the subsystem of a mass m_2 with the frame). The number of possible spring combinations is $i_s = 8$. All connected and disconnected combinations are presented in Figures 1 and 2.

The spring connectedness zone is located between the subsystems with the masses m_1 and m_2 . This zone consists of just one spring k_{12} , which decides whether the basic spring system will be divided into two subsystems or not (in the case of a system with a higher number of degrees-of-freedom, a division into two or more subsystems can occur, of course). The number i_{s-s} of all connected spring combinations is equal to a product of the number of possible arrangements of the springs that decide about connectedness (one) and the number of possible arrangements beyond the spring connectedness zone of the remaining 0, 1, or 2 springs (four); that is to say, $i_{s-s} = 1 \times 4 = 4$ (see Figure 1). The notations of connected spring combinations (S_1 , S_2 , S_5 , and S_6) have been introduced according to the generation procedure presented

FIGURE 1: Connected spring combinations for $n = 2$. FIGURE 2: Disconnected spring combinations for $n = 2$.

by Blazejczyk-Okolewska ([111, Subsection 4.1.1]). One of these systems (S_1) is the basic spring system.

Similarly, the number i_{s-n} of all disconnected spring combinations can be defined. This number is equal to a product of the number of possible arrangements of the springs that decide about the connectedness (one) and the number of possible arrangements beyond the spring connectedness zone of the remaining 0, 1, or 2 springs (four); that is to say, $i_{s-n} = 1 \times 4 = 4$ (see Figure 2). The notations of disconnected spring combinations (S_3 , S_4 , S_7 , and S_8) have been introduced according to the above-mentioned generation procedure. The total number of connected and disconnected spring combinations is of course equal to the number of all spring combinations; that is, $i_s = i_{s-s} + i_{s-n} = 4 + 4 = 8$.

In the case of connected spring combinations (Figure 1), the body of a mass m_1 acts on the body of a mass m_2 (and vice versa, of course), due to an occurrence of the spring connection between these bodies (the spring k_{12}). On the other hand, in the case of disconnected spring combinations (Figure 2), the body of a mass m_1 will never affect the behavior of the body of a mass m_2 (and vice versa, of course). A lack of the spring connection results in the fact that the bodies “will not know” about each other.

Spring adjacency matrices are attributed to all spring combinations. To list them, also the generation procedure has been used. Let us pay attention to the fact that elements and indices of some spring adjacency matrices have been bolded (in Figure 1: S_1 , S_6 ; in Figure 2: S_3 , S_8). This bolding refers to spring systems that are equivalent to each other by inversion (the definition of equivalency has been given by Blazejczyk-Okolewska [111]).

We will conduct a similar analysis for impact connections. The maximal number of fenders (impact connections) is equal to $z = 6$. These are the following connections: z_{1g} is the upper impact connection of the subsystem of a mass m_1 with the frame, z_{1d} the lower impact connection of the subsystem of a mass m_1 with the frame, z_{12g} the upper impact connection of the subsystem of a mass m_1 with the subsystem of a mass m_2 , z_{12d} the lower impact connection of the subsystem of a mass m_1 with the subsystem of a mass m_2 , z_{2g} —upper impact connection of the subsystem of a mass m_2 with the frame, and z_{2d} the lower impact connection of the subsystem of a mass m_2 with the frame. The number of impact connections is equal to $i_z = 64$. All connected and disconnected impact combinations are presented in Figures 3, 4, 5, and 6.

FIGURE 3: Connected impact combinations (fenders z_{12g} and z_{12d}) for $n = 2$.

FIGURE 4: Connected impact combinations (fender z_{12d}) for $n = 2$.

FIGURE 5: Connected impact combinations (fender z_{12g}) for $n = 2$.

FIGURE 6: Disconnected impact combinations (lack of the fenders z_{12g} and z_{12d}) for $n = 2$.

TABLE 1: The way connected and disconnected spring-impact combinations are generated (before elimination) for $n = 2$.

	$i_{s-s} = 4$: connected spring combinations	$i_{s-n} = 4$: disconnected spring combinations
$i_{z-s} = 48$: connected impact combinations	$i_{sz-s} = i_{s-s} \times i_{z-s} = 4 \times 48 = 192$: connected spring-impact combinations	$i_{sz-s} = i_{s-n} \times i_{z-s} = 4 \times 48 = 192$: connected spring-impact combinations
$i_{z-n} = 16$: disconnected impact combinations	$i_{sz-s} = i_{s-s} \times i_{z-n} = 4 \times 16 = 64$: connected spring-impact combinations	$i_{sz-n} = i_{s-n} \times i_{z-n} = 4 \times 16 = 64$: disconnected spring-impact combinations

TABLE 2: Selected classes of spring-impact relations and their representatives.

