Research Article

The Bases of $M_4(\Gamma_0(71))$, $M_6(\Gamma_0(71))$ and the Number of Representation of Integers

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Following a fundamental theorem of Hecke, some bases of $S_4(\Gamma_0(71))$ and $S_6(\Gamma_0(71))$ are determined, and explicit formulas are obtained for the number of representations of positive integers by all possible direct sums (111 different combinations) of seven quadratic forms from the class group of equivalence classes of quadratic forms with discriminant -71 whose representatives are $x_1^2 + x_1x_2 + 18x_2^2$, $2x_1^2 \pm x_1x_2 + 9x_2^2$, $3x_1^2 \pm x_1x_2 + 6x_2^2$, and $4x_1^2 \pm 3x_1x_2 + 5x_2^2$.

1. Introduction

In this work, we will obtain the representation number of certain positive definite quadratic forms by means of the deep theorems of Hecke [1] and Schoeneberg [2]. Lomadze [3] and Vepkhvadze [4] have studied this subject for several quadratic forms. Also the works of Kani [5], Sun and Williams [6] clarify and contribute the subject. Of course, the work of Hijikata et al. [7] for the construction of a general bases of modular forms of $\Gamma_0(N)$ [7] is very important work. Here, we have found bases in special cases $M_4(\Gamma_0(71))$, $M_6(\Gamma_0(71))$ by extending our work on [8, 9] to the case of discriminant -71 including weight 6 case.

There exist 7 equivalence classes of binary quadratic forms of discriminant -71 whose reduced forms are

$$F_1 = x_1^2 + x_1x_2 + 18x_2^2,$$

$$\Phi_1 = 2x_1^2 - x_1x_2 + 9x_2^2,$$

$$\Phi'_1 = 2x_1^2 + x_1x_2 + 9x_2^2,$$

$$\Psi_1 = 3x_1^2 - x_1x_2 + 6x_2^2,$$

$$\Psi'_1 = 3x_1^2 + x_1x_2 + 6x_2^2,$$

$$\Lambda_1 = 4x_1^2 - 3x_1x_2 + 5x_2^2,$$

$$\Lambda'_1 = 4x_1^2 + 3x_1x_2 + 5x_2^2.$$

(1)

Here, Φ'_1 is the inverse of Φ_1 , and they represent the same integers. Similarly, Ψ'_1 is the inverse of Ψ_1 , and they represent the same integers, and Λ'_1 is the inverse of Λ_1 , and they represent the same integers. The group of these quadratic forms is a group of order 7 and can be described, see [10], easily as

$$\begin{aligned} \Phi_1, \Phi_1^2 = \Lambda'_1, \quad \Phi_1^3 = \Psi_1, \quad \Phi_1^4 = \Psi'_1, \\ \Phi_1^5 = \Lambda_1, \quad \Phi_1^6 = \Phi'_1, \quad \Phi_1^7 = F_1. \end{aligned} \quad (2)$$

Therefore, the theta series of Φ_1 , Φ'_1 and Ψ_1 , Ψ'_1 and Λ_1 and Λ'_1 are the same, respectively. F_1 is the identity element. Since 71 is prime, there is only one genus, that is, principal genus.

Let

$$\begin{aligned}
 F_k &= F_1 + \cdots + F_1 \quad (k \text{ times}), \\
 \Phi_k &= \Phi_1 + \cdots + \Phi_1 \quad (k \text{ times}), \\
 \Psi_k &= \Psi_1 + \cdots + \Psi_1 \quad (k \text{ times}), \\
 \Lambda_k &= \Lambda_1 + \cdots + \Lambda_1 \quad (k \text{ times})
 \end{aligned} \tag{3}$$

be k direct sums of the quadratic forms. In this paper, we will obtain the formulas of $r(n; Q)$ for the quadratic forms of 8 variables

$$\begin{aligned}
 H &= F_4, \Phi_4, \Psi_4, \Lambda_4, F_3 \oplus \Phi_1, F_2 \oplus \Phi_2, F_1 \oplus \Phi_3, F_3 \oplus \Psi_1, \\
 &F_2 \oplus \Psi_2, F_1 \oplus \Psi_3, \Phi_3 \oplus \Psi_1, \Phi_2 \oplus \Psi_2, \Phi_1 \oplus \Psi_3, \\
 &F_3 \oplus \Lambda_1, F_2 \oplus \Lambda_2, F_1 \oplus \Lambda_3, \Phi_3 \oplus \Lambda_1, \Phi_2 \oplus \Lambda_2, \\
 &\Phi_1 \oplus \Lambda_3, \Psi_3 \oplus \Lambda_1, \Psi_2 \oplus \Lambda_2, \Psi_1 \oplus \Lambda_3, F_2 \oplus \Phi_1 \oplus \Psi_1, \\
 &F_2 \oplus \Phi_1 \oplus \Lambda_1, F_2 \oplus \Psi_1 \oplus \Lambda_1, F_1 \oplus \Phi_2 \oplus \Psi_1, \\
 &F_1 \oplus \Phi_2 \oplus \Lambda_1, \Psi_1 \oplus \Phi_2 \oplus \Lambda_1, F_1 \oplus \Lambda_1 \oplus \Psi_2, \\
 &F_1 \oplus \Phi_1 \oplus \Psi_2, \Lambda_1 \oplus \Phi_1 \oplus \Psi_2, F_1 \oplus \Lambda_1 \oplus \Psi_2, \\
 &F_1 \oplus \Phi_1 \oplus \Lambda_2, F_1 \oplus \Psi_1 \oplus \Lambda_2, \Phi_1 \oplus \Psi_1 \oplus \Lambda_2, \\
 &F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1,
 \end{aligned} \tag{4}$$

and quadratic form of 12 variables

$$\begin{aligned}
 Q &= F_6, F_5 \oplus \Phi_1, F_5 \oplus \Psi_1, F_5 \oplus \Lambda_1, F_4 \oplus \Phi_2, F_4 \oplus \Psi_2, \\
 &F_4 \oplus \Lambda_2, F_4 \oplus \Phi_1 \oplus \Psi_1, F_4 \oplus \Phi_1 \oplus \Lambda_1, \\
 &F_4 \oplus \Psi_1 \oplus \Lambda_1, F_3 \oplus \Phi_3, F_3 \oplus \Psi_3, F_3 \oplus \Lambda_3, \\
 &F_3 \oplus \Phi_2 \oplus \Psi_1, F_3 \oplus \Phi_2 \oplus \Lambda_1, F_3 \oplus \Psi_2 \oplus \Phi_1, \\
 &F_3 \oplus \Psi_2 \oplus \Lambda_1, F_3 \oplus \Lambda_2 \oplus \Phi_1, F_3 \oplus \Lambda_2 \oplus \Psi_1, \\
 &F_3 \oplus \Psi_2 \oplus \Phi_1, F_3 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1, F_2 \oplus \Phi_4, \\
 &F_2 \oplus \Psi_4, F_2 \oplus \Lambda_4, F_2 \oplus \Phi_3 \oplus \Psi_1, F_2 \oplus \Phi_3 \oplus \Lambda_1, \\
 &F_2 \oplus \Psi_3 \oplus \Phi_1, F_2 \oplus \Psi_3 \oplus \Lambda_1, F_2 \oplus \Lambda_3 \oplus \Phi_1, \\
 &F_2 \oplus \Lambda_3 \oplus \Psi_1, F_2 \oplus \Phi_2 \oplus \Psi_1 \oplus \Lambda_1, F_2 \oplus \Psi_2 \oplus \Phi_1 \oplus \Lambda_1, \\
 &F_2 \oplus \Lambda_2 \oplus \Phi_1 \oplus \Psi_1, F_1 \oplus \Phi_5, F_1 \oplus \Psi_5, F_1 \oplus \Lambda_5,
 \end{aligned}$$

$$\begin{aligned}
 &F_1 \oplus \Phi_4 \oplus \Psi_1, F_1 \oplus \Phi_4 \oplus \Lambda_1, F_1 \oplus \Psi_4 \oplus \Phi_1, \\
 &F_1 \oplus \Psi_4 \oplus \Lambda_1, F_1 \oplus \Lambda_4 \oplus \Phi_1, F_1 \oplus \Lambda_4 \oplus \Psi_1, \\
 &F_1 \oplus \Phi_3 \oplus \Psi_2, F_1 \oplus \Phi_3 \oplus \Lambda_2, F_1 \oplus \Phi_3 \oplus \Psi_1 \oplus \Lambda_1, \\
 &F_1 \oplus \Psi_3 \oplus \Phi_2, F_1 \oplus \Psi_3 \oplus \Lambda_2, F_1 \oplus \Psi_3 \oplus \Phi_1 \oplus \Lambda_1, \\
 &F_1 \oplus \Lambda_3 \oplus \Phi_2, F_1 \oplus \Lambda_3 \oplus \Psi_2, F_1 \oplus \Lambda_3 \oplus \Phi_1 \oplus \Psi_1, \\
 &F_1 \oplus \Phi_2 \oplus \Psi_2 \oplus \Lambda_1, F_1 \oplus \Phi_2 \oplus \Lambda_2 \oplus \Psi_1, \Phi_6, \Phi_5 \oplus \Psi_1, \\
 &\Phi_5 \oplus \Lambda_1, \Phi_4 \oplus \Psi_2, \Phi_4 \oplus \Lambda_2, \Phi_4 \oplus \Psi_1 \oplus \Lambda_1, \Phi_3 \oplus \Psi_3, \\
 &\Phi_3 \oplus \Lambda_3, \Phi_3 \oplus \Psi_2 \oplus \Lambda_1, \Phi_3 \oplus \Lambda_2 \oplus \Psi_1, \Phi_2 \oplus \Psi_4, \\
 &\Phi_2 \oplus \Lambda_4, \Phi_1 \oplus \Psi_5, \Phi_1 \oplus \Lambda_5, \Phi_1 \oplus \Psi_4 \oplus \Lambda_1, \\
 &\Phi_1 \oplus \Lambda_4 \oplus \Psi_1, \Phi_1 \oplus \Psi_3 \oplus \Lambda_2, \Phi_1 \oplus \Lambda_3 \oplus \Psi_2, \Psi_6, \\
 &\Psi_5 \oplus \Lambda_1, \Psi_4 \oplus \Lambda_2, \Psi_3 \oplus \Lambda_3, \Psi_2 \oplus \Lambda_4, \Psi_1 \oplus \Lambda_5, \Lambda_6.
 \end{aligned} \tag{5}$$

In these formulas one can replace Φ_1 , Λ_1 , and Ψ_1 by Φ'_1 , Λ'_1 , and Ψ'_1 , respectively.

