Research Article

Weak and Strong Convergence of an Algorithm for the Split Common Fixed-Point of Asymptotically Quasi-Nonexpansive Operators

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Inspired by the Moudafi (2010), we propose an algorithm for solving the split common fixed-point problem for a wide class of asymptotically quasi-nonexpansive operators and the weak and strong convergence of the algorithm are shown under some suitable conditions in Hilbert spaces. The algorithm and its convergence results improve and develop previous results for split feasibility problems.

1. Introduction

Fixed-point problem is a classical problem in nonlinear analysis and it has application in a wide spectrum of fields such as economics, physics, and applied sciences. In this paper, we are concerned with the split common fixed point problem (SCFP). In fact, the SCFP is an extension of the split feasibility problem (SFP) and the convex feasibility problem (CFP), see [1]. The CFP and SCFP have many applications such as approximation theory [2], image reconstruction, radiation therapy [3, 4], and control [5]. The SFP in finite-dimensional space was first introduced by Censor and Elfving [6] for modeling inverse problems, which arise from phase retrievals, and in medical image reconstruction [7]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [8, 9]. The SFP in an infinite-dimensional Hilbert space can be found in [10–13].

Throughout this paper, we assume that both H_1 and H_2 are real Hilbert spaces, “ $\rightarrow$ ” and “ $\rightharpoonup$ ” denote strong and weak convergence, respectively, $\text{Fix}(T)$ denotes the set of the fixed points of an operator T , that is, $\text{Fix}(T) := \{x \mid x = T(x)\}$. Let $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be two asymptotically quasi-nonexpansive mappings with nonempty fixed-point

sets $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$, respectively. The split common fixed point problem for operators U and T is to find

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \quad (1)$$

where $A : H_1 \rightarrow H_2$ is bounded linear. Denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C \mid Ay \in Q\}. \quad (2)$$

The split common fixed point problem for quasi-nonexpansive mapping in the setting of Hilbert space was first introduced and studied by Moudafi [14]. However, the algorithm presented in [14] has only weak convergence. The purpose of this paper is to propose an algorithm of split common fixed point problem for asymptotically quasi-nonexpanding operator which includes the quasi-nonexpansive mapping and the weak and strong convergence of the algorithm are shown under some suitable conditions in Hilbert spaces. The algorithm and the convergence results improve and develop previously discussed split feasibility problems.

The paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we present an algorithm and show its weak convergence and strong convergence. Section 4 gives some concluding remarks.

2. Preliminaries

Recall that a mapping $T : H \rightarrow H$ is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad (x, y) \in H \times H. \quad (3)$$

A mapping T is called asymptotically nonexpansive if there exists a sequence $\{l_k\} \subset [1, \infty)$ with $l_k \rightarrow 1$ as $k \rightarrow \infty$ such that, for all $(x, y) \in H \times H$,

$$\|T^k(x) - T^k(y)\| \leq l_k \|x - y\|, \quad \forall k \geq 1. \quad (4)$$

A mapping T is called quasi-nonexpansive if for all $(x, z) \in H \times F(T)$

$$\|T(x) - z\| \leq \|x - z\|. \quad (5)$$

A mapping T is called asymptotically quasi-nonexpansive if there exists a sequence $\{l_k\} \subset [1, \infty)$ with $l_k \rightarrow 1$ as $k \rightarrow \infty$ such that, for all $(x, z) \in H \times \text{Fix}(T)$,

$$\|T^k(x) - z\| \leq l_k \|x - z\|, \quad \forall k \geq 1. \quad (6)$$

A mapping T is said to be uniformly L -Lipschitzian if there exists a constant $L > 0$ such that for all $(x, y) \in H \times H$,

$$\|T^k(x) - T^k(y)\| \leq L \|x - y\|, \quad \forall k \geq 1. \quad (7)$$

A mapping T is said to be semicompact if for any bounded sequence $\{x^k\} \subset H$ with $\lim_{k \rightarrow \infty} \|x^k - Tx^k\| = 0$ there exists a subsequence $\{x^{k_p}\}$ of $\{x^k\}$ such that $\{x^{k_p}\}$ converges strongly to a point $x^* \in H$.