Classes of spring-impact relations			
Representatives of the class of relations	Remaining systems from the class of relations	Representatives of the class of relations	Remaining systems from the class of relations
$S_1 Z_1$		$S_5 Z_1$	$S_2 Z_1$
$S_1 Z_{17}$	$S_1 Z_2, S_1 Z_9, S_1 Z_5$	$S_5 Z_9$	$S_5 Z_2, S_2 Z_5, S_2 Z_{17}$
$S_1 Z_{18}$	$S_1 Z_{13}$	$S_5 Z_{10}$	$S_2 Z_{21}$
$S_1 Z_{21}$	$S_1 Z_{10}$	$S_5 Z_{17}$	$S_2 Z_2, S_2 Z_9, S_5 Z_5$
$S_1 Z_{25}$	$S_1 Z_6$	$S_5 Z_{18}$	$S_2 Z_{13}, S_2 Z_{18}, S_5 Z_{13}$
$S_1 Z_{29}$	$S_1 Z_{14}, S_1 Z_{26}, S_1 Z_{22}$	$S_5 Z_{21}$	$S_2 Z_{10}$
$S_1 Z_{30}$		$S_5 Z_{25}$	$S_2 Z_6, S_2 Z_{25}, S_5 Z_6$
$S_1 Z_{33}$	$S_1 Z_3$	$S_5 Z_{26}$	$S_5 Z_{14}, S_2 Z_{22}, S_2 Z_{29}$
$S_1 Z_{35}$		$S_5 Z_{29}$	$S_2 Z_{14}, S_2 Z_{26}, S_5 Z_{22}$
$S_1 Z_{41}$	$S_1 Z_4, S_1 Z_{37}, S_1 Z_{19}$	$S_5 Z_{30}$	$S_2 Z_{30}$
$S_1 Z_{45}$	$S_1 Z_{20}$	$S_5 Z_{33}$	$S_2 Z_3, S_2 Z_{33}, S_5 Z_3$
$S_1 Z_{49}$	$S_1 Z_7, S_1 Z_{34}, S_1 Z_{11}$	$S_5 Z_{34}$	$S_2 Z_7, S_2 Z_{49}, S_5 Z_{11}$
$S_1 Z_{50}$	$S_1 Z_{15}$	$S_5 Z_{35}$	$S_2 Z_{35}$
$S_1 Z_{51}$	$S_1 Z_{36}, S_1 Z_{43}, S_1 Z_{39}$	$S_5 Z_{37}$	$S_2 Z_4, S_2 Z_{41}, S_5 Z_{19}$
$S_1 Z_{52}$	$S_1 Z_{47}$	$S_5 Z_{38}$	$S_2 Z_8, S_2 Z_{57}, S_5 Z_{27}$
$S_1 Z_{53}$	$S_1 Z_{12}, S_1 Z_{42}, S_1 Z_{23}$	$S_5 Z_{41}$	$S_5 Z_4, S_2 Z_{37}, S_2 Z_{19}$
$S_1 Z_{55}$	$S_1 Z_{44}$	$S_5 Z_{42}$	$S_5 Z_{12}, S_2 Z_{53}, S_2 Z_{23}$
$S_1 Z_{57}$	$S_1 Z_8, S_1 Z_{38}, S_1 Z_{27}$	$S_5 Z_{43}$	$S_5 Z_{36}, S_2 Z_{39}, S_2 Z_{51}$
$S_1 Z_{58}$	$S_1 Z_{16}, S_1 Z_{54}, S_1 Z_{31}$	$S_5 Z_{44}$	$S_2 Z_{55}$
$S_1 Z_{59}$	$S_1 Z_{40}$	$S_5 Z_{45}$	$S_2 Z_{20}, S_2 Z_{45}, S_5 Z_{20}$
$S_1 Z_{61}$	$S_1 Z_{24}, S_1 Z_{46}, S_1 Z_{28}$	$S_5 Z_{46}$	$S_2 Z_{24}, S_2 Z_{61}, S_5 Z_{28}$
$S_1 Z_{62}$	$S_1 Z_{32}$	$S_5 Z_{49}$	$S_5 Z_7, S_2 Z_{34}, S_2 Z_{11}$
$S_1 Z_{63}$	$S_1 Z_{48}, S_1 Z_{60}, S_1 Z_{56}$	$S_5 Z_{50}$	$S_2 Z_{15}, S_2 Z_{50}, S_5 Z_{15}$
$S_1 Z_{64}$		$S_5 Z_{51}$	$S_2 Z_{36}, S_2 Z_{43}, S_5 Z_{39}$
		$S_5 Z_{52}$	$S_2 Z_{47}, S_2 Z_{52}, S_5 Z_{47}$
		$S_5 Z_{53}$	$S_2 Z_{12}, S_2 Z_{42}, S_5 Z_{23}$
		$S_5 Z_{54}$	$S_2 Z_{16}, S_2 Z_{58}, S_5 Z_{31}$
		$S_5 Z_{55}$	$S_2 Z_{44}$
		$S_5 Z_{57}$	$S_5 Z_8, S_2 Z_{38}, S_2 Z_{27}$
		$S_5 Z_{58}$	$S_5 Z_{16}, S_2 Z_{54}, S_2 Z_{31}$
		$S_5 Z_{59}$	$S_2 Z_{40}, S_2 Z_{59}, S_5 Z_{40}$
		$S_5 Z_{60}$	$S_5 Z_{48}, S_2 Z_{56}, S_2 Z_{63}$
		$S_5 Z_{61}$	$S_5 Z_{24}, S_2 Z_{46}, S_2 Z_{28}$
		$S_5 Z_{62}$	$S_2 Z_{32}, S_2 Z_{62}, S_5 Z_{32}$
		$S_5 Z_{63}$	$S_2 Z_{48}, S_2 Z_{60}, S_5 Z_{56}$
		$S_5 Z_{64}$	$S_2 Z_{64}$

TABLE 3: Table of spring relations for $n = 2$.