2. The Positive Definite Quadratic Forms

Now we will give some definitions, an important theorem and evaluation of our quadratic forms.

Definition 1. Let $Q : \mathbb{Z}^{2k} \rightarrow \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables

$$Q = \sum_{1 \leq i \leq j \leq 2k}^{2k} b_{ij} x_i x_j, \quad b_{ij} \in \mathbb{Z}. \tag{6}$$

(4) Let D be the determinant of the quadratic form

$$2Q = \sum_{i,j=1}^{2k} a_{ij} x_i x_j, \quad a_{ii} = 2b_{ii}, \quad a_{ji} = a_{ij} = b_{ij} \tag{7}$$

for $i < j$;

And let A_{ij} be the cofactors of a_{ij} for $1 \leq i, j \leq 2k$. If $\delta = \gcd(A_{ii}/2, A_{ij} \text{ for } 1 \leq i, j \leq 2k)$, then $N := D/\delta$ is the smallest positive integer, called the level of Q , for which

NA^{-1} is again an even integral matrix like A . Then,

$\Delta = (-1)^k D$ is called the discriminant of the form Q .

Theorem 2. Let $Q : \mathbb{Z}^{2k} \rightarrow \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables of level N and discriminant Δ . Then (1) the theta function

$$\begin{aligned}
 \Theta_Q(q) &= \sum_{(n_1, n_2, \dots, n_{2k}) \in \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}} q^{Q(n_1, n_2, \dots, n_{2k})} \\
 &= 1 + \sum_{n=1}^{\infty} r(n; Q) q^n, \quad q = e^{2\pi iz}
 \end{aligned} \tag{8}$$

is a modular form on $\Gamma_0(N)$ of weight k and Kronecker character

$$\chi_\Delta(d) = \left(\frac{\Delta}{d}\right) \quad \text{for } d \in \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^x, \quad (9)$$

that is, $\Theta_Q \in M_k(\Gamma_0(N), \chi_\Delta)$. Consider the following:

(2) The homogeneous quadratic polynomials in $2k$ variables

$$\varphi_{ij} = x_i x_j - \frac{1}{2k} \frac{A_{ij}}{D} 2Q, \quad 1 \leq i, j \leq 2k \quad (10)$$

are spherical functions of second order with respect to Q . Consider the following:

(3) The theta series

$$\Theta_{Q, \varphi_{ij}}(q) = \sum_{n=1}^{\infty} \left(\sum_{Q=n} \varphi_{ij} \right) q^n \quad (11)$$

is a cusp form in $S_{k+2}(\Gamma_0(N), \chi_\Delta)$.

(4) If two quadratic forms Q_1, Q_2 have the same level N and the characters $\chi_1(d), \chi_2(d)$ respectively, then the direct sum $Q_1 \oplus Q_2$ of the quadratic forms has the same level N and the character $\chi_1(d) \cdot \chi_2(d)$.

Proof. See [1, 2]. $\square$

Now, let us look at the positive definite quadratic forms of discriminant -71 .

For the quadratic form $F_1 = x_1^2 + x_1 x_2 + 18x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1 x_2 + 36x_2^2 = (x_1, x_2) \begin{pmatrix} 2 & 1 \\ 1 & 36 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (12)$$

the determinant and a cofactor are

$$D = 71, \quad A_{22} = 2. \quad (13)$$

So, $\delta = 1, N = D = 71$, and the discriminant is

$$\Delta = (-1)^{2/2} 71 = -71. \quad (14)$$

The character of F_1 is the Kronecker symbol

$$\chi(d) = \left(\frac{-71}{d}\right) \quad \text{for } d \in \left(\frac{\mathbb{Z}}{71\mathbb{Z}}\right)^x. \quad (15)$$

Similarly, for the quadratic form $\Phi_1 = 2x_1^2 - x_1 x_2 + 9x_2^2$, the determinant and a cofactor are

$$D = 71, \quad A_{21} = -1. \quad (16)$$

So, $\delta = 1, N = D = 79, \Delta = (-1)^{2/2} 71 = -71$, and the character of Φ_1 is (15).

For the quadratic form $\Psi_1 = 3x_1^2 - x_1 x_2 + 6x_2^2$, the determinant and a cofactor are

$$D = 71, \quad A_{21} = -1. \quad (17)$$

So, $\delta = 1, N = D = 71, \Delta = (-1)^{2/2} 71 = -71$, and the character of Ψ_1 is (15).

For the quadratic form $\Lambda_1 = 4x_1^2 - 3x_1 x_2 + 5x_2^2$, the determinant and some cofactors are

$$D = 71, \quad A_{12} = -3, \quad A_{11} = 10. \quad (18)$$

So, $\delta = 1, N = D = 71, \Delta = (-1)^{2/2} 71 = -71$, and the character of Ψ_1 is (15).

Consequently, F_1, Φ_1, Ψ_1 , and Λ_1 are quadratic forms whose theta series are in $M_1(\Gamma_0(71), (-71/d))$; hence, by Theorem 2, $F_2, \Phi_2, \Psi_2, \Lambda_2$ and

$$F_1 \oplus \Phi_1, F_1 \oplus \Psi_1, F_1 \oplus \Lambda_1, \Phi_1 \oplus \Psi_1, \Phi_1 \oplus \Lambda_1, \Psi_1 \oplus \Lambda_1 \quad (19)$$

are quadratic forms whose theta series are in $M_2(\Gamma_0(71))$. Moreover, the theta series of quadratic forms in (4) are in $M_4(\Gamma_0(71))$, and the theta series of quadratic forms in (5) are in $M_6(\Gamma_0(71))$.

We have the following important theorem for the Eisenstein part of theta series associated with the quadratic form.

Theorem 3. Let Q be a positive definite form of 8 or 12 variables whose theta series Θ_Q are in $M_4(\Gamma_0(71))$ and $M_6(\Gamma_0(71))$ respectively. Then, the Eisenstein parts of Θ_Q are

$$E(q; Q) = 1 + \sum_{n=1}^{\infty} (\alpha_1 \sigma_3(n) q^n + \beta_1 \sigma_3(n) q^{71n}) \quad (20)$$

for 8 variables,

$$E(q; Q) = 1 + \sum_{n=1}^{\infty} (\alpha_2 \sigma_5(n) q^n + \beta_2 \sigma_5(n) q^{71n}) \quad (21)$$

for 12 variables,

respectively, where

$$\rho_4 = \frac{3!}{(2\pi)^4} \zeta(4) = \frac{1}{240}, \quad \rho_6 = -\frac{5!}{(2\pi)^6} \zeta(6) = -\frac{1}{504},$$

$$\alpha_1 = 240 \frac{71^2 - 1}{71^4 - 1} = \frac{120}{2521}, \quad \alpha_2 = 504 \frac{71^3 + 1}{71^6 - 1} = \frac{36}{25 \cdot 565},$$

$$\beta_1 = 240 \frac{71^4 - 71^2}{71^4 - 1} = 71^2 \frac{120}{2521},$$

$$\beta_2 = -504 \frac{71^6 + 71^3}{71^6 - 1} = -\frac{12 \cdot 884 \cdot 796}{25 \cdot 565},$$

$$E_{F_4}(q) = \cdots = E_{F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1}(q)$$

$$= 1 + \frac{120}{2521} \sum_{n=1}^{\infty} (q^n + 71^2 q^{71n}) \sigma_3(n)$$

$$= 1 + \frac{120}{2521} \sum_{n=1}^{\infty} \sigma_3^*(n) q^n = 1 + \frac{120}{2521} q$$

$$+ \frac{120 \cdot 9}{2521} q^2 + \frac{120 \cdot 28}{2521} q^3 + \frac{120 \cdot 73}{2521} q^4$$

$$\begin{aligned}
& + \frac{120 \cdot 126}{2521} q^5 + \frac{120 \cdot 252}{2521} q^6 + \frac{120 \cdot 344}{2521} q^7 \\
& + \frac{120 \cdot 585}{2521} q^8 + \frac{120 \cdot 757}{2521} q^9 + \frac{120 \cdot 1134}{2521} q^{10} \\
& + \frac{120 \cdot 1332}{2521} q^{11} + \frac{120 \cdot 2044}{2521} q^{12} + \frac{120 \cdot 2198}{2521} q^{13} \\
& + \frac{120 \cdot 3096}{2521} q^{14} + \frac{120 \cdot 3528}{2521} q^{15} + \frac{120 \cdot 4681}{2521} q^{16} \\
& + \frac{120 \cdot 4914}{2521} q^{17} + \dots,
\end{aligned}$$

$$\sigma_3^*(n) = \begin{cases} \sigma_3(n) & \text{if } n \geq 1, 71 \nmid n \\ \sigma_3(n) + 71^2 \sigma_3\left(\frac{n}{71}\right) & \text{if } 71 \mid n, \end{cases} \quad (22)$$

$$E_{F_6}(q) = \dots = E_{\Lambda_6}$$

$$\begin{aligned}
& = 1 + \frac{36}{25 \cdot 565} \sum_{n=1}^{\infty} (q^n - 71^3 q^{71n}) \sigma_5(n) \\
& = 1 + \frac{36}{25 \cdot 565} \sum_{n=1}^{\infty} \sigma_5^*(n) q^n \\
& = 1 + \frac{36}{25 \cdot 565} q + \frac{1118}{25 \cdot 565} q^2 + \frac{8784}{25 \cdot 565} q^3 \\
& \quad + \frac{38052}{25 \cdot 565} q^4 + \frac{112536}{25 \cdot 565} q^5 + \frac{289872}{25 \cdot 565} q^6 \\
& \quad + \frac{605088}{25 \cdot 565} q^7 + \frac{243540}{5113} q^8 + \frac{2134548}{25 \cdot 565} q^9 \\
& \quad + \frac{3713688}{25 \cdot 565} q^{10} + \frac{5797872}{25 \cdot 565} q^{11} + \frac{9284688}{25 \cdot 565} q^{12} \\
& \quad + \frac{13366584}{25 \cdot 565} q^{13} + \frac{19967904}{25 \cdot 565} q^{14} \\
& \quad + \frac{27458784}{25 \cdot 565} q^{15} + \frac{38966436}{25 \cdot 565} q^{16} \\
& \quad + \frac{51114888}{25 \cdot 565} q^{17} + \frac{70440084}{25 \cdot 565} q^{18} \\
& \quad + \frac{17827920}{5113} q^{19} + \frac{118950552}{25 \cdot 565} q^{20} \\
& \quad + \frac{147641472}{25 \cdot 565} q^{21} + \frac{191329776}{25 \cdot 565} q^{22} \\
& \quad + \frac{231708384}{25 \cdot 565} q^{23} + \frac{59423760}{5113} q^{24} \\
& \quad + \frac{351675036}{25 \cdot 565} q^{25} + \frac{441097272}{25 \cdot 565} q^{26} \\
& \quad + \frac{103739040}{25 \cdot 565} q^{27} + \frac{639578016}{5113} q^{28} \\
& \quad + \frac{147680280}{5113} q^{29} + \dots,
\end{aligned}$$

$$\sigma_5^*(n) = \begin{cases} \sigma_5(n) & \text{if } n \geq 1, 71 \nmid n \\ \sigma_5(n) - 71^3 \sigma_5\left(\frac{n}{71}\right) & \text{if } 71 \mid n. \end{cases} \quad (23)$$

Proof. See [1]. $\square$

3. The Selection of Spherical Functions for Weight 4 and for Weight 6

First, we will be able to select 17 *spherical functions* such that the corresponding cusp forms become a basis of $S_4(\Gamma_0(71))$ since its dimension is 17; see [11].