Let E be a Banach space. A mapping $T : E \rightarrow E$ is said to be demiclosed at origin, if for any sequence $\{x^k\} \subset E$ with $x^k \rightarrow x^*$ and $\|(I - T)x^k\| \rightarrow 0$, $x^* = T(x^*)$.

Lemma 1. Let $T : H \rightarrow H$ be asymptotically quasi-nonexpansive and set $T_\alpha^k := (1 - \alpha)I + \alpha T^k$ for $\alpha \in (0, 1]$. Then, for any $z \in \text{Fix}(T)$ and $x \in H$, one has the following:

- (1) $\langle x - T^k(x), x - z \rangle \geq (1/2)\|x - T^k(x)\|^2 - ((l_k^2 - 1)/2)\|x^k - z\|^2$;
- (2) $\langle x - T^k(x), z - T^k(x) \rangle \leq (1/2)\|x - T^k(x)\|^2 + ((l_k^2 - 1)/2)\|x^k - z\|^2$;
- (3) $\|T_\alpha^k(x) - z\|^2 \leq [1 + \alpha(l_k^2 - 1)]\|x - z\|^2 - \alpha(1 - \alpha)\|x - T^k(x)\|^2$;
- (4) $\langle x - T_\alpha^k(x), x - z \rangle \geq (\alpha/2)\|x - T^k(x)\|^2 - ((\alpha(l_k^2 - 1))/2)\|x^k - z\|^2$.

Lemma 2 (see [15]). Suppose $\{a_k\}$ is a sequence of nonnegative real numbers such that

$$a_{k+1} \leq (1 + \delta_k) a_k + b_k, \quad k \geq 0, \quad (8)$$

if (i) $\sum_{k=0}^{\infty} \delta_k < \infty$ and (ii) $\sum_{k=1}^{\infty} b_k < \infty$. Then, $\lim_{k \rightarrow \infty} a_k$ exists. In particular, if $\{a_k\}$ has a subsequence which converges strongly to zero, then $\lim_{k \rightarrow \infty} a_k = 0$.

3. The Algorithm and Its Asymptotic Convergence

We now give a description of an algorithm.

Algorithm 3. Initialization: let $x^0 \in H_1$ be arbitrary.

Iterative step: for $k \in N$, set $u^k = x^k + \gamma A^T(T^k - I)A(x^k)$, let

$$x^{k+1} = (1 - \alpha_k)u^k + \alpha_k U^k(u^k), \quad k \in N, \quad (9)$$

where $\alpha_k \in [\alpha, 1 - \alpha]$, $\alpha \in (0, 1)$, and $\gamma \in (0, 1/\lambda)$, $\lambda = \|A\|^2$.

In what follows, we establish the weak convergence and strong convergence of Algorithm 3.

Lemma 4 (Opial [16]). Let H be a Hilbert space and let $\{x^k\}$ be a sequence in H such that there exists a nonempty set $S \subset H$ satisfying the following.

- (1) For every x^* , $\lim_k \|x^k - x^*\|$ exists.
- (2) Any weak cluster point of the sequence $\{x^k\}$ belongs to S .

Then, there exists $z \in S$ such that $\{x^k\}$ weakly converges to z .

Theorem 5. Let $U : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be two asymptotically quasi-nonexpansive operators with nonempty $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$. Assume that $U - I$ and $T - I$ are demiclosed at 0 and $\Gamma \neq \emptyset$, $\sum_{k=1}^{\infty} (l_k^2 - 1) < \infty$, and T and U are uniformly L -Lipschitzian. Then,

- (I) any sequence $\{x^k\}$ generated by Algorithm 3 converges weakly to a point $x^* \in \Gamma$;
- (II) if U is also semi-compact, then both $\{x^k\}$ and $\{u^k\}$ generated by Algorithm 3 converge strongly to a point $x^* \in \Gamma$.