Fields of spring information	
S_i	S_i^P
1	1
2	5
3	3
4	7
5	2
6	6
7	4
8	8

The impact connectedness zone is located between the subsystems of masses m_1 and m_2 . This zone consists of two fenders z_{12g} and z_{12d} that decide whether a division of the basic impact system into two subsystems will occur or not. The number i_{z-s} of all connected impact combinations is equal to a product of the number of possible arrangements of the impact connections that decide about the connectedness (three) and the number of possible arrangements, beyond the impact adjacency zone, of the remaining impact connections ($2^4 = 16$); that is to say, $i_{z-s} = 3 \times 16 = 48$. Figure 3 shows the systems: $Z_1, Z_2, Z_5, Z_6, Z_9, Z_{10}, Z_{13}, Z_{14}, Z_{17}, Z_{18}, Z_{21}, Z_{22}, Z_{25}, Z_{26}, Z_{29}$, and Z_{30} , that is, all such connected impact combinations that the connection between subsystems occurs both via the upper fender z_{12g} and via the lower fender z_{12d} . One of these systems (Z_1) is the basic impact system. In Figure 4 the following systems are marked: $Z_3, Z_4, Z_7, Z_8, Z_{11}, Z_{12}, Z_{15}, Z_{16}, Z_{19}, Z_{20}, Z_{23}, Z_{24}, Z_{27}, Z_{28}, Z_{31}$, and Z_{32} , that is, all connected impact combinations in which the connection between subsystems occurs only via the lower fender z_{12d} . Figure 5 presents the systems: $Z_{33}, Z_{34}, Z_{37}, Z_{38}, Z_{41}, Z_{42}, Z_{45}, Z_{46}, Z_{49}, Z_{50}, Z_{53}, Z_{54}, Z_{57}, Z_{58}, Z_{61}$, and Z_{62} , that is, all connected impact combinations in which the connection between subsystems occurs only via the upper fender z_{12g} . The notations of connected impact combinations have been introduced according to the generation procedure discussed by Blazejczyk-Okolewska ([11], Section 4.1.1]). The number i_{z-n} of all disconnected impact combinations is equal to a product of the number of possible arrangements of the impact connections that decide about the disconnectedness (one) and the number of possible arrangements, beyond the impact connectedness zone, of the remaining impact connections ($2^4 = 16$); that is to say, $i_{z-n} = 1 \times 16 = 16$. In Figure 6, the following systems are depicted: $Z_{35}, Z_{36}, Z_{39}, Z_{40}, Z_{43}, Z_{44}, Z_{47}, Z_{48}, Z_{51}, Z_{52}, Z_{55}, Z_{56}, Z_{59}, Z_{60}, Z_{63}$, and Z_{64} , that is, all disconnected impact combinations. The total number of connected and disconnected impact combinations is equal to the number of all impact combinations $i_z = i_{z-s} + i_{z-n} = 48 + 16 = 64$. The impact adjacency matrices are attributed to all impact combinations.

Let us notice that the elements and indices of some impact adjacency matrices, similarly as spring adjacency matrices, have been bolded (in Figure 3: $Z_1, Z_6, Z_{13}, Z_{18}, Z_{25}$, and Z_{30} ; in Figure 4: Z_3, Z_{15}, Z_{20} , and Z_{32} ; in Figure 5: Z_{33}, Z_{45}, Z_{50} ,

TABLE 4: Table of impact relations for $n = 2$.

Fields of impact information			
Z_i	Z_i^T	Z_i^P	Z_i^{PT}
1	1	1	1
2	9	5	17
3	33	33	3
4	41	37	19
5	17	2	9
6	25	6	25
7	49	34	11
8	57	38	27
9	2	17	5
10	10	21	21
11	34	49	7
12	42	53	23
13	18	18	13
14	26	22	29
15	50	50	15
16	58	54	31
17	5	9	2
18	13	13	18
19	37	41	4
20	45	45	20
21	21	10	10
22	29	14	26
23	53	42	12
24	61	46	28
25	6	25	6
26	14	29	22
27	38	57	8
28	46	61	24
29	22	26	14
30	30	30	30
31	54	58	16
32	62	62	32
33	3	3	33
34	11	7	49
35	35	35	35
36	43	39	51
37	19	4	41
38	27	8	57
39	51	36	43
40	59	40	59
41	4	19	37
42	12	23	53
43	36	51	39
44	44	55	55
45	20	20	45

TABLE 4: Continued.

Fields of impact information			
Z_i	Z_i^T	Z_i^P	Z_i^{PT}
46	28	24	61
47	52	52	47
48	60	56	63
49	7	11	34
50	15	15	50
51	39	43	36
52	47	47	52
53	23	12	42
54	31	16	58
55	55	44	44
56	63	48	60
57	8	27	38
58	16	31	54
59	40	59	40
60	48	63	56
61	24	28	46
62	32	32	62
63	56	60	48
64	64	64	64

and Z_{62} ; in Figure 6: Z_{35} , Z_{40} , Z_{47} , Z_{52} , Z_{59} , and Z_{64}). These are all impact systems equivalent to each other via inversion or translocation.

In the case of forty-eight connected impact combinations (Figures 3, 4, and 5), the body of a mass m_1 acts on the body of a mass m_2 due to an occurrence of the impact connection between the subsystems (fenders z_{12g} and z_{12d} in Figure 3, one of these fenders in Figures 4 and 5). On the other hand, in the case of sixteen disconnected impact combinations (Figure 6), the body of a mass m_1 will never affect the behavior of the body of a mass m_2 (and vice versa, of course). A lack of the impact connection (a lack of both fenders z_{12g} and z_{12d}) will cause that the bodies “will not know” about each other.