For the quadratic form

$$\begin{aligned}
2F_2 & = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 + 36x_4^2 \\
& = (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & & \\ 1 & 36 & & \\ & & 2 & 1 \\ & & 1 & 36 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad (24)
\end{aligned}$$

the determinant and a cofactor are $D = 71^2$, $A_{11} = 2556$. By putting $2k = 4$, $Q = F_2$, and appropriate i, j in Theorem 2, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{2556}{71^2} 2F_2 = x_1^2 - \frac{18}{71} F_2, \quad (25)$$

which will be a spherical function of second order with respect to F_2 . Similarly,

$$\begin{aligned}
2F_4 & = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 + 36x_4^2 + 2x_5^2 \\
& \quad + 2x_5x_6 + 36x_6^2 + 2x_7^2 + 2x_7x_8 + 36x_8^2, \\
D & = 71^4, \quad A_{11} = 12 \, 884 \, 796, \\
\varphi_{11} & = x_1^2 - \frac{1}{8} \frac{12884796}{71^4} 2F_4 = x_1^2 - \frac{9}{71} F_4, \\
2\Phi_2 & = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 4x_3^2 - 2x_3x_4 + 18x_4^2, \\
D & = 71^2, \quad A_{11} = 1278, \quad A_{12} = 71, \\
\varphi_{11} & = x_1^2 - \frac{1}{4} \frac{1278}{71^2} 2\Phi_2 = x_1^2 - \frac{9}{71} \Phi_2, \\
\varphi_{12} & = x_1^2 - \frac{1}{4} \frac{71}{71^2} 2\Phi_2 = x_1x_2 - \frac{1}{142} \Phi_2,
\end{aligned}$$

$$2(F_1 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 4x_3^2 - 2x_3x_4 + 18x_4^2,$$

$$D = 71^2, \quad A_{11} = 2556, \quad A_{12} = -71,$$

$$A_{33} = 1278, \quad A_{34} = 71,$$

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{2556}{71^2} 2(F_1 \oplus \Phi_1) = x_1^2 - \frac{18}{71} (F_1 \oplus \Phi_1),$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4} \frac{71}{71^2} 2(F_1 \oplus \Phi_1) = x_1^2 + \frac{1}{142} (F_1 \oplus \Phi_1),$$

$$\varphi_{33} = x_3^2 - \frac{1}{4} \frac{1278}{71^2} 2(F_1 \oplus \Phi_1) = x_3^2 - \frac{9}{71} (F_1 \oplus \Phi_1),$$

$$\varphi_{34} = x_3x_4 - \frac{1}{4} \frac{71}{71^2} 2(F_1 \oplus \Phi_1) = x_3x_4 - \frac{1}{142} (F_1 \oplus \Phi_1),$$

$$2\Psi_2 = 6x_1^2 - 2x_1x_2 + 12x_2^2 + 6x_3^2 - 2x_3x_4 + 12x_4^2,$$

$$D = 71^2, \quad A_{11} = 852,$$

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{852}{71^2} 2\Psi_2 = x_1^2 - \frac{6}{71} \Psi_2,$$

$$2(F_1 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 6x_3^2 - 2x_3x_4 + 12x_4^2,$$

$$D = 71^2, \quad A_{11} = 2556, \quad A_{12} = -71,$$

$$A_{33} = 852, \quad A_{44} = 426,$$

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{2556}{71^2} 2(F_1 \oplus \Psi_1) = x_1^2 - \frac{18}{71} (F_1 \oplus \Psi_1),$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4} \frac{71}{71^2} 2(F_1 \oplus \Psi_1) = x_2x_3 + \frac{1}{142} (F_1 \oplus \Psi_1),$$

$$\varphi_{33} = x_3^2 - \frac{1}{4} \frac{852}{71^2} 2(F_1 \oplus \Psi_1) = x_3^2 - \frac{6}{71} (F_1 \oplus \Psi_1),$$

$$\varphi_{44} = x_4^2 - \frac{1}{4} \frac{426}{71^2} 2(F_1 \oplus \Psi_1) = x_4^2 - \frac{3}{71} (F_1 \oplus \Psi_1)$$

$$2(\Phi_1 \oplus \Psi_1) = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 6x_3^2 - 2x_3x_4 + 12x_4^2,$$

$$D = 71^2, \quad A_{11} = 1278, \quad A_{34} = 71,$$

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{1278}{71^2} 2(\Phi_1 \oplus \Psi_1) = x_1^2 - \frac{9}{71} (\Phi_1 \oplus \Psi_1),$$

$$\varphi_{34} = x_3x_4 - \frac{1}{4} \frac{71}{71^2} 2(\Phi_1 \oplus \Psi_1) = x_3x_4 - \frac{1}{142} (\Phi_1 \oplus \Psi_1),$$

$$2\Lambda_2 = 8x_1^2 - 6x_1x_2 + 10x_2^2 + 8x_3^2 - 6x_3x_4 + 10x_4^2,$$

$$D = 71^2, \quad A_{22} = 568,$$

$$\varphi_{22} = x_2^2 - \frac{1}{4} \frac{568}{71^2} 2\Lambda_2 = x_2^2 - \frac{4}{71} \Lambda_2,$$

$$2(F_1 \oplus \Lambda_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 8x_3^2 - 6x_3x_4 + 10x_4^2,$$

$$D = 71^2, \quad A_{11} = 2556,$$

$$\varphi_{11} = x_1^2 - \frac{1}{4} \frac{2556}{71^2} 2(F_1 \oplus \Lambda_1) = x_1^2 - \frac{18}{71} (F_1 \oplus \Lambda_1),$$

$$2(\Phi_1 \oplus \Lambda_1) = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 8x_3^2 - 6x_3x_4 + 10x_4^2,$$

$$D = 71^2, \quad A_{34} = 213,$$

$$\varphi_{34} = x_3x_4 - \frac{1}{4} \frac{213}{71^2} 2(\Phi_1 \oplus \Lambda_1) = x_3x_4 - \frac{3}{142} (\Phi_1 \oplus \Lambda_1). \quad (26)$$

Similarly, we will be able to select 29 *spherical functions* such that the corresponding cusp forms become a basis of $S_6(\Gamma_0(71))$ since its dimension is 29; see [11],

$$2F_4 = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 + 36x_4^2 + 2x_5^2 + 2x_5x_6 + 36x_6^2 + 2x_7^2 + 2x_7x_8 + 36x_8^2,$$

$$D = 71^4, \quad A_{11} = 12\,884\,796,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{12884796}{71^4} 2F_4 = x_1^2 - \frac{9}{71} F_4,$$

$$2\Phi_4 = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 4x_3^2 - 2x_3x_4 + 18x_4^2$$

$$+ 4x_5^2 - 2x_5x_6 + 18x_6^2 + 4x_7^2 - 2x_7x_8 + 18x_8^2,$$

$$D = 71^4, \quad A_{11} = 6442\,398, \quad A_{22} = 1431\,644,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{6442398}{71^4} 2\Phi_4 = x_1^2 - \frac{9}{142} \Phi_4,$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{1431644}{71^4} 2\Phi_4 = x_2^2 - \frac{1}{71} \Phi_4,$$

$$2(F_3 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 + 36x_4^2 + 2x_5^2 + 2x_5x_6 + 36x_6^2 + 4x_7^2 - 2x_7x_8 + 18x_8^2,$$

$$D = 71^4, \quad A_{11} = 12\,884\,796, \quad A_{22} = 715\,822,$$

$$A_{77} = 6442\,398, \quad A_{88} = 1431644,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{12884796}{71^4} 2(F_3 \oplus \Phi_1) = x_1^2 - \frac{9}{71} (F_3 \oplus \Phi_1),$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{715822}{71^4} 2(F_3 \oplus \Phi_1) = x_2^2 - \frac{1}{142} (F_3 \oplus \Phi_1),$$

$$\varphi_{77} = x_7^2 - \frac{1}{8} \frac{6442398}{71^4} 2(F_3 \oplus \Phi_1) = x_7^2 - \frac{9}{142} (F_3 \oplus \Phi_1),$$

$$\varphi_{88} = x_8^2 - \frac{1}{8} \frac{1431644}{71^4} 2(F_3 \oplus \Phi_1) = x_8^2 - \frac{1}{71} (F_3 \oplus \Phi_1),$$

$$2(F_2 \oplus \Phi_2) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 + 36x_4^2 + 4x_5^2 - 2x_5x_6 + 18x_6^2 + 4x_7^2 - 2x_7x_8 + 18x_8^2,$$

$$A_{22} = 715822, \quad A_{11} = 12884796, \quad A_{55} = 6442398,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{12884796}{71^4} 2(F_2 \oplus \Phi_2) = x_1^2 - \frac{9}{71} (F_2 \oplus \Phi_2),$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{715822}{71^4} 2(F_2 \oplus \Phi_2) = x_2^2 - \frac{1}{142} (F_2 \oplus \Phi_2),$$

$$\varphi_{55} = x_5^2 - \frac{1}{8} \frac{6442398}{71^4} 2(F_2 \oplus \Phi_2) = x_5^2 - \frac{9}{142} (F_2 \oplus \Phi_2),$$

