Research Article

Spectral Method with the Tensor-Product Nodal Basis for the Steklov Eigenvalue Problem

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This paper discusses spectral method with the tensor-product nodal basis at the Legendre-Gauss-Lobatto points for solving the Steklov eigenvalue problem. A priori error estimates of spectral method are discussed, and based on the work of Melenk and Wohlmuth (2001), a posterior error estimator of the residual type is given and analyzed. In addition, this paper combines the shifted-inverse iterative method and spectral method to establish an efficient scheme. Finally, numerical experiments with MATLAB program are reported.

1. Introduction

Since Steklov eigenvalue problems have important physical background and many applications, for instance, they appear in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [1] and the references therein), in the analysis of the antiplane shearing on a system of collinear faults under slip-dependent friction law (see [2]), in the study of surface waves (see [3]), in the study of the vibration modes of a structure in contact with an incompressible fluid (see [4]) and vibration of a pendulum (see [5]), and in the eigen-oscillations of mechanical systems with boundary conditions containing frequency [6], and thus, numerical methods for solving Steklov eigenvalue problems have received increasing attention in recent years. Bramble and Osborn [7] studied the Galerkin approximation of a Steklov eigenvalue problem of nonself-adjoint second order elliptic operators in smooth domain, Andreev and Todorov [8] discussed the isoparametric finite element method for the approximation of the Steklov eigenvalue problem of second-order self-adjoint elliptic differential operators, Armentano and Padra [9] introduced and analyzed the conforming linear finite element approximation of the Steklov eigenvalue problem in a bounded polygonal domain, Alonso and Russo [10], Yang et al. [11], and Li et al. [12] studied nonconforming finite elements approximation of the Steklov eigenvalue problem, Li and Yang [13] and Bi and

Yang [14] discussed a two-grid method of the conforming and non-conforming finite element method, respectively, Li et al. [15] studied the extrapolation and superconvergence of the Steklov eigenvalue problem, Tang et al. [16] studied the boundary element approximation, and Cao et al. [17] discussed multiscale asymptotic method for Steklov eigenvalue equations in composite medias. However, to the best of our knowledge, there have been no reports on spectral method for Steklov eigenvalue problems.

Spectral method is an important numerical method for solving differential equations developed after finite-difference and finite-element methods. Because spectral method has the characteristics of superior accuracy, it is widely used in the field of meteorology, physics, and mechanics (see [18, 19]). This paper discusses spectral method with the tensor-product nodal basis at the Legendre-Gauss-Lobatto (LGL for short) points for solving the Steklov eigenvalue problem. We use spectral approximation theory (see [20]) to give a priori error estimates. Based on the work of [21], we discuss a posterior error estimator of the residual type. Moreover, inspired by the work of [22] this paper combines the shifted-inverse iterative method and spectral method to propose an efficient computation scheme. Finally, by using program with Matlab, we implement the numerical experiments and get satisfactory results.

The rest of the paper is organized as follows. In Section 2, some preliminaries needed in this paper are presented. In Section 3, we discuss a priori and a posteriori error estimates for spectral method. In Section 4, we establish an efficient scheme combining the shifted-inverse iteration and spectral method. In Section 5, the numerical experiments on the square domain are reported.

2. Preliminaries

We consider the following boundary value problem:

$$\begin{aligned} -\Delta u + u &= 0, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= f, \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where $\Omega \subset \mathbb{R}^2$ is a rectangular domain and $\partial/\partial n$ is the outward normal derivative on $\partial\Omega$. We denote the Sobolev space with norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$ by $H^m(\Omega)$. Let $\|\cdot\|_0$ and $\|\cdot\|_{0,\partial\Omega}$ be the norms in the space $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively.

Throughout this paper, C denotes a generic positive constant independent of the polynomial degrees N , which may not be the same at each occurrence.

The variational problem associated with (1) is given by the following.

Find $u \in H^1(\Omega)$, such that

$$a(u, v) = b(f, v), \quad \forall v \in H^1(\Omega), \quad (2)$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} uv, \quad \forall u, v \in H^1(\Omega), \\ b(f, v) &= \int_{\partial\Omega} f v, \quad \forall f, v \in L^2(\partial\Omega). \end{aligned} \quad (3)$$

Obviously, the bilinear form $a(\cdot, \cdot)$ is continuous and $H^1(\Omega)$ -elliptic, that is, there exists constants A and B independent of u, v , such that

$$\begin{aligned} |a(u, v)| &\leq A \|u\|_1 \|v\|_1, \quad \forall u, v \in H^1(\Omega), \\ a(v, v) &\geq B \|v\|_1^2, \quad \forall u, v \in H^1(\Omega). \end{aligned} \quad (4)$$

From the Lax-Milgram theorem we know that there exists a unique solution to (2).

The following regularity results (see [4]) will be needed in the sequel: for $f \in L^2(\partial\Omega)$, the solution of problem (1) $u \in H^{1+r/2}(\Omega)$, and

$$\|u\|_{1+r/2} \leq M_1 \|f\|_{0,\partial\Omega}; \quad (5)$$

furthermore, if $f \in H^{1/2}(\partial\Omega)$, then $u \in H^{1+r}(\Omega)$ and

$$\|u\|_{1+r} \leq M_2 \|f\|_{1/2,\partial\Omega}, \quad (6)$$

where M_1, M_2 are constants independent of f , and $r = 1$ if Ω is convex and $r < \pi/\omega$ if Ω is concave (with ω being the largest inner angle of Ω).