According to the assumed principle (*Phase I*, Blazejczyk-Okolewska [111]), we will match now each spring combination with each impact combination. The total number of connected spring-impact combinations (for $n = 2$ before elimination) is equal to $i_{sz-s} = 448$, whereas the total number of disconnected spring-impact combinations (before elimination) is equal to $i_{sz-n} = 64$. Table 1 shows *Phase I* for the system of $n = 2$. As a result of matching all connected spring combinations (Figure 1) with all connected impact combinations (Figures 3, 4, and 5), connected spring-impact combinations arise. Their total number equals $i_{sz-s} = i_{s-s} \times i_{z-s} = 4 \times 48 = 192$. As a result of matching all connected spring combinations (Figure 1) with all disconnected impact combinations (Figure 6), connected spring-impact combinations arise. Their total number is $i_{sz-s} = i_{s-s} \times i_{z-n} = 4 \times 16 = 64$. As a result of matching all disconnected spring combinations (Figure 2) with all connected impact combinations (Figures 3, 4, and 5), connected spring-impact combinations arise. Their

number is equal to $i_{sz-s} = i_{s-n} \times i_{z-s} = 4 \times 48 = 192$. As a result of matching all disconnected spring combinations (Figure 2) with all disconnected impact combinations (Figure 6), disconnected spring-impact combinations arise. Their number equals $i_{sz-n} = i_{s-n} \times i_{z-n} = 4 \times 16 = 64$. The number of connected spring-impact combinations added to the number of disconnected spring-impact combinations is equal to the number of all spring-impact combinations; that is to say, $i_{sz} = i_{sz-s} + i_{sz-n} = 448 + 64 = 512$.

Due to the convenience of presentation, spring, impact and spring-impact combinations will be referred to as spring, impact, and spring-impact systems up to the end of this section.

The next stage consists in the elimination of equivalent combinations (subphase I of *Phase II*, Blazejczyk-Okolewska [111]). For $n = 2$, it can be conducted without a computer, but it is a very time-consuming procedure.

Let us begin with the generation of the adjacency matrix of all spring-impact systems. We take the first one, and we generate a transposed, inverted, and translocated matrix to it. Next, we check the equivalency of the taken matrix to itself and to the adjacency matrices of the subsequent spring-impact systems. If any equivalency is identified (see equations (2) ÷ (4), Blazejczyk-Okolewska [111]), we make a class of relations, and we choose the system with the highest member as its representative (see the principles of selection, Blazejczyk-Okolewska [111]). The above-mentioned activities should be repeated for each spring-impact system which has not been included in any class of relations yet.

As an example, let us consider the spring-impact system S_2Z_{23} . It is equivalent to other systems $S_2Z_{53}S_5Z_{12}$ and S_5Z_{42} . The class of relations in this case has four elements. We choose S_5Z_{42} as its representative as it is the system with the highest number, and the systems S_2Z_{23} , S_2Z_{53} , and S_5Z_{12} are thus eliminated.

Table 2 presents a list of selected classes of relations and their representatives. The first two columns include all spring-impact systems that arise as a result of matching the spring system S_1 with all impact systems $Z_1 \div Z_{64}$. An analogous list can be given for the remaining spring systems equivalent to one another via inversion, that is, S_3 , S_6 , and S_8 (in Table 2, we replace S_1 by the subsequent systems S_3 , S_6 , and S_8). The last two columns in the table include all spring-impact systems that result from matching the spring system S_5 with all impact systems $Z_1 \div Z_{64}$. An analogous list can be prepared for the spring system S_7 (in Table 2, we replace S_5 by the spring system S_7 and S_2 by the spring system S_4). For $n = 2$, there are 168 classes of relations. Among them, we can distinguish 24 classes of relations in which spring-impact systems are disconnected (see Table 5 discussed further in this section as well). Classes of spring-impact relations can be one-element (one spring-impact system which is simultaneously the representative of the class of relations), two elements (two equivalent spring-impact systems and one of them is the representative of the class of relations), or four elements (four equivalent spring-impact systems and one of them is the representative of the class of relations).

Below, in the context of the two-degrees-of-freedom system, an application of the procedures developed by

TABLE 5: List of connected and disconnected representatives of classes of connection-impact (spring-impact) relations.