$$2\Psi_4 = 6x_1^2 - 2x_1x_2 + 12x_2^2 + 6x_3^2 - 2x_3x_4 \\ + 12x_4^2 + 6x_5^2 - 2x_5x_6 + 12x_6^2 + 6x_7^2 \\ - 2x_7x_8 + 12x_8^2,$$

$$D = 71^4, \quad A_{11} = 4294 \ 932, \quad A_{12} = 357 \ 911,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{4294 \ 932}{71^4} 2\Psi_4 = x_1^2 - \frac{3}{71} \Psi_4,$$

$$\varphi_{12} = x_1x_2 - \frac{1}{8} \frac{357911}{71^4} 2\Psi_4 = x_1x_2 - \frac{1}{284} \Psi_4,$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{2147466}{71^4} 2\Psi_4 = x_2^2 - \frac{3}{142} \Psi_4,$$

$$2(F_3 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 \\ + 36x_4^2 + 2x_5^2 + 2x_5x_6 + 36x_6^2 + 6x_7^2 \\ - 2x_7x_8 + 12x_8^2,$$

$$A_{11} = 12 \ 884 \ 796,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{12 \ 884 \ 796}{71^4} 2(F_3 \oplus \Psi_1) \\ = x_1^2 - \frac{9}{71} (F_3 \oplus \Psi_1),$$

$$2(F_2 \oplus \Psi_2) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 \\ + 36x_4^2 + 6x_5^2 - 2x_5x_6 + 12x_6^2 + 6x_7^2 \\ - 2x_7x_8 + 12x_8^2,$$

$$A_{22} = 715 \ 822, \quad A_{66} = 2147 \ 466,$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{715822}{71^4} 2(F_2 \oplus \Psi_2) \\ = x_2^2 - \frac{1}{142} (F_2 \oplus \Psi_2),$$

$$\varphi_{66} = x_6^2 - \frac{1}{8} \frac{2147 \ 466}{71^4} 2(F_2 \oplus \Psi_2) \\ = x_6^2 - \frac{3}{142} (F_2 \oplus \Psi_2),$$

$$2(\Phi_3 \oplus \Psi_1) = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 4x_3^2 - 2x_3x_4 \\ + 18x_4^2 + 4x_5^2 - 2x_5x_6 + 18x_6^2 + 6x_7^2 \\ - 2x_7x_8 + 12x_8^2,$$

$$D = 71^4, \quad A_{11} = 6442 \ 398, \quad A_{22} = 1431 \ 644,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{6442398}{71^4} 2(\Phi_3 \oplus \Psi_1) = x_1^2 - \frac{9}{71} (\Phi_3 \oplus \Psi_1),$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{1431 \ 644}{71^4} 2(\Phi_3 \oplus \Psi_1) \\ = x_2^2 - \frac{1}{71} (\Phi_3 \oplus \Psi_1),$$

$$2(\Phi_1 \oplus \Psi_3) = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 6x_3^2 - 2x_3x_4 + 12x_4^2 \\ + 6x_5^2 - 2x_5x_6 + 12x_6^2 + 6x_7^2 - 2x_7x_8 + 12x_8^2,$$

$$D = 71^4, \quad A_{22} = 1431 \ 644, \quad A_{44} = 2147 \ 466,$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{1431 \ 644}{71^4} 2(\Phi_1 \oplus \Psi_3) \\ = x_2^2 - \frac{1}{71} (\Phi_1 \oplus \Psi_3),$$

$$\varphi_{44} = x_4^2 - \frac{1}{8} \frac{2147 \ 466}{71^4} 2(\Phi_1 \oplus \Psi_3) \\ = x_4^2 - \frac{3}{142} (\Phi_1 \oplus \Psi_3),$$

$$2(F_3 \oplus \Lambda_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 \\ + 36x_4^2 + 2x_5^2 + 2x_5x_6 + 36x_6^2 + 8x_7^2 \\ - 6x_7x_8 + 10x_8^2,$$

$$D = 71^4, \quad A_{11} = 12 \ 884 \ 796, \quad A_{22} = 715 \ 822,$$

$$\varphi_{11} = x_1^2 - \frac{1}{8} \frac{12 \ 884 \ 796}{71^4} 2(F_3 \oplus \Lambda_1) \\ = x_1^2 - \frac{9}{71} (F_3 \oplus \Lambda_1),$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{715 \ 822}{71^4} 2(F_3 \oplus \Lambda_1) \\ = x_2^2 - \frac{1}{142} (F_3 \oplus \Lambda_1),$$

$$2(\Phi_1 \oplus \Lambda_3) = 4x_1^2 - 2x_1x_2 + 18x_2^2 + 8x_3^2 - 6x_3x_4 \\ + 10x_4^2 + 8x_5^2 - 6x_5x_6 + 10x_6^2 + 8x_7^2 \\ - 6x_7x_8 + 10x_8^2,$$

$$D = 71^4, \quad A_{12} = 357 \ 911, \quad A_{22} = 1431 \ 644,$$

$$A_{44} = 2863 \ 288,$$

$$\varphi_{12} = x_1x_2 - \frac{1}{8} \frac{357 \ 911}{71^4} 2(\Phi_1 \oplus \Lambda_3) \\ = x_1x_2 - \frac{1}{284} (\Phi_1 \oplus \Lambda_3),$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{1431 \ 644}{71^4} 2(\Phi_1 \oplus \Lambda_3) \\ = x_2^2 - \frac{1}{71} (\Phi_1 \oplus \Lambda_3),$$

$$\varphi_{44} = x_4^2 - \frac{1}{8} \frac{2863 \ 288}{71^4} 2(\Phi_1 \oplus \Lambda_3) \\ = x_4^2 - \frac{2}{71} (\Phi_1 \oplus \Lambda_3),$$

$$2(\Psi_3 \oplus \Lambda_1) = 6x_1^2 - 2x_1x_2 + 12x_2^2 + 6x_3^2 - 2x_3x_4 \\ + 12x_4^2 + 6x_5^2 - 2x_5x_6 + 12x_6^2 + 8x_7^2 \\ - 6x_7x_8 + 10x_8^2,$$

$$D = 71^4, \quad A_{22} = 2147 \ 466,$$

$$\varphi_{22} = x_2^2 - \frac{1}{8} \frac{2147 \ 466}{71^4} 2(\Psi_3 \oplus \Lambda_1) \\ = x_2^2 - \frac{3}{142} (\Psi_3 \oplus \Lambda_1),$$

$$2(F_2 \oplus \Phi_1 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 36x_2^2 + 2x_3^2 + 2x_3x_4 \\ + 36x_4^2 + 4x_5^2 - 2x_5x_6 + 18x_6^2 + 6x_7^2 \\ - 2x_7x_8 + 12x_8^2,$$

$$D = 71^4, \quad A_{55} = 6442 \ 398, \quad A_{56} = 357 \ 911,$$

$$A_{78} = 357 \ 911,$$

$$\varphi_{55} = x_5^2 - \frac{1}{8} \frac{6442 \ 398}{71^4} 2(F_2 \oplus \Phi_1 \oplus \Psi_1) \\ = x_5^2 - \frac{9}{142} (F_2 \oplus \Phi_1 \oplus \Psi_1),$$

$$\varphi_{56} = x_5x_6 - \frac{1}{8} \frac{357 \ 911}{71^4} 2(F_2 \oplus \Phi_1 \oplus \Psi_1) \\ = x_5x_6 - \frac{1}{284} (F_2 \oplus \Phi_1 \oplus \Psi_1),$$

$$\varphi_{78} = x_7x_8 - \frac{1}{8} \frac{357 \ 911}{71^4} 2(F_2 \oplus \Phi_1 \oplus \Psi_1) \\ = x_7x_8 - \frac{1}{284} (F_2 \oplus \Phi_1 \oplus \Psi_1).$$

(27)

4. The Solutions of $Q = n$ and the Theta Series Associated with the Quadratic Forms

(1) $F_1 = x_1^2 + x_1x_2 + 18x_2^2 = n$ has the following solutions:
Consider the following:

$$n = 1 \implies x_1 = \pm 1, x_2 = 0,$$

$$n = 4 \implies x_1 = \pm 2, x_2 = 0,$$

$$n = 9 \implies x_1 = \pm 3, x_2 = 0,$$

$$n = 16 \implies x_1 = \pm 4, x_2 = 0.$$

(28)

There is no integral solutions for $n = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17$.

Consider the following:

(2) $\Phi_1 = 2x_1^2 - x_1x_2 + 9x_2^2 = n$ has the following solutions:

$$n = 2 \implies x_1 = \pm 1, x_2 = 0, \quad n = 8 \implies x_1 = \pm 2, x_2 = 0,$$

$$n = 9 \implies x_1 = 0, x_2 = \pm 1,$$

$$n = 10 \implies x_1 = 1, x_2 = -1 \text{ or } x_1 = -1, x_2 = 1,$$

$$n = 12 \implies x_1 = 1, x_2 = 1 \text{ or } x_1 = -1, x_2 = -1,$$

$$n = 15 \implies x_1 = 2, x_2 = -1 \text{ or } x_1 = -2, x_2 = 1.$$

(29)

There is no integral solutions for $n = 1, 3, 4, 5, 6, 7, 11, 13, 14, 16, 17$.

Consider the following:

(3) $\Psi_1 = 3x_1^2 - x_1x_2 + 6x_2^2 = n$ has the following solutions:

$$n = 3 \implies x_1 = \pm 1, x_2 = 0, \quad n = 6 \implies x_1 = 0, x_2 = \pm 1,$$

$$n = 8 \implies x_1 = 1, x_2 = -1, \text{ or } x_1 = -1, x_2 = 1,$$

$$n = 10 \implies x_1 = 1, x_2 = 1, \text{ or } x_1 = -1, x_2 = -1, \quad (30)$$

$$n = 12 \implies x_1 = \pm 2, x_2 = 0,$$

$$n = 16 \implies x_1 = -2, x_2 = 1, x_1 = 2, x_2 = -1.$$

There is no integral solutions for $n = 1, 2, 4, 5, 7, 9, 11, 13, 14, 15, 17$.