Define

$$S_N(\Omega) = \text{span} \{x^i y^j \mid 0 \leq i, j \leq N\}. \quad (7)$$

The spectral-Galerkin approximation to (2) is the following.

Find $u_N \in S_N(\Omega)$, such that

$$a(u_N, v_N) = b(f, v_N), \quad \forall v_N \in S_N(\Omega). \quad (8)$$

For $m \geq 3/2$ and each integer $N \geq 1$, we define the interpolation operator

$$I_N : H^m(\Omega) \longrightarrow S_N(\Omega) \quad (9)$$

with the LGL interpolation nodes in Ω . We quote from [18] the following interpolation estimates for spectral method at the tensor-product LGL points

$$\|u - I_N u\|_1 \leq C(m) N^{1-m} \|u\|_m. \quad (10)$$

Let $u \in H^m(\Omega)$ and u_N be the solution of (2) and (8), respectively. Then we derive from Céa Lemma that

$$\|u_N - u\|_1 \leq C(m) N^{1-m} \|u\|_m. \quad (11)$$

Notice that (11) is also valid for special method with modal basis (see [18, 19]).

Using Aubin-Nitsche technique, we deduce the a priori error estimate from (5) and (10)-(11):

$$\|u_N - u\|_{0,\partial\Omega} \leq C N^{1-m-r/2} \|u\|_m. \quad (12)$$

3. Spectral Method and Its Error Estimates for the Steklov Eigenvalue Problem

3.1. Spectral Method for the Steklov Eigenvalue Problem. We consider the following Steklov eigenvalue problem:

$$\begin{aligned} -\Delta u + u &= 0, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda u, \quad \text{on } \partial\Omega, \end{aligned} \quad (13)$$

where $\Omega \subset \mathbb{R}^2$ is a rectangular domain.

The variational problem associated with (13) is given by the following.

Find $\lambda \in \mathbb{R}$, $0 \neq u \in H^1(\Omega)$, such that

$$\begin{aligned} a(u, v) &= \lambda b(u, v), \quad \forall v \in H^1(\Omega), \\ \|u\|_{0,\partial\Omega} &= 1. \end{aligned} \quad (14)$$

Let (λ, u) be an eigenpair of (14); then from (6) we know that $u \in H^{1+r}(\Omega)$.

The spectral-Galerkin approximation to (14) is as the follows.

Find $\lambda_N \in \mathbb{R}$, $0 \neq u_N \in S_N(\Omega)$, such that

$$\begin{aligned} a(u_N, v_N) &= \lambda_N b(u_N, v_N), \quad \forall v_N \in S_N(\Omega), \\ \|u_N\|_{0,\partial\Omega} &= 1. \end{aligned} \quad (15)$$

Let $T : L^2(\partial\Omega) \rightarrow H^1(\Omega)$ and $T_N : L^2(\partial\Omega) \rightarrow S_N(\Omega)$ be the solution operators defined by

$$\begin{aligned} a(Tf, v) &= b(f, v), \quad \forall v \in H^1(\Omega), \\ a(T_N f, v) &= b(f, v), \quad \forall v_N \in S_N(\Omega). \end{aligned} \quad (16)$$

Obviously, $T : H^1(\Omega) \rightarrow H^1(\Omega)$ is completely continuous and $T_N : H^1(\Omega) \rightarrow S_N(\Omega)$ is a finite ranked operator. Equations (14) and (15) have the following equivalent operator forms, respectively:

$$\begin{aligned} Tu &= \lambda^{-1}u, \\ T_N u_N &= \lambda_N^{-1}u_N. \end{aligned} \quad (17)$$

3.2. A Prior Error Estimates of Spectral Method for the Steklov Eigenvalue Problem. Let (λ, u) be an eigenpair of (14), let $V(\lambda)$ be the space spanned by all eigenfunctions corresponding to the eigenvalue λ , and let $V_N(\lambda)$ be the direct sum of the eigenspace corresponding to the eigenvalues λ_N of (15) that converge to λ .

Denote

$$\begin{aligned} \varepsilon_N &= \varepsilon_N(\lambda) = \sup_{u \in V(\lambda), \|u\|_1=1} \inf_{v \in S_N(\Omega)} \|u - v\|_1, \\ \eta_a(N) &= \sup_{f \in H^1(\Omega), \|f\|_1=1} \inf_{v \in S_N(\Omega)} \|Tf - v\|_1, \\ \rho_N &= \sup_{f \in L^2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \inf_{v \in S_N(\Omega)} \|Tf - v\|_1, \end{aligned} \quad (18)$$

where ε_N characterizes the degree of the space $S_N(\Omega)$ approximating $V(\lambda)$, $\eta_a(N)$ and ρ_N characterize the degrees of $S_N(\Omega)$ approximating the generalized solution Tf of (2).