Connected representatives of classes of relations																	
$P_1 = S_1$	Z_1 1	Z_{17} 4	Z_{18} 2	Z_{21} 2	Z_{25} 2	Z_{29} 4	Z_{30} 1										
	Z_{33} 2	Z_{41} 4	Z_{45} 2	Z_{49} 4	Z_{50} 2	Z_{53} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 2							
	Z_{35} 1	Z_{51} 4	Z_{52} 2	Z_{55} 2	Z_{59} 2	Z_{63} 4	Z_{64}^* 1										
$P_3 = S_3$	Z_1 1	Z_{17} 4	Z_{18} 2	Z_{21} 2	Z_{25} 2	Z_{29} 4	Z_{30} 1										
	Z_{33} 2	Z_{41} 4	Z_{45} 2	Z_{49} 4	Z_{50} 2	Z_{53} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 2							
$P_5 = S_5$	Z_1 2	Z_9 4	Z_{10} 2	Z_{17} 4	Z_{18} 4	Z_{21} 2	Z_{25} 4	Z_{26} 4	Z_{29} 4	Z_{30} 2							
	Z_{33} 4	Z_{34} 4	Z_{37} 4	Z_{38} 4	Z_{41} 4	Z_{42} 4	Z_{45} 4	Z_{46} 4	Z_{49} 4	Z_{50} 4	Z_{53} 4	Z_{54} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 4	
	Z_{35} 2	Z_{43} 4	Z_{44} 2	Z_{51} 4	Z_{52} 4	Z_{55} 2	Z_{59} 4	Z_{60} 4	Z_{63} 4	Z_{64}^* 2							
$P_6 = S_6$	Z_1 1	Z_{17} 4	Z_{18} 2	Z_{21} 2	Z_{25} 2	Z_{29} 4	Z_{30} 1										
	Z_{33} 2	Z_{41} 4	Z_{45} 2	Z_{49} 4	Z_{50} 2	Z_{53} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 2							
	Z_{35} 1	Z_{51} 4	Z_{52} 2	Z_{55} 2	Z_{59} 2	Z_{63} 4	Z_{64}^* 1										
$P_7 = S_7$	Z_1 2	Z_9 4	Z_{10} 2	Z_{17} 4	Z_{18} 4	Z_{21} 2	Z_{25} 4	Z_{26} 4	Z_{29} 4	Z_{30} 2							
	Z_{33} 4	Z_{34} 4	Z_{37} 4	Z_{38} 4	Z_{41} 4	Z_{42} 4	Z_{45} 4	Z_{46} 4	Z_{49} 4	Z_{50} 4	Z_{53} 4	Z_{54} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 4	
$P_8 = S_8^{\#}$	Z_1 1	Z_{17} 4	Z_{18} 2	Z_{21} 2	Z_{25} 2	Z_{29} 4	Z_{30} 1										
	Z_{33} 2	Z_{41} 4	Z_{45} 2	Z_{49} 4	Z_{50} 2	Z_{53} 4	Z_{57} 4	Z_{58} 4	Z_{61} 4	Z_{62} 2							
Disconnected representatives of classes of relations																	
$P_3 = S_3$	Z_{35} 1	Z_{51} 4	Z_{52} 2	Z_{55} 2	Z_{59} 2	Z_{63} 4	Z_{64}^* 1										
$P_7 = S_7$	Z_{35} 2	Z_{43} 4	Z_{44} 2	Z_{51} 4	Z_{52} 4	Z_{55} 2	Z_{59} 4	Z_{60} 4	Z_{63} 4	Z_{64}^* 2							
$P_8 = S_8^{\#}$	Z_{35} 1	Z_{51} 4	Z_{52} 2	Z_{55} 2	Z_{59} 2	Z_{63} 4	Z_{64}^* 1										

*Intensively shaded.

#Slightly shaded.

Blazejczyk-Okolewska [111] will be discussed. It is worth noticing that, whenever we take the subsequent adjacency matrix, we will have the full information on equivalencies to other adjacency matrices (both for spring and impact systems).

Table 3 is a table of spring relations for $n = 2$, that is to say, a list of all spring systems (the first column S_i) together with the information on their inversed equivalency (the second column S_i^P). This table has been made on the basis of the spring table generation procedure (Blazejczyk-Okolewska [111, Section 4.1.2]). Whenever we take the subsequent spring adjacency matrix, we have the full information on it. It follows from Table 3 that four spring systems are equivalent to one another via inversion (S_1 , S_3 , S_6 , and S_8), and there are two

pairs of spring systems equivalent via inversion: $S_5 \sim S_2$ and $S_7 \sim S_4$ ($5 = S_5 > S_2 = 2$, $7 = S_7 > S_4 = 4$).

Table 4 is a table of impact relations for $n = 2$, that is to say, a list of all impact systems (the first column Z_i) with the information on their transposed (the second column Z_i^T), inversed (the third column Z_i^P), and translocated (the fourth column Z_i^{PT}) equivalency. This table has been made according to the impact table generation procedure (Blazejczyk-Okolewska [111, Section 4.1.2]). Whenever we take the next impact adjacency matrix, we have the full information on it. We can read from Table 4 that for twenty-four impact systems, the number of the matrix Z_i (it refers to the lower index) fulfils condition (6) described by Blazejczyk-Okolewska [111]. All the systems (Z_1 , Z_{17} , Z_{18} , Z_{21} , Z_{25} ,

TABLE 6: Examples from the references ($n = 1$) and the notations of the structural pattern (S_p) of the systems considered in them.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
1		Cempel [5]	$S_p = \mathbf{P}_2 \mathbf{Z}_3$
2		Krivtsov and Wiercigroch [45]	$S_p = \mathbf{P}_2 \mathbf{Z}_3$
3		Pavlovskaja et al. [31]	$S_p = \mathbf{P}_2 \mathbf{Z}_3$
4		Ajibose et al. [46]	$S_p = \mathbf{P}_2 \mathbf{Z}_3$
5		Babickii [47]; Li and Lu [25]; Peterka [33]; Shaw and Holmes [37]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
6		Chin et al. [6, 7]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
7		de Weger et al. [8]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
8		Hinrichs et al. [14]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
9		Bichri et al. [3]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
10		Di Bernardo et al. [9]; Foale and Bishop [13]; Isomäki et al. [16]; Ivanov [18]; Peterka and Vacík [34]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$