(4) $\Lambda_1 = 4x_1^2 - 3x_1x_2 + 5x_2^2$ has the following solutions:

Consider the following:

$$n = 4 \implies x_1 = \pm 1, x_2 = 0, \quad n = 5 \implies x_1 = 0, x_2 = \pm 1,$$

$$n = 6 \implies x_1 = 1, x_2 = -1, \text{ or } x_1 = -1, x_2 = 1,$$

$$n = 12 \implies x_1 = 1, x_2 = 1, \text{ or } x_1 = -1, x_2 = -1, \quad (31)$$

$$n = 15 \implies x_1 = 2, x_2 = 1, \text{ or } x_1 = -2, x_2 = 1,$$

$$n = 16 \implies x_1 = \pm 2, x_2 = 0.$$

There is no integral solutions for $n = 1, 2, 3, 7, 8, 9, 10, 11, 13, 14, 17$.

Hence, for quadratic forms of 8 variables, we have

$$\begin{aligned}
\Theta_{F_1}(q) &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 4q^{18} \\
&\quad + 4q^{20} + 4q^{24} + 2q^{25} + \dots, \\
\Theta_{\Phi_1}(q) &= 1 + 2q^2 + 2q^8 + 2q^9 + 2q^{10} + 2q^{12} \\
&\quad + 2q^{15} + 2q^{18} + 2q^{19} + 2q^{24} + \dots, \\
\Theta_{\Psi_1}(q) &= 1 + 2q^3 + 2q^6 + 2q^8 + 2q^{10} + 2q^{12} \\
&\quad + 2q^{16} + 2q^{20} + 2q^{24} + 2q^{25} + 2q^{27} \\
&\quad + 2q^{29} + \dots, \\
\Theta_{\Lambda_1}(q) &= 1 + 2q^4 + 2q^5 + 2q^6 + 2q^{12} + 2q^{15} \\
&\quad + 2q^{16} + 2q^{18} + 2q^{20} + 2q^{24} \\
&\quad + 2q^{27} + \dots, \\
\Theta_{F_4}(q) &= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 \\
&\quad + 96q^6 + 64q^7 + 24q^8 + 104q^9 \\
&\quad + 144q^{10} + 96q^{11} + 96q^{12} \\
&\quad + 112q^{13} + 192q^{14} + 192q^{15} \\
&\quad + 24q^{16} + 144q^{17} + \dots, \\
\Theta_{\Phi_4}(q) &= 1 + 8q^2 + 24q^4 + 32q^6 + 24q^8 + 8q^9 \\
&\quad + 56q^{10} + 48q^{11} + 152q^{12} + 96q^{13} \\
&\quad + 208q^{14} + 72q^{15} + 184q^{16} \\
&\quad + 96q^{17} + \dots, \\
\Theta_{\Psi_4}(q) &= 1 + 8q^3 + 32q^6 + 8q^8 + 80q^9 + 8q^{10} \\
&\quad + 48q^{11} + 144q^{12} + 48q^{13} + 144q^{14} \\
&\quad + 208q^{15} + 176q^{16} + 256q^{17} + \dots, \\
\Theta_{\Lambda_4}(q) &= 1 + 8q^4 + 8q^5 + 8q^6 + 24q^8 + 48q^9 \\
&\quad + 72q^{10} + 48q^{11} + 64q^{12} + 96q^{13} \\
&\quad + 192q^{14} + 232q^{15} + 264q^{16} \\
&\quad + 208q^{17} + \dots, \\
\Theta_{F_3 \oplus \Phi_1}(q) &= 1 + 6q + 14q^2 + 20q^3 + 30q^4 + 40q^5 \\
&\quad + 36q^6 + 48q^7 + 62q^8 + 44q^9 + 86q^{10} \\
&\quad + 136q^{11} + 110q^{12} + 160q^{13} \\
&\quad + 196q^{14} + 162q^{15} + 198q^{16} + 204q^{17} \\
&\quad + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_2 \oplus \Phi_2}(q) &= 1 + 4q + 8q^2 + 16q^3 + 24q^4 + 24q^5 \\
&\quad + 32q^6 + 32q^7 + 24q^8 + 56q^9 + 68q^{10} \\
&\quad + 88q^{11} + 156q^{12} + 152q^{13} + 168q^{14} \\
&\quad + 196q^{15} + 184q^{16} + 216q^{17} + \dots, \\
\Theta_{F_1 \oplus \Phi_3}(q) &= 1 + 2q + 6q^2 + 12q^3 + 14q^4 + 24q^5 \\
&\quad + 20q^6 + 16q^{17} + 30q^8 + 20q^9 + 58q^{10} \\
&\quad + 96q^{11} + 114q^{12} + 168q^{13} + 156q^{14} \\
&\quad + 166q^{15} + 158q^{16} + 180q^{17} + \dots, \\
\Theta_{F_3 \oplus \Psi_1}(q) &= 1 + 6q + 12q^2 + 10q^3 + 18q^4 + 48q^5 \\
&\quad + 42q^6 + 24q^7 + 86q^8 + 106q^9 + 62q^{10} \\
&\quad + 124q^{11} + 154q^{12} + 148q^{13} + 204q^{14} \\
&\quad + 140q^{15} + 188q^{16} + 312q^{17} + \dots, \\
\Theta_{F_2 \oplus \Psi_2}(q) &= 1 + 4q + 4q^2 + 4q^3 + 20q^4 + 24q^5 \\
&\quad + 8q^6 + 48q^7 + 72q^8 + 28q^9 + 92q^{10} \\
&\quad + 136q^{11} + 88q^{12} + 176q^{13} + 184q^{14} \\
&\quad + 168q^{15} + 292q^{16} + 248q^{17} + \dots, \\
\Theta_{F_1 \oplus \Psi_3}(q) &= 1 + 2q + 6q^3 + 14q^4 + 18q^6 + 48q^7 \\
&\quad + 6q^8 + 46q^9 + 106q^{10} + 36q^{11} \\
&\quad + 114q^{12} + 172q^{13} + 108q^{14} + 228q^{15} \\
&\quad + 248q^{16} + 240q^{17} + \dots, \\
\Theta_{\Phi_3 \oplus \Psi_1}(q) &= 1 + 6q^2 + 2q^3 + 12q^4 + 12q^5 + 10q^6 \\
&\quad + 24q^7 + 20q^8 + 22q^9 + 68q^{10} \\
&\quad + 36q^{11} + 120q^{12} + 84q^{13} + 160q^{14} \\
&\quad + 126q^{15} + 198q^{16} + 204q^{17} + \dots, \\
\Theta_{\Phi_2 \oplus \Psi_2}(q) &= 1 + 4q^2 + 4q^3 + 4q^4 + 16q^5 + 8q^6 \\
&\quad + 16q^7 + 40q^8 + 12q^9 + 64q^{10} \\
&\quad + 64q^{11} + 68q^{12} + 120q^{13} + 128q^{14} \\
&\quad + 156q^{15} + 196q^{16} + 224q^{17} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_3}(q) &= 1 + 2q^2 + 6q^3 + 12q^5 + 18q^6 + 44q^8 \\
&\quad + 34q^9 + 20q^{10} + 100q^{11} + 68q^{12} \\
&\quad + 84q^{13} + 168q^{14} + 146q^{15} \\
&\quad + 210q^{16} + 220q^{17} + \dots,
\end{aligned}$$