Lemma 1. Let $V(\lambda) \subset H^m(\Omega)$; then

$$\begin{aligned} \varepsilon_N &\leq CN^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m, \\ \eta_a(N) &\leq CN^{-r}, \\ \rho_N &\leq CN^{-r/2}. \end{aligned} \quad (19)$$

Proof. We deduce from the interpolation estimate (10) that

$$\begin{aligned} \varepsilon_N &\leq \sup_{u \in V(\lambda), \|u\|_1=1} \|u - I_N u\|_1 \\ &\leq CN^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m. \end{aligned} \quad (20)$$

Using (10) and (6) we have

$$\begin{aligned} \eta_a(N) &\leq \sup_{f \in H^1(\Omega), \|f\|_1=1} \|Tf - I_N Tf\|_1 \\ &\leq CN^{-r} \sup_{f \in H^1(\Omega), \|f\|_1=1} \|Tf\|_{1+r} \\ &\leq CN^{-r}. \end{aligned} \quad (21)$$

From (10) and regularity estimate (5) we get

$$\begin{aligned} \rho_N &\leq \sup_{f \in L^2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \|Tf - I_N Tf\|_1 \\ &\leq CN^{-r/2} \sup_{f \in L^2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \|Tf\|_{1+r/2,\Omega} \\ &\leq CN^{-r/2}. \end{aligned} \quad (22)$$

The proof is completed. $\square$

Theorem 2. Let $V(\lambda) \subset H^m(\Omega)$; then

$$\lambda_N - \lambda \leq C \left\{ N^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m \right\}^2; \quad (23)$$

let $u_N \in V_N(\lambda)$ and $\|u_N\|_1 = 1$; then there exists $u \in V(\lambda)$, such that

$$\|u_N - u\|_1 \leq CN^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m, \quad (24)$$

$$\|u_N - u\|_{0,\partial\Omega} \leq CN^{1-m-r/2} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m; \quad (25)$$

let $u \in V(\lambda)$ and $\|u\|_1 = 1$; then there exists $u_N \in V_N(\lambda)$, such that

$$\|u_N - u\|_1 \leq CN^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m. \quad (26)$$

Proof. From [20, 23] we have that

$$\lambda_N - \lambda \leq C\varepsilon_N^2; \quad (27)$$

let $u_N \in V_N(\lambda)$ and $\|u_N\|_1 = 1$; then there exists $u \in V(\lambda)$, such that

$$\begin{aligned} \|u_N - u\|_1 &\leq C\varepsilon_N, \\ \|u_N - u\|_{0,\partial\Omega} &\leq C\rho_N\varepsilon_N; \end{aligned} \quad (28)$$

let $u \in V(\lambda)$ and $\|u\|_1 = 1$; then there exists $u_N \in V_N(\lambda)$, such that

$$\|u_N - u\|_1 \leq C\varepsilon_N. \quad (29)$$

Combining Lemma 1 we obtain the desired results. $\square$

3.3. A Posterior Error Estimates for the Steklov Eigenvalue Problem. Theorem 2.1 in [24] gave the following results.

Lemma 3. There exists a positive constant C and a linear bounded operator $I : L^1(\Omega) \rightarrow S_N(\Omega) \subset L^1(\Omega)$, such that for all $v \in H^1(\Omega)$ and an arbitrary edge e of $\partial\Omega$, and there holds

$$\begin{aligned} \|v - Iv\|_0 &\leq C \frac{1}{N} \|\nabla v\|_0, \\ \|v - Iv\|_{0,e} &\leq C \frac{1}{\sqrt{N}} \|\nabla v\|_0. \end{aligned} \quad (30)$$

Analysis of the error indicators requires polynomial inverse estimates in weighted Sobolev spaces. Define on the interval $S = (0, 1)$ the weight function $\Phi_S := x(1 - x)$; then there holds the following polynomial inverse estimates.

Lemma 4. *Let $-1 < \alpha < \beta$, $\sigma \in [0, 1]$; then there exists C_1 , $C_2 = C(\alpha, \beta)$, and $C_3 = C_\sigma$ such that for every univariate polynomial π_N of degree N , there holds*

$$\int_S \Phi_S(x) (\pi'_N(x))^2 dx \leq C_1 N^2 \int_S \pi_N^2(x) dx, \quad (31)$$

$$\int_S \Phi_S^\alpha(x) \pi_N^2(x) dx \leq C_2 N^{2(\beta-\alpha)} \int_S \Phi_S^\beta(x) \pi_N^2(x) dx, \quad (32)$$

$$\int_S \Phi_S^{2\sigma}(x) (\pi'_N(x))^2 dx \leq C_3 N^{2(2-\sigma)} \int_S \Phi_S^\sigma(x) \pi_N^2(x) dx. \quad (33)$$

Proof. See Lemma 2.4 in [21]. $\square$

Consider the domain $\kappa = (0, 1)^2$ and the weight function $\Phi_\kappa := \text{dist}(x, \partial\kappa)$. From [21] we have the two-dimensional polynomial inverse estimates.