TABLE 6: Continued.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
11		Stefanski and Kapitaniak [40, 41]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
12		Awrejcewicz et al. [2]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
13		Ma et al. [48]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
14		Nordmark [30]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
15		Ing et al. [49]; Shaw and Holmes [37]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
16		Ma et al. [48]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
17		Andreaus et al. [1]; Kundu et al. [22]; Ma et al. [48]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
18		Püst and Peterka [35]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
19		Ma et al. [48]	$S_p = \mathbf{P}_1 \mathbf{Z}_3$
20		Babickii [47]; Cempel [5]; Lee and Yan [23]	$S_p = \mathbf{P}_1 \mathbf{Z}_1$

TABLE 6: Continued.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
21		Luo and Menon [27]	$S_p = \mathbf{P}_1 \mathbf{Z}_1$
22		Blankenship and Kahraman [4]; Kahraman and Singh [19]	$S_p = \mathbf{P}_1 \mathbf{Z}_1$
23		Lin and Bapat [26]; Wiercigroch and Sin [50]	$S_p = \mathbf{P}_1 \mathbf{Z}_1$
24		Leine and van Campen [24]	$S_p = \mathbf{P}_1 \mathbf{Z}_1$

$\mathbf{Z}_{29}$, $\mathbf{Z}_{30}$, $\mathbf{Z}_{33}$, $\mathbf{Z}_{35}$, $\mathbf{Z}_{41}$, $\mathbf{Z}_{45}$, $\mathbf{Z}_{49}$, $\mathbf{Z}_{50}$, $\mathbf{Z}_{51}$, $\mathbf{Z}_{52}$, $\mathbf{Z}_{53}$, $\mathbf{Z}_{55}$, $\mathbf{Z}_{57}$, $\mathbf{Z}_{58}$, $\mathbf{Z}_{59}$, $\mathbf{Z}_{61}$, $\mathbf{Z}_{62}$, $\mathbf{Z}_{63}$, and $\mathbf{Z}_{64}$) will be matched to those spring systems which are equivalent to one another via inversion ($\mathbf{S}_1$, $\mathbf{S}_3$, $\mathbf{S}_6$, and $\mathbf{S}_8$). After matching, we will obtain 96 spring-impact systems that are the representatives of classes of relations. Among them, 48 systems are equivalent to one another via inversion or translocation. Their lower indices will be bolded.

The remaining two spring systems ($\mathbf{S}_5$ and $\mathbf{S}_7$) should be matched to impact systems fulfilling condition (6) and, additionally, to impact systems from the classes of relations that fulfill condition (7) of the numbers equal to the higher number out of the two numbers $\mathbf{Z}^P$ and $\mathbf{Z}^{PT}$. All the impact systems satisfying these conditions, and we have 36 such systems altogether ($\mathbf{Z}_1$, $\mathbf{Z}_9$, $\mathbf{Z}_{10}$, $\mathbf{Z}_{17}$, $\mathbf{Z}_{18}$, $\mathbf{Z}_{21}$, $\mathbf{Z}_{25}$, $\mathbf{Z}_{26}$, $\mathbf{Z}_{29}$, $\mathbf{Z}_{30}$, $\mathbf{Z}_{33}$, $\mathbf{Z}_{34}$, $\mathbf{Z}_{35}$, $\mathbf{Z}_{37}$, $\mathbf{Z}_{38}$, $\mathbf{Z}_{41}$, $\mathbf{Z}_{42}$, $\mathbf{Z}_{43}$, $\mathbf{Z}_{44}$, $\mathbf{Z}_{45}$, $\mathbf{Z}_{46}$, $\mathbf{Z}_{49}$, $\mathbf{Z}_{50}$, $\mathbf{Z}_{51}$, $\mathbf{Z}_{52}$, $\mathbf{Z}_{53}$, $\mathbf{Z}_{54}$, $\mathbf{Z}_{55}$, $\mathbf{Z}_{57}$, $\mathbf{Z}_{58}$, $\mathbf{Z}_{59}$, $\mathbf{Z}_{60}$, $\mathbf{Z}_{61}$, $\mathbf{Z}_{62}$, $\mathbf{Z}_{63}$, and $\mathbf{Z}_{64}$), after matching to the spring systems ($\mathbf{S}_5$ and $\mathbf{S}_7$), yield 72 spring-impact systems that make another group of the representatives of classes of relations.

The total number of spring-impact systems which remain after the elimination of equivalent systems is thus $96 + 72 = 168$.

Table 5 shows all spring-impact systems that have not been eliminated. They are divided into two groups: connected systems and disconnected systems.

The list has been obtained with the identity procedures (Blażejczyk-Okolewska [111, Subsections 4.1 and 4.2]), which eliminate equivalent systems and investigate their connectedness. In the first column of Table 5, there are spring systems. In the remaining part of the table, there are impact systems. The digit written under the symbol of the impact system corresponds to the number of systems in the given class of spring-impact relations (including the representative of this class). The bolding of this digit and of the index of the adjacency matrix $\mathbf{S}_i \mathbf{Z}_j$ means that the given spring-impact system is equivalent to itself via inversion of translocation. On such a list as the one prepared, an actual spring-impact system can be found easily. There are altogether 144 connected spring-impact systems (including 3 combinations without impacts), whereas the number of all disconnected spring-impact systems is equal to 24. It means that, as a result of the elimination of equivalent and disconnected systems, the original number of systems (512) has decreased more than three times (144).

Table 5 presents these spring-impact systems that have been selected as the representatives of classes of relations. Its comparison to Tables 3 and 4 provides the information on all the systems which belong to the given class of relations.