Proof. First, we prove that for each $z \in \Gamma$, $\lim_{k \rightarrow \infty} \|x^k - z\|$ and $\lim_{k \rightarrow \infty} \|u^k - z\|$ exist. Also $\lim_{k \rightarrow \infty} \|(T^k - I)(Ax^k)\| = 0$ and $\lim_{k \rightarrow \infty} \|U^k(u^k) - u^k\| = 0$.

By using (1) in Lemma 1, from (9) we obtain

$$\begin{aligned} \|x^{k+1} - z\|^2 &= \|(1 - \alpha_k)u^k + \alpha_k U^k(u^k) - z\|^2 \\ &\leq [1 + \alpha_k(l_k^2 - 1)]\|u^k - z\|^2 \\ &\quad - \alpha_k(1 - \alpha_k)\|u^k - U^k(u^k)\|^2. \end{aligned} \quad (10)$$

By deducing, it follows that

$$\begin{aligned} \|u^k - z\|^2 &= \|x^k + \gamma A^T(T^k - I)(Ax^k) - z\|^2 \\ &= \|x^k - z\|^2 + \gamma^2 \|A^T(T^k - I)(Ax^k)\|^2 \\ &\quad + 2\gamma \langle x^k - z, A^T(T^k - I)(Ax^k) \rangle \\ &\leq \|x^k - z\|^2 + \lambda \gamma^2 \|(T^k - I)(Ax^k)\|^2 \\ &\quad + 2\gamma \langle Ax^k - Az, (T^k - I)(Ax^k) \rangle; \end{aligned} \quad (11)$$

that is,

$$\begin{aligned} \|u^k - z\|^2 &\leq \|x^k - z\|^2 + \lambda\gamma^2 \|(T^k - I)(Ax^k)\|^2 \\ &\quad + 2\gamma \langle Ax^k - Az, (T^k - I)(Ax^k) \rangle. \end{aligned} \quad (12)$$

Setting $\theta := 2\gamma \langle Ax^k - Az, (T - I)(Ax^k) \rangle$ and using (1) of Lemma 1, we obtain

$$\begin{aligned} \theta &= 2\gamma \langle Ax^k - Az, (T^k - I)(Ax^k) \rangle \\ &= 2\gamma \langle Ax^k - Az + (T^k - I)(Ax^k) \\ &\quad - (T^k - I)(Ax^k), (T^k - I)(Ax^k) \rangle \\ &= 2\gamma \left(\langle Ax^k - Az, (T - I)(Ax^k) \rangle - \|(T - I)(Ax^k)\|^2 \right) \\ &\leq 2\gamma \left(\frac{1}{2} \|(T^k - I)(Ax^k)\|^2 + \frac{l_k^2 - 1}{2} \|Ax^k - Az\|^2 \right. \\ &\quad \left. - \|(T^k - I)(Ax^k)\|^2 \right) \\ &= -\gamma \|(T^k - I)(Ax^k)\|^2 + \gamma\lambda(l_k^2 - 1) \|x^k - z\|^2. \end{aligned} \quad (13)$$

The key inequality mentioned above, combined with (12), yields

$$\begin{aligned} \|u^k - z\|^2 &\leq (1 + \gamma\lambda(l_k^2 - 1)) \|x^k - z\|^2 \\ &\quad - \gamma(1 - \lambda\gamma) \|(T^k - I)(Ax^k)\|^2. \end{aligned} \quad (14)$$

Substituting (14) into (10) yields

$$\begin{aligned} \|x^{k+1} - z\|^2 &\leq [1 + \alpha_k(l_k^2 - 1)] \\ &\quad \times [(1 + \gamma\lambda(l_k^2 - 1)) \|x^k - z\|^2 \\ &\quad - \gamma(1 - \lambda\gamma) \|(T^k - I)(Ax^k)\|^2] \\ &\quad - \alpha_k(1 - \alpha_k) \|u^k - U^k(u^k)\|^2 \\ &= [1 + (\alpha_k + \gamma\lambda)(l_k^2 - 1) + \alpha_k\gamma\lambda(l_k^2 - 1)^2] \|x^k - z\|^2 \\ &\quad - [1 + \alpha_k(l_k^2 - 1)] \gamma(1 - \lambda\gamma) \|(T^k - I)(Ax^k)\|^2 \\ &\quad - \alpha_k(1 - \alpha_k) \|u^k - U^k(u^k)\|^2 \\ &= [1 + \delta_k] \|x^k - z\|^2 - [1 + \alpha_k(l_k^2 - 1)] \gamma(1 - \lambda\gamma) \\ &\quad \times \|(T^k - I)(Ax^k)\|^2 \\ &\quad - \alpha_k(1 - \alpha_k) \|u^k - U^k(u^k)\|^2 + b_k, \end{aligned} \quad (15)$$