Lemma 5. *Let $-1 < \alpha < \beta$, $\sigma \in [0, 1]$; then there exists C_1 , $C_2 = C(\alpha, \beta)$, and $C_3 = C_\sigma$ such that for all polynomials $\pi_N \in S_N(\Omega)$ there hold*

$$\int_\kappa \Phi_\kappa |\nabla \pi_N|^2 dx dy \leq C_1 N^2 \int_\kappa |\pi_N|^2 dx dy, \quad (34)$$

$$\int_\kappa \Phi_\kappa^\alpha \pi_N^2 dx dy \leq C_2 N^{2(\beta-\alpha)} \int_\kappa \Phi_\kappa^\beta \pi_N^2 dx dy, \quad (35)$$

$$\int_\kappa \Phi_\kappa^{2\sigma} |\nabla \pi_N|^2 dx dy \leq C_3 N^{2(2-\sigma)} \int_\kappa \Phi_\kappa^\sigma \pi_N^2 dx dy. \quad (36)$$

Proof. See Lemma 2.5 in [21]. $\square$

Lemma 6. *Let $\alpha \in (1/2, 1]$, set $\hat{e} := (0, 1) \times \{0\}$, and let $\Phi_{\hat{e}}$ be defined as the same as Φ_S , that is, $\Phi_{\hat{e}} = x(1 - x)$. Then there exists $C_\alpha > 0$, such that for every $\varepsilon \in (0, 1)$ and every univariate polynomial π_N of degree N , there exists an extension $v_{\hat{e}} \in H^1(\kappa)$, such that the following holds:*

$$v_{\hat{e}}|_{\hat{e}} = \pi_N \cdot \Phi_{\hat{e}}^\alpha, \quad v_{\hat{e}}|_{\partial\kappa \setminus \hat{e}} = 0, \quad (37)$$

$$\|v_{\hat{e}}\|_{0,\kappa}^2 \leq C_\alpha \varepsilon \|\pi_N \Phi_{\hat{e}}^{\alpha/2}\|_{0,\hat{e}}^2, \quad (38)$$

$$\|\nabla v_{\hat{e}}\|_{0,\kappa}^2 \leq C_\alpha (\varepsilon N^{2(2-\alpha)} + \varepsilon^{-1}) \|\pi_N \Phi_{\hat{e}}^{\alpha/2}\|_{0,\hat{e}}^2. \quad (39)$$

Proof. See Lemma 2.6 in [21]. $\square$

Remark 7. Because there has an affine transformation from rectangular domain to the unit square, Lemma 5 is suited to rectangular domain, and Lemmas 4 and 6 are valid for all edges of rectangular.

Define the global error indicator η_α , $\alpha \in [0, 1]$, which is expressed as the sum of two terms:

$$\eta_\alpha^2 := \eta_{\alpha;B}^2 + \eta_{\alpha;E}^2, \quad (40)$$

where the first term $\eta_{\alpha;B}^2$ is the weighted internal residual given by

$$\eta_{\alpha;B}^2 := \frac{1}{N^2} \|(\Delta u_N - u_N) \Phi_\Omega^{\alpha/2}\|_0^2, \quad (41)$$

and the second term $\eta_{\alpha;E}^2$ is the weighted edge residual given by

$$\eta_{\alpha;E}^2 := \frac{1}{N} \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,\partial\Omega}^2, \quad (42)$$

where the weight functions Φ_Ω and Φ_e are scaled transformations of the weight functions Φ_κ and $\Phi_{\hat{e}}$, respectively.

By using the proof of [21], we can get the following results.

Theorem 8. *Let $\alpha \in [0, 1]$. Then there exists $C > 0$ independent of N , such that*

$$\|u - u_N\|_1^2 \leq C \|\lambda u - \lambda_N u_N\|_{0,\partial\Omega}^2 + C N^{2\alpha} (\eta_{\alpha;B}^2 + \eta_{\alpha;E}^2). \quad (43)$$

Proof. We denote $w := u - u_N - I(u - u_N)$ where I is an interpolation operator satisfying Lemma 3. We derive from $H^1(\Omega)$ -ellipticity of $a(\cdot, \cdot)$ that

$$\begin{aligned} C \|u - u_N\|_1^2 &\leq a(u - u_N, u - u_N) \\ &= a(u - u_N, w) + a(u - u_N, I(u - u_N)) \\ &= \int_{\partial\Omega} \lambda u w - a(u_N, w) \\ &\quad + \int_{\partial\Omega} (\lambda u - \lambda_N u_N) I(u - u_N). \end{aligned} \quad (44)$$

Since

$$\begin{aligned} a(u_N, w) &= \int_\Omega \nabla u_N \cdot \nabla w + \int_\Omega u_N w \\ &= - \int_\Omega \Delta u_N w + \int_{\partial\Omega} \frac{\partial u_N}{\partial n} w + \int_\Omega u_N w, \end{aligned} \quad (45)$$

thus

$$\begin{aligned} C \|u - u_N\|_1^2 &\leq \int_{\partial\Omega} (\lambda u - \lambda_N u_N) w + \int_{\partial\Omega} \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) w \\ &\quad + \int_\Omega (\Delta u_N - u_N) w \\ &\quad + \int_{\partial\Omega} (\lambda u - \lambda_N u_N) I(u - u_N) \\ &= \int_{\partial\Omega} (\lambda u - \lambda_N u_N) (u - u_N) \\ &\quad + \int_{\partial\Omega} \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) w + \int_\Omega (\Delta u_N - u_N) w. \end{aligned} \quad (46)$$