The spring-impact systems (structural patterns) presented in Table 5 are written in such a way that the given spring connection and the first line starting with $\mathbf{Z}_1$ correspond to the connections of the impact systems having both

FIGURE 7: Spring combinations for $n = 1$. FIGURE 8: Impact combinations for $n = 1$.

fenders in the connectedness zone, the second line beginning with Z_{33} to the connections of the impact systems that have one fender in the connectedness zone, and the third line starting with Z_{33} to the connections of the impact system that do not have fenders in the connectedness zone. The shaded cells describe systems in which there is no impact connection (the cell intensively shaded), they have no spring connection (the cell lightly shaded), or there is no impact or spring connection (connection of the cell slightly shaded to the cell very intensively shaded). Such systems will be considered in Sections 3 and 4.

One of the spring-impact systems described in Table 5, that is, $S_1 Z_1$, includes the full configuration of springs and fenders (the maximum number of springs and the maximum number of fenders). It is thus the basic spring-impact system for $n = 2$. All the remaining spring-impact systems can be made out of it via removing a certain number of springs or fenders.

The analysis of spring-impact systems is definitely simplified for $n = 1$ and becomes much more difficult for $n \geq 3$. A classification of spring-impact systems with one degree-of-freedom is presented below.

3. Systems with One Degree-of-Freedom

The maximum number of springs for the system with one degree-of-freedom ($n = 1$) is equal to $s = 1$ (k_1 is the spring connecting the body of a mass m_1 with the frame, see the first system in Figure 7). The number of possible spring combinations $i_s = 2$. In Figure 7, all combinations with their adjacency matrices are presented.

In this case it is difficult to talk about the spring connectedness (subphase II of *Phase II*): either the system has no spring connecting it to the frame (S_2 in Figure 7), or has such a spring (S_1 in Figure 7), but it does not belong to the spring connectedness zone.

The maximum number of fenders (impact connections) for $n = 1$ is $z = 2$ (z_{1g} is the upper impact connection of the body of a mass m_1 with the frame and z_{1d} and the lower impact connection of the body of a mass m_1 with the frame). Both impact connections and their notations are marked in Figure 8. The number of possible impact combinations $i_z = 4$. Figure 8 shows all such combinations along with their adjacency matrices. In the case of impact combinations of $n = 1$, it is also difficult to talk about impact connectedness

FIGURE 9: Connection-impact (spring-impact) combinations for $n = 1$.

(subphase of *Phase II*): either the system does not have any fenders connecting it to the frame (Z_4 in Figure 8), has one fender (lower Z_2 in Figure 8 or upper Z_3 in Figure 8) or has two such fenders (Z_1 in Figure 8), but they do not belong to the impact connectedness zone. The following relations take place in them: the impact combination Z_1 is equivalent to itself, the Z_4 combination is equivalent to itself, and the Z_2 combination is equivalent to the combination Z_3 via transposition.

According to the assumed principle (*Phase I*), we match now each spring combination with each impact combination. As a result of the elimination of redundant equivalent combinations (via transposition), we obtain the spring-impact systems presented in Figure 9. As all spring and impact combinations with one degree-of-freedom are connected; thus all spring-impact systems formed as a result of matching such combinations are connected. In Figure 9 the systems in which there are no impact connections ($S_1 Z_4$), spring connections ($S_2 Z_1$ and $S_2 Z_3$), impact connections, and spring connections ($S_2 Z_4$) are marked.

Let us notice that systems with one degree-of-freedom can be treated as subsystems of disconnected systems with two degrees-of-freedom. The subsystems of matching spring systems would correspond to the systems shown in Figure 9: S_7 (the spring connection of the system of a mass m_1 with the frame) and S_8 (there is no spring connection of the subsystem of a mass m_1 with the frame), with the following disconnected impact systems: Z_{55} (the impact connection via the fenders z_{1g} and z_{1d} of the subsystem of a mass m_1 with the frame), Z_{63} (the impact connection via the fender z_{1g} of the subsystem of a mass m_1 with the frame), and Z_{64} (there is no impact connection of the subsystem of a mass m_1 with the frame).

4. Examples of Systems Considered in the Literature and Their Structural Patterns

Let us notice that instead of a spring connection, we can introduce any other connection that describes the action of at least one force (linear or nonlinear) that depends on displacement or velocity in the system. It cannot only be an elasticity force, but also a viscous damping force, a friction force, or an elastic-damping force or even a triple combination of these forces. As the occurrence of a fender does not determine the way the impact is modeled, thus the approach described here enables identification of all structural patterns of connection-impact systems for the given n . Let us draw attention to the fact that their number does not grow with an appearance of subsequent qualitatively different forces (dependent on displacement or velocity) or with a change in the impact modeling method. All the considerations conducted for spring-impact systems can be thus repeated without any obstacles for connection-impact systems. The obtained adjacency matrices of PZ connection-impact systems are identical to the adjacency matrices of SZ spring-impact systems. Hence, we can state that for $n = 1$ ($P_1 Z_1$, $P_1 Z_3$, $P_2 Z_1$, and $P_2 Z_3$ in Figure 9) and $n = 2$ (all connected representatives of classes of relations without $P_1 Z_{64}$, $P_5 Z_{64}$, and $P_6 Z_{64}$; see Table 5); we have identified all structural patterns of connection-impact systems. Tables 6 and 7 include selected examples of mechanical one- and two-degrees-of-freedom systems chosen from the literature. For all systems, their structural patterns S_p have been assigned according to the classification method proposed in [111]. Similarly, structural patterns for other examples from the literature can be defined.