$$\begin{aligned}\Theta_{F_3 \oplus \Lambda_1}(q) = & 1 + 6q + 12q^2 + 8q^3 + 8q^4 + 38q^5 \\ & + 62q^6 + 52q^7 + 64q^8 + 106q^9 \\ & + 132q^{10} + 120q^{11} + 82q^{12} \\ & + 120q^{13} + 204q^{14} + 174q^{15} \\ & + 144q^{16} + 244q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2 \oplus \Lambda_2}(q) = & 1 + 4q + 4q^2 + 8q^4 + 28q^5 + 36q^6 \\ & + 32q^7 + 40q^8 + 76q^9 + 116q^{10} \\ & + 120q^{11} + 120q^{12} + 168q^{13} \\ & + 208q^{14} + 180q^{15} + 176q^{16} \\ & + 224q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Lambda_3}(q) = & 1 + 2q + 8q^4 + 18q^5 + 18q^6 + 12q^7 \\ & + 24q^8 + 62q^9 + 96q^{10} + 96q^{11} \\ & + 98q^{12} + 136q^{13} + 180q^{14} \\ & + 218q^{15} + 256q^{16} + 276q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_3 \oplus \Lambda_1}(q) = & 1 + 6q^2 + 14q^4 + 2q^5 + 22q^6 + 12q^7 \\ & + 42q^8 + 30q^9 + 70q^{10} + 40q^{11} \\ & + 84q^{12} + 48q^{13} + 144q^{14} + 128q^{15} \\ & + 278q^{16} + 264q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_2 \oplus \Lambda_2}(q) = & 1 + 4q^2 + 8q^4 + 4q^5 + 20q^6 + 16q^7 \\ & + 40q^8 + 28q^9 + 56q^{10} + 48q^{11} \\ & + 100q^{12} + 96q^{13} + 168q^{14} \\ & + 136q^{15} + 192q^{16} + 168q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_1 \oplus \Lambda_3}(q) = & 1 + 2q^2 + 6q^4 + 6q^5 + 18q^6 + 12q^7 \\ & + 26q^8 + 26q^9 + 62q^{10} + 72q^{11} \\ & + 112q^{12} + 96q^{13} + 136q^{14} + 136q^{15} \\ & + 222q^{16} + 256q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_3 \oplus \Lambda_1}(q) = & 1 + 6q^3 + 2q^4 + 2q^5 + 20q^6 + 12q^7 \\ & + 18q^8 + 44q^9 + 42q^{10} + 60q^{11} \\ & + 92q^{12} + 100q^{13} + 136q^{14} \\ & + 186q^{15} + 212q^{16} + 228q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_2 \oplus \Lambda_2}(q) = & 1 + 4q^3 + 4q^4 + 4q^5 + 12q^6 + 16q^7 \\ & + 24q^8 + 32q^9 + 48q^{10} + 64q^{11} \\ & + 96q^{12} + 104q^{13} + 136q^{14} \\ & + 188q^{15} + 220q^{16} + 232q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_1 \oplus \Lambda_3}(q) = & 1 + 2q^3 + 6q^4 + 6q^5 + 8q^6 + 12q^7 \\ & + 26q^8 + 36q^9 + 50q^{10} + 100q^{12} \\ & + 108q^{13} + 144q^{14} + 174q^{15} \\ & + 248q^{16} + 252q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1}(q) = & 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 \\ & + 4q^9 + 8q^{10} + 8q^{13} + 4q^{16} \\ & + 8q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2 \oplus \Phi_1 \oplus \Lambda_1}(q) = & 1 + 4q + 6q^2 + 8q^3 + 14q^4 + 18q^5 \\ & + 30q^6 + 52q^7 + 58q^8 + 70q^9 + 90q^{10} \\ & + 88q^{11} + 96q^{12} + 120q^{13} + 152q^{14} \\ & + 196q^{15} + 246q^{16} + 280q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2 \oplus \Psi_1 \oplus \Lambda_1}(q) = & 1 + 4q + 4q^2 + 2q^3 + 14q^4 + 26q^5 \\ & + 20q^6 + 36q^7 + 66q^8 + 72q^9 + 86q^{10} \\ & + 100q^{11} + 124q^{12} + 172q^{13} + 168q^{14} \\ & + 170q^{15} + 256q^{16} + 300q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_2 \oplus \Psi_1}(q) = & 1 + 2q + 4q^2 + 10q^3 + 10q^4 + 16q^5 \\ & + 26q^6 + 16q^7 + 38q^8 + 50q^9 + 42q^{10} \\ & + 100q^{11} + 102q^{12} + 116q^{13} + 172q^{14} \\ & + 168q^{15} + 212q^{16} + 224q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_2 \oplus \Lambda_1}(q) = & 1 + 2q + 4q^2 + 8q^3 + 8q^4 + 14q^5 \\ & + 22q^6 + 28q^7 + 48q^8 + 58q^9 + 64q^{10} \\ & + 72q^{11} + 78q^{12} + 96q^{13} + 132q^{14} \\ & + 202q^{15} + 248q^{16} + 292q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_1 \oplus \Phi_2 \oplus \Lambda_1}(q) &= 1 + 4q^2 + 2q^3 + 6q^4 + 10q^5 + 12q^6 \\ &\quad + 20q^7 + 34q^8 + 32q^9 + 58q^{10} \\ &\quad + 52q^{11} + 88q^{12} + 76q^{13} + 152q^{14} \\ &\quad + 142q^{15} + 224q^{16} + 236q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Lambda_1 \oplus \Psi_2}(q) &= 1 + 2q + 4q^3 + 12q^4 + 6q^5 + 14q^6 \\ &\quad + 36q^7 + 32q^8 + 46q^9 + 72q^{10} \\ &\quad + 80q^{11} + 114q^{12} + 136q^{13} + 148q^{14} \\ &\quad + 206q^{15} + 260q^{16} + 252q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_2}(q) &= 1 + 2q + 2q^2 + 8q^3 + 10q^4 + 8q^5 \\ &\quad + 28q^6 + 24q^7 + 22q^8 + 72q^9 + 50q^{10} \\ &\quad + 64q^{11} + 142q^{12} + 104q^{13} + 148q^{14} \\ &\quad + 210q^{15} + 210q^{16} + 260q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Lambda_1 \oplus \Phi_1 \oplus \Psi_2}(q) &= 1 + 2q^2 + 4q^3 + 2q^4 + 10q^5 + 14q^6 \\ &\quad + 12q^7 + 34q^8 + 34q^9 + 46q^{10} \\ &\quad + 64q^{11} + 88q^{12} + 96q^{13} + 136q^{14} \\ &\quad + 172q^{15} + 202q^{16} + 240q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Lambda_1 \oplus \Psi_2}(q) &= 1 + 2q + 4q^3 + 12q^4 + 6q^5 + 14q^6 \\ &\quad + 36q^7 + 32q^8 + 46q^9 + 72q^{10} \\ &\quad + 80q^{11} + 114q^{12} + 136q^{13} \\ &\quad + 148q^{14} + 206q^{15} + 260q^{16} \\ &\quad + 252q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1 \oplus \Lambda_2}(q) &= 1 + 2q + 2q^2 + 4q^3 + 6q^4 + 12q^5 \\ &\quad + 24q^6 + 32q^7 + 38q^8 + 48q^9 + 66q^{10} \\ &\quad + 88q^{11} + 118q^{12} + 144q^{13} + 156q^{14} \\ &\quad + 174q^{15} + 182q^{16} + 212q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Psi_1 \oplus \Lambda_2}(q) &= 1 + 2q + 2q^3 + 10q^4 + 12q^5 + 14q^6 \\ &\quad + 24q^7 + 38q^8 + 54q^9 + 66q^{10} \\ &\quad + 84q^{11} + 122q^{12} + 156q^{13} + 156q^{14} \\ &\quad + 184q^{15} + 252q^{16} + 264q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_1 \oplus \Psi_1 \oplus \Lambda_2}(q) &= 1 + 2q^2 + 2q^3 + 4q^4 + 8q^5 + 14q^6 \\ &\quad + 16q^7 + 28q^8 + 34q^9 + 52q^{10} \\ &\quad + 60q^{11} + 100q^{12} + 100q^{13} \\ &\quad + 152q^{14} + 150q^{15} + 206q^{16} \\ &\quad + 220q^{17} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1}(q) &= 1 + 2q + 2q^2 + 6q^3 + 8q^4 + 10q^5 \\ &\quad + 24q^6 + 28q^7 + 36q^8 + 60q^9 + 64q^{10} \\ &\quad + 76q^{11} + 102q^{12} + 124q^{13} + 148q^{14} \\ &\quad + 188q^{15} + 234q^{16} + 256q^{17} + \dots.\end{aligned}\tag{32}$$

For quadratic forms of 12 variables, we have

$$\begin{aligned}\Theta_{F_6}(q) &= 1 + 12q + 60q^2 + 160q^3 + 252q^4 \\ &\quad + 312q^5 + 544q^6 + 960q^7 \\ &\quad + 1020q^8 + 876q^9 + 1560q^{10} \\ &\quad + 2400q^{11} + 2080q^{12} + 2040q^{13} \\ &\quad + 3264q^{14} + 4160q^{15} + 4092q^{16} \\ &\quad + 3480q^{17} + 4404q^{18} + 7440q^{19} \\ &\quad + 7536q^{20} + 6768q^{21} + 11280q^{22} \\ &\quad + 15168q^{23} + 16168q^{24} + 18420q^{25} \\ &\quad + 21720q^{26} + 28720q^{27} + 32880q^{28} \\ &\quad + 32232q^{29} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_6}(q) &= 1 + 12q^2 + 60q^4 + 160q^6 \\ &\quad + 252q^8 + 12q^9 + 324q^{10} + 120q^{11} \\ &\quad + 676q^{12} + 480q^{13} + 1560q^{14} \\ &\quad + 972q^{15} + 2460q^{16} + 1200q^{17} \\ &\quad + 2976q^{18} + 1956q^{19} + 4524q^{20} \\ &\quad + 5040q^{21} + 8664q^{22} + 9240q^{23} \\ &\quad + 13312q^{24} + 11544q^{25} + 16080q^{26} \\ &\quad + 14920q^{27} + 22224q^{28} + 25944q^{29} + \dots,\end{aligned}$$

$$\Theta_{\Psi_6}(q)$$

$$\begin{aligned} &= 1 + 12q^3 + 72q^6 + 12q^8 + 280q^9 \\ &\quad + 12q^{10} + 120q^{11} + 792q^{12} \\ &\quad + 120q^{13} + 600q^{14} + 1752q^{15} \\ &\quad + 672q^{16} + 1920q^{17} + 3344q^{18} \\ &\quad + 2520q^{19} + 4512q^{20} + 6144q^{21} \\ &\quad + 6960q^{22} + 8664q^{23} + 11632q^{24} \\ &\quad + 14796q^{25} + 15600q^{26} + 21476q^{27} \\ &\quad + 26280q^{28} + 27852q^{29} + \dots, \end{aligned}$$

$$\Theta_{\Lambda_6}(q)$$

$$\begin{aligned} &= 1 + 12q^4 + 12q^5 + 12q^6 + 60q^8 + 120q^9 \\ &\quad + 180q^{10} + 120q^{11} + 232q^{12} \\ &\quad + 480q^{13} + 960q^{14} + 1132q^{15} \\ &\quad + 1332q^{16} + 1560q^{17} + 2692q^{18} \\ &\quad + 3960q^{19} + 5484q^{20} + 6000q^{21} \\ &\quad + 6960q^{22} + 8280q^{23} + 12136q^{24} \\ &\quad + 15576q^{25} + 18744q^{26} + 19172q^{27} \\ &\quad + 22320q^{28} + 26904q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_5 \oplus \Phi_1}(q)$$

$$\begin{aligned} &= 1 + 10q + 42q^2 + 100q^3 + 170q^4 \\ &\quad + 272q^5 + 420q^6 + 544q^7 + 682q^8 + 912q^9 \\ &\quad + 1062q^{10} + 1320q^{11} + 1942q^{12} \\ &\quad + 2264q^{13} + 2564q^{14} + 3586q^{15} \\ &\quad + 4050q^{16} + 4244q^{17} + 5382q^{18} \\ &\quad + 6222q^{19} + 7536q^{20} + 9540q^{21} \\ &\quad + 10924q^{22} + 13604q^{23} + 16206q^{24} \\ &\quad + 17730q^{25} + 20900q^{26} + 24080q^{27} \\ &\quad + 27964q^{28} + 33848q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_4 \oplus \Phi_2}(q)$$

$$\begin{aligned} &= 1 + 8q + 28q^2 + 64q^3 + 124q^4 \\ &\quad + 208q^5 + 288q^6 + 384q^7 + 508q^8 + 588q^9 \\ &\quad + 764q^{10} + 1096q^{11} + 1356q^{12} \end{aligned}$$

$$\begin{aligned} &\quad + 1872q^{13} + 2568q^{14} + 2884q^{15} \\ &\quad + 3580q^{16} + 23384q^{17} + 4696q^{18} \\ &\quad + 5900q^{19} + 7444q^{20} + 8912q^{21} \\ &\quad + 11144q^{22} + 13320q^{23} + 15584q^{24} \\ &\quad + 17568q^{25} + 19936q^{26} + 23384q^{27} \\ &\quad + 26800q^{28} + 32344q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_5 \oplus \Psi_1}(q)$$