By using Cauchy-Schwartz inequality, the trace theorem, and Lemma 3, we get

$$\begin{aligned} \int_{\partial\Omega} (\lambda u - \lambda_N u_N) (u - u_N) &\leq \|\lambda u - \lambda_N u_N\|_{0,\partial\Omega} \|u - u_N\|_1, \\ \int_{\partial\Omega} \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) w &\leq C \left\| \lambda_N u_N - \frac{\partial u_N}{\partial n} \right\|_{0,\partial\Omega} \frac{1}{\sqrt{N}} \|\nabla (u - u_N)\|_0, \\ \int_{\Omega} (\Delta u_N - u_N) w &\leq C \|\Delta u_N - u_N\|_0 \frac{1}{N} \|\nabla (u - u_N)\|_0. \end{aligned} \quad (47)$$

Plugging (47) into (46) gives

$$\begin{aligned} C \|u - u_N\|_1 &\leq \|\lambda u - \lambda_N u_N\|_{0,\partial\Omega} + C \frac{1}{\sqrt{N}} \left\| \lambda_N u_N - \frac{\partial u_N}{\partial n} \right\|_{0,\partial\Omega} \\ &\quad + C \frac{1}{N} \|\Delta u_N - u_N\|_0. \end{aligned} \quad (48)$$

Setting $\alpha = 0$, $\beta = \alpha$ in (32) and (35), respectively, we obtain $\eta_{0;B} \leq CN^\alpha \eta_{\alpha;B}$ and $\eta_{0;E} \leq CN^\alpha \eta_{\alpha;E}$, and thus

$$C \|u - u_N\|_1 \leq \|\lambda u - \lambda_N u_N\|_{0,\partial\Omega} + CN^\alpha \eta_{\alpha;E} + CN^\alpha \eta_{\alpha;B}. \quad (49)$$

The proof is completed. $\square$

Lemma 9. Let $\alpha \in [0, 1]$. Then there exists $C > 0$ independent of N , such that

$$\eta_{\alpha;B}^2 \leq CN^{2(1-\alpha)} \|u_N - u\|_1^2. \quad (50)$$

Proof. Denote $v_\Omega := (u_N - \Delta u_N) \Phi_\Omega^\alpha \in H_0^1(\Omega)$, $\alpha \in (0, 1]$; then

$$\begin{aligned} \|v_\Omega \Phi_\Omega^{-\alpha/2}\|_0^2 &= \int_{\Omega} (u_N - \Delta u_N) v_\Omega \\ &= \int_{\Omega} u_N v_\Omega + \int_{\Omega} \nabla u_N \cdot \nabla v_\Omega - \int_{\partial\Omega} \frac{\partial u_N}{\partial n} v_\Omega \\ &= a(u_N - u, v_\Omega) \\ &\leq C \|u_N - u\|_1 |v_\Omega|_1. \end{aligned} \quad (51)$$

Now, we consider $|v_\Omega|_1$. Using (35)-(36), we find for $\alpha > 1/2$ that

$$\begin{aligned} |v_\Omega|_1^2 &= \|\nabla ((u_N - \Delta u_N) \Phi_\Omega^\alpha)\|_0^2 \\ &\leq 2 \int_{\Omega} \Phi_\Omega^{2\alpha} |\nabla (u_N - \Delta u_N)|^2 + 2 \int_{\Omega} |\nabla \Phi_\Omega^\alpha|^2 (u_N - \Delta u_N)^2 \\ &\leq CN^{2(2-\alpha)} \int_{\Omega} \Phi_\Omega^\alpha (u_N - \Delta u_N)^2 \\ &\quad + C \int_{\Omega} \Phi_\Omega^{2(\alpha-1)} (u_N - \Delta u_N)^2 \\ &\leq CN^{2(2-\alpha)} \int_{\Omega} \Phi_\Omega^\alpha (u_N - \Delta u_N)^2 \\ &= CN^{2(2-\alpha)} \|v_\Omega \Phi_\Omega^{-\alpha/2}\|_0^2, \end{aligned} \quad (52)$$

noticing that (35) is valid since the assumption $\alpha > 1/2$ implies $2\alpha - 2 > -1$, hence we take $\beta = \alpha$, $\alpha = 2(\alpha - 1)$ in (35) and the above third inequality holds. By the definition $\|v_\Omega \Phi_\Omega^{-\alpha/2}\|_0 = N \eta_{\alpha;B}$, we obtain

$$\eta_{\alpha;B} \leq CN^{1-\alpha} \|u_N - u\|_1. \quad (53)$$

To get an upper bound of the error indicator $\eta_{\alpha;B}$ in the case of $0 \leq \alpha \leq 1/2$, using (35) we find for $\beta > 1/2$ that

$$\begin{aligned} N \eta_{\alpha;B} &= \|(u_N - \Delta u_N) \Phi_\Omega^{\alpha/2}\|_0 \\ &\leq CN^{\beta-\alpha} \|(u_N - \Delta u_N) \Phi_\Omega^{\beta/2}\|_0 \\ &\leq CN^{\beta-\alpha} \cdot N \cdot N^{1-\beta} \|u_N - u\|_1. \end{aligned} \quad (54)$$