TABLE 7: Examples from the references ($n = 2$) and the notations of the structural pattern (S_p) of the systems considered in them.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
1		Aidanpää and Gupta [51]; Luo and Xie [52]	$S_p = \mathbf{P}_5 \mathbf{Z}_{63}$
2		Wiercigroch et al. [53]	$S_p = \mathbf{P}_5 \mathbf{Z}_{63}$
3		Valente et al. [54]	$S_p = \mathbf{P}_1 \mathbf{Z}_{63}$
4		Peterka [32]	$S_p = \mathbf{P}_1 \mathbf{Z}_{63}$
5		Bazhenov et al. [55]	$S_p = \mathbf{P}_5 \mathbf{Z}_{62}$
6		Ho et al. [56]	$S_p = \mathbf{P}_5 \mathbf{Z}_{62}$
7		Czolczynski [57]	$S_p = \mathbf{P}_3 \mathbf{Z}_{62}$
8		Czolczynski and Kapitaniak [58]	$S_p = \mathbf{P}_3 \mathbf{Z}_{62}$

TABLE 7: Continued.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
9		Blaziejczyk-Okolewska et al. [59]; Luo et al. [60]	$S_p = \mathbf{P}_3 \mathbf{Z}_{62}$
10		Luo [61]	$S_p = \mathbf{P}_1 \mathbf{Z}_{62}$
11		Luo and Lv [62]; Luo et al. [60]	$S_p = \mathbf{P}_1 \mathbf{Z}_{62}$
12		Luo et al. [63]	$S_p = \mathbf{P}_5 \mathbf{Z}_{55}$
13		Wagg [64]	$S_p = \mathbf{P}_5 \mathbf{Z}_{52}$
14		de Souza et al. [65]; de Souza and Caldas [66]; Han et al. [67]; Luo [68]	$S_p = \mathbf{P}_8 \mathbf{Z}_{30}$
15		De Souza and Caldas [69]; Masri and Caughey [70]; Sadek [71]	$S_p = \mathbf{P}_7 \mathbf{Z}_{30}$
16		Chatterjee et al. [72]	$S_p = \mathbf{P}_7 \mathbf{Z}_{30}$
17		Masri and Ibrahim [73]	$S_p = \mathbf{P}_7 \mathbf{Z}_{30}$

TABLE 7: Continued.

No.	Model of the system	Considered by, for example,	Structural pattern (S_p)
18		Blazejczyk-Okolewska [74]; Peterka and Blazejczyk-Okolewska [75]	$S_p = \mathbf{P}_5 \mathbf{Z}_{30}$
19		Blazejczyk-Okolewska and Kapitaniak [76, 77]	$S_p = \mathbf{P}_5 \mathbf{Z}_{30}$
20		Yue and Xie [78]	$S_p = \mathbf{P}_3 \mathbf{Z}_{30}$

An introduction of arbitrary connections instead of spring connections also leads to the determination of all structural patterns for mechanical systems without impacts (referred to as oscillators) of $n = 1$ ($\mathbf{P}_1 \mathbf{Z}_4$ in Figure 9) and $n = 2$ ($\mathbf{P}_1 \mathbf{Z}_{64}$, $\mathbf{P}_5 \mathbf{Z}_{64}$, and $\mathbf{P}_6 \mathbf{Z}_{64}$ in Table 5). Such systems are commonly analyzed. As an example, we can take the $\mathbf{P}_5 \mathbf{Z}_{64}$ pattern, which is considered to be a dynamic vibration damper in the majority of books on theory of vibrations of machines and devices (e.g., Den Hartog [112], Timoshenko [113]). On the other hand, with the $\mathbf{P}_1 \mathbf{Z}_4$ (Figure 9) pattern, all systems with one degree-of-freedom can be described. The analysis of various oscillators of $n = 1$ (linear and nonlinear) can be found, for example, in monographs by Den Hartog [112], Timoshenko [113], Kapitaniak [114], and Guckenheimer and Holmes [101].

5. Conclusions

In the study, the classification method developed by Blazejczyk-Okolewska [111] has been illustrated in the example of mechanical systems with one and two degrees-of-freedom. The relations occurring in the system have been described in terms of the matrix representation. The identification and the elimination of redundant adjacency matrices and the identification of connected systems have been conducted with the previously developed procedures. According to the principles of selection, the connected representatives of connection-impact classes of relations have been determined. They constitute a set of all structural patterns of vibroimpact systems. Four patterns of impact oscillators with one degree-of-freedom and 141 patterns of impact oscillators with two degrees-of-freedom have been distinguished. The knowledge of classification principles enables classification of each mechanical system with impacts with arbitrary connections. The procedure for

classification of the given system has been presented in numerous examples taken from the literature devoted to the subject.

The considerations conducted for systems with impacts have allowed for defining the number of patterns of systems without impacts, commonly referred to as oscillators. In this case, one pattern with one degree-of-freedom and three patterns with two degrees-of-freedom have been distinguished.

The proposed classification of mechanical systems with impacts according to characteristic properties of their structure seems to be a natural classification. It reflects the relationships between the system structures, tells us about their way of evolution, and presents their genesis. It allows us to rearrange the knowledge on systems with impacts and is the basis for understanding the sources of their diversity. Providing a full set of objects to be analyzed, it gives hints for new ideas and directions in designing technical devices.

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