where $\delta_k = (\alpha_k + \gamma\lambda)(l_k^2 - 1) + \alpha_k\gamma\lambda(l_k^2 - 1)^2$, $b_k = 0$. Since $\sum_{k=1}^{\infty} (l_k^2 - 1) < \infty$, we have $\sum_{k=1}^{\infty} \delta_k < \infty$. From Lemma 1, it follows that $\lim_{k \rightarrow \infty} \|x^k - z\|$ exists. By the virtue of (14), we know that $\lim_{k \rightarrow \infty} \|u^k - z\|$ exists. Therefore, from (15), we have

$$\begin{aligned} &[1 + \alpha_k(l_k^2 - 1)] \gamma(1 - \lambda\gamma) \|(T^k - I)(Ax^k)\|^2 \\ &\quad + \alpha_k(1 - \alpha_k) \|u^k - U^k(u^k)\|^2 \\ &\leq \|x^k - z\|^2 - \|x^{k+1} - z\|^2 \\ &\quad - (\alpha_k + \gamma\lambda)(l_k^2 - 1) \\ &\quad + \alpha_k\gamma\lambda(l_k^2 - 1)^2 \|x^k - z\|^2 \longrightarrow 0. \end{aligned} \quad (16)$$

By the assumptions on l_k and α_k , we get

$$\begin{aligned} \lim_{k \rightarrow +\infty} \|(T^k - I)(Ax^k)\| &= 0, \\ \lim_{k \rightarrow +\infty} \|U^k(u^k) - u^k\| &= 0. \end{aligned} \quad (17)$$

Step 1. Now, we prove that $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ and $\lim_{k \rightarrow \infty} \|u^{k+1} - u^k\| = 0$. As a matter of a fact, it follows from (9) that

$$\begin{aligned} \|x^{k+1} - x^k\| &= \|(1 - \alpha_k)u^k + \alpha_k U^k(u^k) - x^k\| \\ &= \|(1 - \alpha_k)(x^k + \gamma A^T(T^k - I)Ax^k) + \alpha_k U^k(u^k) - x^k\| \\ &= \|(1 - \alpha_k)\gamma A^T(T^k - I)Ax^k + \alpha_k(U^k(u^k) - x^k)\| \\ &= \|(1 - \alpha_k)\gamma A^T(T^k - I)Ax^k \\ &\quad + \alpha_k(U^k(u^k) - u^k) + \alpha_k(u^k - x^k)\| \\ &= \|(1 - \alpha_k)\gamma A^T(T^k - I)Ax^k \\ &\quad + \alpha_k(U^k(u^k) - u^k) + \alpha_k\gamma A^T(T^k - I)Ax^k\| \\ &= \|\gamma A^T(T^k - I)Ax^k + \alpha_k(U^k(u^k) - u^k)\|. \end{aligned} \quad (18)$$

In view of (17), we have that

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \quad (19)$$

Similarly, it follows from (9), (17), and (19) that

$$\begin{aligned}
 & \|u^{k+1} - u^k\| \\
 &= \|(x^{k+1} + \gamma A^T (T^{k+1} - I) A x^{k+1}) \\
 &\quad - (x^k + \gamma A^T (T^k - I) A x^k)\| \\
 &\leq \|x^{k+1} - x^k\| + \gamma \|A^T (T^{k+1} - I) A x^{k+1}\| \\
 &\quad + \gamma \|A^T (T^k - I) A x^k\| \rightarrow 0, \quad (k \rightarrow \infty).
 \end{aligned} \tag{20}$$