Therefore,

$$\eta_{\alpha;B}^2 \leq CN^{2(1-\alpha)} \|u_N - u\|_1^2. \quad (55)$$

We obtain the desired result immediately. $\square$

Lemma 10. Let $\alpha \in [0, 1]$, $\varepsilon > 0$. Then there exists $C > 0$ independent of N , such that

$$\begin{aligned} \eta_{\alpha;E}^2 &\leq CN^{\max\{1-2\alpha+2\varepsilon, 0\}} \\ &\quad \times \left\{ N \|u_N - u\|_1^2 + \frac{1}{N} \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega}^2 \right\}. \end{aligned} \quad (56)$$

Proof. In order to obtain an upper bound for the error indicator $\eta_{\alpha;E}$, we will use weight function and a proper extension operator. Let e be an edge of Ω ; we construct a function $w_e \in H^1(\Omega)$ and $w_e|_e = (\lambda_N u_N - \partial u_N / \partial n) \Phi_e^\alpha$.

Let $v_e = (\lambda_N u_N - \partial u_N / \partial n) \Phi_e^\alpha$. We extend v_e to $\partial\Omega$ by using Lemma 6 where the polynomial π_N corresponds to

$\lambda_N u_N - (\partial u_N / \partial n)$, and from (37) we know that v_e is zero on $\partial\Omega \setminus e$. We extend w_e to $\partial\Omega$ such that $w_e = 0$ on $\partial\Omega \setminus e$; then

$$\begin{aligned} \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,e}^2 &= \int_{\partial\Omega} \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) w_e \\ &= \int_{\partial\Omega} (\lambda_N u_N - \lambda u) w_e \\ &\quad + a(u - u_N, w_e) \\ &\quad + \int_{\Omega} (u_N - \Delta u_N) w_e \\ &\leq \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega} \|w_e\|_{0,\partial\Omega} \\ &\quad + C \|u - u_N\|_1 \|w_e\|_1 \\ &\quad + \|u_N - \Delta u_N\|_0 \|w_e\|_0. \end{aligned} \quad (57)$$

Now we estimate $\|w_e\|_{0,\partial\Omega}$, $\|w_e\|_0$, $\|w_e\|_1$ in turn. First,

$$\|w_e\|_{0,\partial\Omega}^2 \leq C \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,e}^2. \quad (58)$$

In the case of $\alpha \in (1/2, 1]$ we have

$$\begin{aligned} \|w_e\|_0^2 &\leq C \varepsilon \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,e}^2, \\ \|w_e\|_1^2 &\leq C (\varepsilon N^{2(2-\alpha)} + \varepsilon^{-1}) \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,e}^2. \end{aligned} \quad (59)$$

Combining (58) and (59), we get

$$\begin{aligned} \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,e} &\leq C \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega} + C [\varepsilon N^{2(2-\alpha)} + \varepsilon^{-1}]^{1/2} \\ &\quad \times \|u - u_N\|_1 + C \varepsilon^{1/2} \|u_N - \Delta u_N\|_0. \end{aligned} \quad (60)$$

By the definition of $\eta_{\alpha,E}^2$ and setting $\alpha = 0$ in Lemma 9, we have that

$$\begin{aligned} \eta_{\alpha,E}^2 &\leq C \frac{1}{N} \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega}^2 \\ &\quad + C \left\{ \frac{1}{N} [\varepsilon N^{2(2-\alpha)} + \varepsilon^{-1}] + N^3 \varepsilon \right\} \|u - u_N\|_1^2. \end{aligned} \quad (61)$$

Setting $\varepsilon = 1/N^2$ yields the assertion for $\alpha > 1/2$ in (61).

For the case $\alpha \in [0, 1/2]$, setting $\beta = 1/2 + \varepsilon$ and using (35), we find

$$\begin{aligned} \eta_{\alpha,E} &= \frac{1}{\sqrt{N}} \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\alpha/2} \right\|_{0,\partial\Omega} \\ &\leq C \frac{1}{\sqrt{N}} N^{\beta-\alpha} \left\| \left(\lambda_N u_N - \frac{\partial u_N}{\partial n} \right) \Phi_e^{\beta/2} \right\|_{0,\partial\Omega} \\ &= C N^{\beta-\alpha} \eta_{\beta,E}; \end{aligned} \quad (62)$$

therefore,

$$\eta_{\alpha,E}^2 \leq C N^{1+2\varepsilon-2\alpha} \eta_{\beta,E}^2. \quad (63)$$

From the above discussion we complete the proof. $\square$

Combining Lemmas 9 and 10, we obtain the following theorem.