$$\begin{aligned} &= 1 + 10q + 40q^2 + 82q^3 + 110q^4 + 192q^5 \\ &\quad + 402q^6 + 520q^7 + 506q^8 + 910q^9 \\ &\quad + 1462q^{10} + 1364q^{11} + 1642q^{12} \\ &\quad + 2724q^{13} + 3060q^{14} + 3284q^{15} \\ &\quad + 4032q^{16} + 4584q^{17} + 6060q^{18} \\ &\quad + 6680q^{19} + 6314q^{20} + 9344q^{21} \\ &\quad + 11700q^{22} + 11624q^{23} + 14422q^{24} \\ &\quad + 17316q^{25} + 21064q^{26} + 25562q^{27} \\ &\quad + 26304q^{28} + 33066q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_4 \oplus \Psi_2}(q)$$

$$\begin{aligned} &= 1 + 8q + 24q^2 + 36q^3 + 56q^4 + 144q^5 \\ &\quad + 232q^6 + 224q^7 + 412q^8 + 784q^9 \\ &\quad + 756q^{10} + 936q^{11} + 1800q^{12} \\ &\quad + 1976q^{13} + 2152q^{14} + 3336q^{15} \\ &\quad + 3720q^{16} + 4464q^{17} + 5880q^{18} \\ &\quad + 5896q^{19} + 7728q^{20} + 9984q^{21} \\ &\quad + 9872q^{22} + 12360q^{23} + 15104q^{24} \\ &\quad + 16796q^{25} + 20608q^{26} + 22412q^{27} \\ &\quad + 26456q^{28} + 34132q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_5 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 10q + 40q^2 + 80q^3 + 92q^4 + 134q^5 \\ &\quad + 342q^6 + 580q^7 + 620q^8 + 814q^9 + 1444q^{10} \\ &\quad + 1904q^{11} + 1922q^{12} + 2120q^{13} + 2900q^{14} \\ &\quad + 3862q^{15} + 3972q^{16} + 3844q^{17} + 5502q^{18} \\ &\quad + 7320q^{19} + 7118q^{20} + 7644q^{21} \end{aligned}$$

$$\begin{aligned}
& + 10240q^{22} + 12584q^{23} + 14366q^{24} \\
& + 16938q^{25} + 21624q^{26} + 27222q^{27} \\
& + 29780q^{28} + 33104q^{29} + \dots, \\
\Theta_{F_4 \oplus \Lambda_2}(q) \\
& = 1 + 8q + 24q^2 + 32q^3 + 28q^4 \\
& + 84q^5 + 228q^6 + 320q^7 + 348q^8 + 560q^9 \\
& + 988q^{10} + 1352q^{11} + 1544q^{12} \\
& + 1904q^{13} + 2784q^{14} + 3620q^{15} \\
& + 3748q^{16} + 4088q^{17} + 5396q^{18} \\
& + 6504q^{19} + 6996q^{20} + 8176q^{21} \\
& + 10256q^{22} + 12360q^{23} + 14152q^{24} \\
& + 16384q^{25} + 20280q^{26} + 24012q^{27} \\
& + 26576q^{28} + 32632q^{29} + \dots, \\
\Theta_{F_4 \oplus \Phi_1 \oplus \Psi_1}(q) \\
& = 1 + 8q + 26q^2 + 50q^3 + 88q^4 + 164q^5 \\
& + 242q^6 + 320q^7 + 496q^8 + 650q^9 \\
& + 776q^{10} + 1204q^{11} + 1532q^{12} \\
& + 1700q^{13} + 2552q^{14} + 3130q^{15} \\
& + 3282q^{16} + 4516q^{17} + 5358q^{18} \\
& + 5942q^{19} + 7726q^{20} + 8924q^{21} \\
& + 10624q^{22} + 13060q^{23} + 14720q^{24} \\
& + 17478q^{25} + 20380q^{26} + 23554q^{27} \\
& + 27980q^{28} + 32174q^{29} + \dots, \\
\Theta_{F_4 \oplus \Phi_1 \oplus \Lambda_1}(q) \\
& = 1 + 8q + 26q^2 + 48q^3 + 74q^4 + 130q^5 \\
& + 214q^6 + 324q^7 + 510q^8 + 714q^9 \\
& + 914q^{10} + 1264q^{11} + 1592q^{12} \\
& + 1800q^{13} + 2272q^{14} + 2876q^{15} \\
& + 3474q^{16} + 4360q^{17} + 5300q^{18} \\
& + 6390q^{19} + 7758q^{20} + 8824q^{21} \\
& + 10076q^{22} + 12300q^{23} + 14696q^{24} \\
& + 17188q^{25} + 20988q^{26} + 24926q^{27}
\end{aligned}$$

$$\begin{aligned}
& + 28496q^{28} + 33100q^{29} + \dots, \\
\Theta_{F_4 \oplus \Psi_1 \oplus \Lambda_1}(q) \\
& = 1 + 8q + 24q^2 + 34q^3 + 42q^4 + 114q^5 \\
& + 228q^6 + 260q^7 + 366q^8 + 716q^9 \\
& + 966q^{10} + 1092q^{11} + 1516q^{12} \\
& + 2068q^{13} + 2600q^{14} + 3082q^{15} \\
& + 3492q^{16} + 4660q^{17} + 5958q^{18} \\
& + 6148q^{19} + 7116q^{20} + 9356q^{21} \\
& + 10636q^{22} + 11720q^{23} + 14020q^{24} \\
& + 16930q^{25} + 20576q^{26} + 23724q^{27} \\
& + 26388q^{28} + 32762q^{29} + \dots, \\
\Theta_{F_3 \oplus \Phi_3}(q) \\
& = 1 + 6q + 18q^2 + 44q^3 + 90q^4 + 144q^5 \\
& + 212q^6 + 288q^7 + 330q^8 + 424q^9 \\
& + 570q^{10} + 720q^{11} + 1130q^{12} \\
& + 1584q^{13} + 1980q^{14} + 2694q^{15} \\
& + 3114q^{16} + 3516q^{17} + 4402q^{18} \\
& + 5154q^{19} + 6540q^{20} + 8436q^{21} \\
& + 10596q^{22} + 12876q^{23} + 14726q^{24} \\
& + 17382q^{25} + 19212q^{26} + 21800q^{27} \\
& + 26940q^{28} + 31440q^{29} + \dots, \\
\Theta_{F_3 \oplus \Psi_3}(q) \\
& = 1 + 6q + 12q^2 + 14q^3 + 144q^4 + 96q^5 \\
& + 90q^6 + 144q^7 + 378q^8 + 386q^9 \\
& + 402q^{10} + 1020q^{11} + 1170q^{12} \\
& + 1116q^{13} + 2244q^{14} + 2604q^{15} \\
& + 2748q^{16} + 4464q^{17} + 4784q^{18} \\
& + 5616q^{19} + 8106q^{20} + 8360q^{21} \\
& + 10116q^{22} + 13080q^{23} + 14246q^{24} \\
& + 17292q^{25} + 19848q^{26} + 22758q^{27} \\
& + 28176q^{28} + 31254q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_3 \oplus \Lambda_3}(q)$$

$$\begin{aligned} &= 1 + 6q + 12q^2 + 8q^3 + 12q^4 + 66q^5 \\ &\quad + 138q^6 + 156q^7 + 180q^8 + 354q^9 \\ &\quad + 672q^{10} + 936q^{11} + 1090q^{12} \\ &\quad + 1464q^{13} + 2220q^{14} + 2898q^{15} \\ &\quad + 3300q^{16} + 4020q^{17} + 5214q^{18} \\ &\quad + 6288q^{19} + 7146q^{20} + 8212q^{21} \\ &\quad + 9744q^{22} + 11496q^{23} + 13318q^{24} \\ &\quad + 16182q^{25} + 20016q^{26} + 23034q^{27} \\ &\quad + 26316q^{28} + 32352q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_2 \oplus \Psi_1}(q)$$

$$\begin{aligned} &= 1 + 6q + 16q^2 + 34q^3 + 70q^4 + 112q^5 \\ &\quad + 162q^6 + 256q^7 + 330q^8 + 422q^9 \\ &\quad + 658q^{10} + 812q^{11} + 1062q^{12} \\ &\quad + 1620q^{13} + 1876q^{14} + 2384q^{15} \\ &\quad + 3288q^{16} + 3600q^{17} + 4520q^{18} \\ &\quad + 5852q^{19} + 6742q^{20} + 8424q^{21} \\ &\quad + 10316q^{22} + 12392q^{23} + 14446q^{24} \\ &\quad + 16888q^{25} + 20360q^{26} + 22810q^{27} \\ &\quad + 27144q^{28} + 32578q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_2 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 6q + 16q^2 + 32q^3 + 60q^4 + 94q^5 \\ &\quad + 142q^6 + 236q^7 + 348q^8 + 494q^9 \\ &\quad + 720q^{10} + 912q^{11} + 1134q^{12} \\ &\quad + 1472q^{13} + 1772q^{14} + 2258q^{15} \\ &\quad + 3020q^{16} + 3828q^{17} + 4826q^{18} \\ &\quad + 6012q^{19} + 7170q^{20} + 8316q^{21} \\ &\quad + 9824q^{22} + 11608q^{23} + 13814q^{24} \\ &\quad + 17094q^{25} + 20472q^{26} + 24166q^{27} \\ &\quad + 28492q^{28} + 32200q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_1 \oplus \Psi_2}(q)$$

$$\begin{aligned} &= 1 + 6q + 14q^2 + 24q^3 + 54q^4 + 96q^5 \\ &\quad + 124q^6 + 216q^7 + 338q^8 + 380q^9 \\ &\quad + 626q^{10} + 928q^{11} + 966q^{12} \\ &\quad + 1536q^{13} + 2148q^{14} + 2162q^{15} \\ &\quad + 3198q^{16} + 4156q^{17} + 4362q^{18} \\ &\quad + 5982q^{19} + 7264q^{20} + 8268q^{21} \\ &\quad + 10476q^{22} + 12228q^{23} + 14486q^{24} \\ &\quad + 17010q^{25} + 19988q^{26} + 23700q^{27} \\ &\quad + 26876q^{28} + 31980q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Psi_2 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 6q + 12q^2 + 12q^3 + 32q^4 + 86q^5 \\ &\quad + 102q^6 + 132q^7 + 316q^8 + 450q^9 \\ &\quad + 504q^{10} + 840q^{11} + 1202q^{12} \\ &\quad + 1464q^{13} + 1996q^{14} + 2486q^{15} \\ &\quad + 3192q^{16} + 4284q^{17} + 4834q^{18} \\ &\quad + 5776q^{19} + 7686q^{20} + 8844q^{21} \\ &\quad + 9920q^{22} + 12304q^{23} + 14558q^{24} \\ &\quad + 16810q^{25} + 20032q^{26} + 22658q^{27} \\ &\quad + 26764q^{28} + 32596q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_1 \oplus \Lambda_2}(q)$$