Step 2. We prove that $\lim_{k \rightarrow \infty} \|A x^k - T(A x^k)\| = 0$ and $\lim_{k \rightarrow \infty} \|u^k - U(u^k)\| = 0$. Setting $\eta_k := \|u^k - U(u^k)\|$, since U is uniformly L -Lipschitzian continuous, it follows from (14) and (17) that

$$\begin{aligned}
 \|u^k - U(u^k)\| &\leq \|u^k - U^k(u^k)\| + \|U^k(u^k) - U(u^k)\| \\
 &\leq \eta_k + L \|U^{k-1}(u^k) - u^k\| \\
 &\leq \eta_k + L (\|U^{k-1}(u^k) - U^{k-1}(u^{k-1})\| \\
 &\quad + \|U^{k-1}(u^{k-1}) - u^k\|) \\
 &\leq \eta_k + L^2 \|u^k - u^{k-1}\| \\
 &\quad + L (\|U^{k-1}(u^{k-1}) - u^{k-1}\| + \|u^{k-1} - u^k\|) \\
 &\leq \eta_k + L(L+1) \|u^k - u^{k-1}\| \\
 &\quad + L\eta_{k-1} \rightarrow 0, \quad (k \rightarrow \infty);
 \end{aligned} \tag{21}$$

in other words,

$$\lim_{k \rightarrow \infty} \|u^k - U(u^k)\| = 0. \tag{22}$$

Similarly, we have

$$\lim_{k \rightarrow \infty} \|A x^k - T(A x^k)\| = 0. \tag{23}$$

Step 3. Finally, we prove that $x^k \rightharpoonup x^*$ and $x^k \rightarrow x^*$, where $x^* \in \Gamma$. Denote by x^* a weak-cluster point of $\{x^k\}$, and denote by $\{x^{k_\sigma}\}$ a subsequence of $\{x^k\}$. Obviously,

$$w - \lim_{\sigma} y^{k_\sigma} = w - \lim_{\sigma} x^{k_\sigma} = x^*. \tag{24}$$

Then, from (23) and the demiclosedness of $T - I$ at 0, we obtain

$$T(A x^*) = A x^*; \tag{25}$$

it follows that, $A x^* \in Q$.

Noticing $u^k = x^k + \gamma A^*(T - I)(A x^k)$, it follows that $w - \lim_{\sigma} u^{k_\sigma} = x^*$. By the demiclosedness of $U - I$ at 0, we have

$$U(x^*) = x^*. \tag{26}$$

Hence, $x^* \in C$, and therefore $x^* \in \Gamma$.

Since there is no more than one weak cluster point, the weak convergence of the whole sequence $\{x^k\}$ follows by applying Lemma 4 with $S = \Gamma$.

The Proof of Conclusion (II). Since U is semi-compact, it follows from (22) that there exists a subsequence $\{u^{k_h}\}$ of $\{u^k\}$ such that $\{u^{k_h}\} \rightarrow u^* \in H$ (some point in H). Since $x^k = u^k - \gamma A^T(T^k - I)A(x^k)\{u^{k_h}\} \rightarrow x^*$, this implies that $x^* = u^*$. Therefore, $\{u^{k_h}\} \rightarrow x^* \in \Gamma$ as $h \rightarrow \infty$. Since for any $z \in \Gamma$, $\lim_{k \rightarrow \infty} \|x^k - z\|$ and $\lim_{k \rightarrow \infty} \|u^k - z\|$ exist, we know that $\lim_{k \rightarrow \infty} \|x^k - x^*\| = \lim_{k \rightarrow \infty} \|u^k - x^*\| = 0$. This implies that $\{x^k\}$ and $\{u^k\}$ both converge strongly to a point $x^* \in \Gamma$. The proof is completed. $\square$

4. Concluding Remarks

In this paper, we have proposed an algorithm for solving the SCFP in the wide class of asymptotically quasi-nonexpansive operators and obtained its weak and strong convergence in general Hilbert spaces in a new way. Next, we will improve the algorithm to solve the multiple split common fixed point problem in infinite Hilbert spaces.

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