Theorem 11. Let $\alpha \in [0, 1]$. Then there exists $C > 0$ independent of N , such that

$$\begin{aligned} \eta_{\alpha}^2 &\leq C N^{\max\{1-2\alpha+2\varepsilon, 0\}} \\ &\quad \times \left\{ N \|u_N - u\|_1^2 + \frac{1}{N} \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega}^2 \right\}. \end{aligned} \quad (64)$$

Remark 12. The term $\|\lambda_N u_N - \lambda u\|_{0,\partial\Omega}^2$ is a term of higher order. In fact, for any $v \in H^1(\Omega)$, we have

$$\begin{aligned} \int_{\partial\Omega} (\lambda_N u_N - \lambda u) v &= \int_{\partial\Omega} \lambda (u_N - u) v + \int_{\partial\Omega} (\lambda_N - \lambda) u_N v \\ &\leq (\lambda \|u_N - u\|_{0,\partial\Omega} + (\lambda_N - \lambda) \|u_N\|_{0,\partial\Omega}) \|v\|_{0,\partial\Omega}. \end{aligned} \quad (65)$$

Thus, taking $v = \lambda_N u_N - \lambda u$ and using the a priori estimates (23) and (25), we conclude that

$$\begin{aligned} \|\lambda_N u_N - \lambda u\|_{0,\partial\Omega} &\leq C N^{1-m-r/2} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m \\ &\quad + C \left\{ N^{1-m} \sup_{u \in V(\lambda), \|u\|_1=1} \|u\|_m \right\}^2. \end{aligned} \quad (66)$$

4. Spectral Method Based on the Shifted-Inverse Iteration

High efficient finite element schemes have been researched further in recent years; for example, see [22, 25–31]. Based on these works, this paper combines the shifted-inverse iteration and spectral method to propose an efficient scheme.

Scheme 13 (spectral method based on the shifted-inverse iteration).

Step 1. Solve (13) in $S_{N_1}(\Omega)$: Find $\lambda_{N_1} \in R$, $u_{N_1} \in S_{N_1}(\Omega)$, such that

$$a(u_{N_1}, v) = \lambda_{N_1} b(u_{N_1}, v), \quad \forall v \in S_{N_1}(\Omega). \quad (67)$$

Step 2. Solve a linear system in $S_{N_2}(\Omega)$ ($N_2 > N_1$). Find $u^{N_2} \in S_{N_2}(\Omega)$, such that

$$a(u^{N_2}, v) - \lambda_{N_1} b(u^{N_2}, v) = b(u_{N_1}, v), \quad \forall v \in S_{N_2}(\Omega). \quad (68)$$

Step 3. Compute the Rayleigh quotient

$$\lambda^{N_2} = \frac{a(u^{N_2}, u^{N_2})}{b(u^{N_2}, u^{N_2})}. \quad (69)$$

TABLE 1: The eigenvalues of LGL spectral method.

N	DOF	λ_1	λ_2	λ_4
3	16	0.24007936507936	1.49252393896074	2.08264705882353
4	25	0.24007908542791	1.49230719085030	2.08264705882353
5	36	0.24007908542792	1.49230316584831	2.08264705403320
6	49	0.24007908542722	1.49230313468790	2.08264705403320
7	64	0.24007908542723	1.49230313453462	2.08264705403320
8	81	0.24007908542723	1.49230313453360	2.08264705403320
9	100	0.24007908542722	1.49230313453358	2.08264705403318
10	121	0.24007908542721	1.49230313453357	2.08264705403318
11	144	0.24007908542722	1.49230313453359	2.08264705403321
12	169	0.24007908542721	1.49230313453357	2.08264705403318
13	196	0.24007908542723	1.49230313453360	2.08264705403323
14	225	0.24007908542722	1.49230313453357	2.08264705403318
15	256	0.24007908542724	1.49230313453359	2.08264705403320
16	289	0.24007908542723	1.49230313453359	2.08264705403319
17	324	0.24007908542728	1.49230313453361	2.08264705403321
18	361	0.24007908542725	1.49230313453359	2.08264705403319
19	400	0.24007908542721	1.49230313453356	2.08264705403317
20	441	0.24007908542718	1.49230313453349	2.08264705403311
21	484	0.24007908542705	1.49230313453341	2.08264705403305
22	529	0.24007908542721	1.49230313453356	2.08264705403317
23	576	0.24007908542727	1.49230313453364	2.08264705403326
24	625	0.24007908542700	1.49230313453339	2.08264705403301
25	676	0.24007908542723	1.49230313453356	2.08264705403317
27	784	0.24007908542747	1.49230313453375	2.08264705403338
29	900	0.24007908542735	1.49230313453373	2.08264705403335
30	961	0.24007908542716	1.49230313453351	2.08264705403311
31	1024	0.24007908542735	1.49230313453364	2.08264705403327
34	1225	0.24007908542735	1.49230313453361	2.08264705403323

Let (λ_{N_1}, u_{N_1}) be the k th eigenpair of (67) then (λ^{N_2}, u^{N_2}) is regarded as the k th approximate eigenpair of (13).

Theorem 14. Suppose that $V(\lambda) \subset H^m(\Omega)$ and N_1 is properly large. Let (λ^{N_2}, u^{N_2}) be the k th approximate eigenpair obtained by Scheme 13; then there exists the k th eigenpair (λ, u) of (13), such that

$$\begin{aligned} \|u^{N_2} - u\|_1 &\leq C(N_1^{-r+3(1-m)} + N_2^{1-m}), \\ |\lambda^{N_2} - \lambda| &\leq C(N_1^{-2r+6(1-m)} + N_2^{2(1-m)}). \end{aligned} \quad (70)$$

Proof. From the proof of Theorem 4.1 in [22] we can deduce that

$$\begin{aligned} \|u^{N_2} - u\|_1 &\leq C(\eta_a(N_1)\varepsilon_{N_1}^3 + \varepsilon_{N_2}), \\ |\lambda^{N_2} - \lambda| &\leq C(\eta_a(N_1)^2\varepsilon_{N_1}^6 + \varepsilon_{N_2}^2), \end{aligned} \quad (71)$$

which together with Lemma 1 yields (70). $\square$

Theorem 14 tells us, when $N_1 < N_2 \leq N_1^{(r+3(m-1))/(m-1)}$, the degree of precision (DOP) of λ^{N_2} and u^{N_2} obtained by Scheme 13 is $O(N_2^{2(1-m)})$ and $O(N_2^{1-m})$, respectively; that is to say, (λ^{N_2}, u^{N_2}) have the same DOP with (λ_{N_2}, u_{N_2}) .

5. Numerical Experiments

Consider the Steklov eigenvalue problem (13) where $\Omega = [0, 1]^2$.

We use spectral method with tensor-product nodal basis at the LGL points (LGL spectral method) to solve (13). We construct two-dimensional basis functions by using the tensor product of one-dimensional basis functions. More precisely, let $\{h_k\}_{k=1}^{N+1}$ be the Lagrange basis polynomials associated with the LGL points; then $S_N(\Omega) = \text{span}\{\phi_i : \phi_i = h_{i1}(x)h_{j1}(y), i, j = 1, 2, \dots, N+1, i = 1, 2, \dots, (N+1)^2\}$, and let $u_N = \sum_i u_i \phi_i$; then (15) is equivalent to the following linear system:

$$AU = \lambda_N BU, \quad (72)$$

where A is called stiffness matrix, B is called mass matrix, and

$$\begin{aligned} U &= (u_1, u_2, \dots, u_{(N+1)^2})^T, \\ A &= (a_{i,j}), \quad a_{i,j} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j + \int_{\Omega} \phi_i \phi_j, \\ &\quad 1 \leq i, j \leq (N+1)^2, \end{aligned} \quad (73)$$

$$B = (b_{i,j}), \quad b_{i,j} = \int_{\partial\Omega} \phi_i \phi_j, \quad 1 \leq i, j \leq (N+1)^2.$$

TABLE 2: The eigenvalues of the bilinear finite element.

h	DOF	λ_1	λ_2	λ_4
1/8	81	0.24022930645534	1.49577435506528	2.08269976804992
1/16	289	0.24011662732054	1.49317640860702	2.08266035463158
1/32	1089	0.24008847006616	1.49252179566534	2.08265038681122
1/64	4225	0.24008143153493	1.49235782124229	2.08264788770449
1/128	16641	0.24007967195002	1.49231680754965	2.08264726248054
1/150	22801	0.24007951252139	1.49231309100913	2.08264720582238
1/200	40401	0.24007932566556	1.49230873510836	2.08264713941521
1/256	66049	0.24007923205310	1.49230655286905	2.08264710614529
1/300	90601	0.24007919219220	1.49230562369281	2.08264709197866
1/400	160801	0.24007914547718	1.49230453468660	2.08264707537565
1/512	263169	0.24007912210763	1.49230398913452	2.08264706706984

TABLE 3: The eigenvalues of shifted-inverse iterative method.

N_1	N_2	$\lambda_1^{N_2}$	$\lambda_2^{N_2}$	$\lambda_4^{N_2}$
4	5	0.24007908542792	1.49230316584831	2.08264705403320
4	6	0.24007908542722	1.49230313468790	2.08264705403320
4	7	0.24007908542722	1.49230313453463	2.08264705403320
4	8	0.24007908542723	1.49230313453360	2.08264705403320

We program with Matlab to solve (72), and the numerical results are presented in Table 1.

In the tables in this section, DOF denotes the degree of freedom (number of nodes).

From Table 1 we can see that

$$\lambda_1 \approx 0.240079085427,$$

$$\lambda_2 = \lambda_3 \approx 1.492303134533, \quad \lambda_4 \approx 2.082647054033. \quad (74)$$

We also use the bilinear finite element to solve (13), and the numerical results are listed in Table 2.

From Table 2 we see that

$$\lambda_1 \approx 0.2400791, \quad \lambda_2 = \lambda_3 \approx 1.49230, \quad (75)$$

$$\lambda_4 \approx 2.0826470.$$

We use Scheme 13 to solve (13), and let $\lambda_i^{N_2}$ be the i th approximate eigenvalue ($i = 1, 2, 4$), and the numerical results are presented in Table 3.

From Table 3 we find the eigenvalues obtained by Scheme 13 have the same accuracy with λ_{i,N_2} which is computed directly in $S_{N_2}(\Omega)$ by LGL spectral method.

Since we do not know the exact eigenvalues of the Steklov eigenvalue problem, many research studies reported the approximate eigenvalues; for example, see [15, 22]. This paper uses the LGL spectral method to solve (13), and in Table 1 we give

$$\lambda_1 \approx 0.240079085427,$$

$$\lambda_2 = \lambda_3 \approx 1.492303134533, \quad \lambda_4 \approx 2.082647054033. \quad (76)$$

We think these approximate eigenvalues have high accuracy.

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