$$\begin{aligned} &= 1 + 6q + 14q^2 + 20q^3 + 34q^4 + 68q^5 \\ &\quad + 120q^6 + 208q^7 + 322q^8 + 436q^9 \\ &\quad + 626q^{10} + 904q^{11} + 1198q^{12} \\ &\quad + 1576q^{13} + 2060q^{14} + 2590q^{15} \\ &\quad + 3186q^{16} + 3820q^{17} + 4542q^{18} \\ &\quad + 5614q^{19} + 6948q^{20} + 8300q^{21} \\ &\quad + 10180q^{22} + 12476q^{23} + 14510q^{24} \\ &\quad + 16918q^{25} + 19780q^{26} + 22876q^{27} \\ &\quad + 27020q^{28} + 31976q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Psi_1 \oplus \Lambda_2}(q)$$

$$\begin{aligned} &= 1 + 8q + 12q^2 + 10q^3 + 22q^4 + 76q^5 \\ &\quad + 118q^6 + 136q^7 + 250q^8 + 442q^9 \\ &\quad + 602q^{10} + 796q^{11} + 1114q^{12} \\ &\quad + 1612q^{13} + 2164q^{14} + 2544q^{15} \\ &\quad + 3152q^{16} + 4264q^{17} + 5132q^{18} \\ &\quad + 5736q^{19} + 7094q^{20} + 8864q^{21} \\ &\quad + 10308q^{22} + 11952q^{23} + 13958q^{24} \\ &\quad + 16728q^{25} + 20008q^{26} + 22398q^{27} \\ &\quad + 26128q^{28} + 32226q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 6q + 14q^2 + 22q^3 + 44q^4 + 82q^5 \\ &\quad + 120q^6 + 204q^7 + 328q^8 + 432q^9 \\ &\quad + 648q^{10} + 924q^{11} + 1102q^{12} \\ &\quad + 1492q^{13} + 2004q^{14} + 2396q^{15} \\ &\quad + 3094q^{16} + 3944q^{17} + 4772q^{18} \\ &\quad + 5938q^{19} + 7144q^{20} + 8352q^{21} \\ &\quad + 10136q^{22} + 12180q^{23} + 14034q^{24} \\ &\quad + 16904q^{25} + 20396q^{26} + 23340q^{27} \\ &\quad + 27600q^{28} + 32194q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_4}(q)$$

$$\begin{aligned} &= 1 + 4q + 12q^2 + 32q^3 + 60q^4 + 104q^5 \\ &\quad + 160q^6 + 192q^7 + 252q^8 + 300q^9 \\ &\quad + 352q^{10} + 592q^{11} + 824q^{12} \\ &\quad + 1224q^{13} + 1808q^{14} + 2120q^{15} \\ &\quad + 2652q^{16} + 3048q^{17} + 3452q^{18} \\ &\quad + 4456q^{19} + 5608q^{20} + 7696q^{21} \\ &\quad + 9696q^{22} + 11984q^{23} + 14504q^{24} \\ &\quad + 15596q^{25} + 18312q^{26} + 21152q^{27} \\ &\quad + 24528q^{28} + 31144q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Psi_4}(q)$$

$$\begin{aligned} &= 1 + 4q + 4q^2 + 8q^3 + 36q^4 + 40q^5 \\ &\quad + 32q^6 + 160q^7 + 204q^8 + 116q^9 \\ &\quad + 496q^{10} + 688q^{11} + 432q^{12} \\ &\quad + 1272q^{13} + 1712q^{14} + 1360q^{15} \\ &\quad + 2900q^{16} + 3496q^{17} + 3436q^{18} \\ &\quad + 5856q^{19} + 6528q^{20} + 7344q^{21} \\ &\quad + 10352q^{22} + 11664q^{23} + 13912q^{24} \\ &\quad + 16740q^{25} + 19656q^{26} + 23688q^{27} \\ &\quad + 26752q^{28} + 31264q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Lambda_4}(q)$$

$$\begin{aligned} &= 1 + 4q + 4q^2 + 12q^4 + 48q^5 + 72q^6 \\ &\quad + 64q^7 + 92q^8 + 244q^9 + 464q^{10} \\ &\quad + 592q^{11} + 672q^{12} + 1000q^{13} \\ &\quad + 1632q^{14} + 2248q^{15} + 2764q^{16} \\ &\quad + 3488q^{17} + 4420q^{18} + 5536q^{19} \\ &\quad + 6696q^{20} + 8048q^{21} + 9648q^{22} \\ &\quad + 11280q^{23} + 13192q^{24} + 15948q^{25} \\ &\quad + 18840q^{26} + 21320q^{27} + 25168q^{28} \\ &\quad + 31304q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_3 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 4q + 10q^2 + 26q^3 + 48q^4 + 76q^5 \\ &\quad + 130q^6 + 168q^7 + 224q^8 + 346q^9 \\ &\quad + 404q^{10} + 604q^{11} + 912q^{12} \\ &\quad + 1076q^{13} + 1608q^{14} + 2070q^{15} \\ &\quad + 2386q^{16} + 3260q^{17} + 3858q^{18} \\ &\quad + 4770q^{19} + 6114q^{20} + 7540q^{21} \\ &\quad + 9512q^{22} + 11124q^{23} + 13912q^{24} \\ &\quad + 16258q^{25} + 18316q^{26} + 23010q^{27} \\ &\quad + 26244q^{28} + 30766q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_3 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 4q + 10q^2 + 24q^3 + 42q^4 + 66q^5 \\ &\quad + 110q^6 + 156q^7 + 238q^8 + 370q^9 \\ &\quad + 494q^{10} + 688q^{11} + 892q^{12} \\ &\quad + 1072q^{13} + 1400q^{14} + 1816q^{15} \\ &\quad + 2418q^{16} + 3256q^{17} + 4264q^{18} \\ &\quad + 5314q^{19} + 6314q^{20} + 7576q^{21} \\ &\quad + 8764q^{22} + 10444q^{23} + 13056q^{24} \\ &\quad + 15928q^{25} + 19692q^{26} + 23822q^{27} \\ &\quad + 27672q^{28} + 31628q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_3}(q)$$

$$\begin{aligned} &= 1 + 4q + 6q^2 + 14q^3 + 36q^4 + 44q^5 \\ &\quad + 74q^6 + 160q^7 + 168q^8 + 262q^9 \\ &\quad + 516q^{10} + 492q^{11} + 764q^{12} \\ &\quad + 1348q^{13} + 1248q^{14} + 1898q^{15} \\ &\quad + 2910q^{16} + 2844q^{17} + 4098q^{18} \\ &\quad + 5478q^{19} + 5982q^{20} + 7868q^{21} \\ &\quad + 9648q^{22} + 11500q^{23} + 13616q^{24} \\ &\quad + 16454q^{25} + 20004q^{26} + 22230q^{27} \\ &\quad + 27188q^{28} + 32178q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Psi_3 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 4q + 4q^2 + 6q^3 + 30q^4 + 42q^5 \\ &\quad + 36q^6 + 124q^7 + 206q^8 + 192q^9 \\ &\quad + 394q^{10} + 636q^{11} + 700q^{12} \\ &\quad + 1100q^{13} + 1528q^{14} + 1850q^{15} \\ &\quad + 2680q^{16} + 3324q^{17} + 3978q^{18} \\ &\quad + 5484q^{19} + 6676q^{20} + 7652q^{21} \\ &\quad + 9820q^{22} + 12008q^{23} + 13644q^{24} \\ &\quad + 16738q^{25} + 19760q^{26} + 22488q^{27} \\ &\quad + 27620q^{28} + 31358q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Lambda_3}(q)$$

$$\begin{aligned} &= 1 + 4q + 6 + 8q^3 + 18q^4 + 38q^5 \\ &\quad + 74q^6 + 124q^7 + 174q^8 + 254q^9 \\ &\quad + 406q^{10} + 624q^{11} + 888q^{12} \\ &\quad + 1200q^{13} + 1568q^{14} + 2032q^{15} \\ &\quad + 2634q^{16} + 3328q^{17} + 4204q^{18} \\ &\quad + 5286q^{19} + 6346q^{20} + 7544q^{21} \\ &\quad + 9140q^{22} + 11052q^{23} + 13360q^{24} \\ &\quad + 16232q^{25} + 19492q^{26} + 23202q^{27} \\ &\quad + 27368q^{28} + 31548q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Psi_1 \oplus \Lambda_3}(q)$$

$$\begin{aligned} &= 1 + 4q + 4q^2 + 2q^3 + 18q^4 + 46q^5 \\ &\quad + 56q^6 + 76q^7 + 150q^8 + 264q^9 \\ &\quad + 386q^{10} + 524q^{11} + 772q^{12} \\ &\quad + 1172q^{13} + 1568q^{14} + 1950q^{15} \\ &\quad + 2636q^{16} + 3572q^{17} + 4374q^{18} \\ &\quad + 5284q^{19} + 6592q^{20} + 8100q^{21} \\ &\quad + 9692q^{22} + 11320q^{23} + 13452q^{24} \\ &\quad + 16454q^{25} + 19368q^{26} + 22184q^{27} \\ &\quad + 26476q^{28} + 31474q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_2 \oplus \Psi_1 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 4q + 8q^2 + 18q^3 + 34q^4 + 50q^5 \\ &\quad + 92q^6 + 148q^7 + 206q^8 + 340q^9 \\ &\quad + 474q^{10} + 612q^{11} + 888q^{12} \\ &\quad + 1124q^{13} + 1432q^{14} + 1966q^{15} \\ &\quad + 2500q^{16} + 3188q^{17} + 4122q^{18} \\ &\quad + 5168q^{19} + 6232q^{20} + 7636q^{21} \\ &\quad + 9340q^{22} + 10976q^{23} + 13500q^{24} \\ &\quad + 16310q^{25} + 19176q^{26} + 23204q^{27} \\ &\quad + 27108q^{28} + 31322q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_2 \oplus \Lambda_1}(q)$$

$$\begin{aligned} &= 1 + 4q + 6q^2 + 12q^3 + 30q^4 + 42q^5 \\ &\quad + 70q^6 + 140q^7 + 182q^8 + 278q^9 \\ &\quad + 470q^{10} + 568q^{11} + 800q^{12} \\ &\quad + 1200q^{13} + 1440q^{14} + 1924q^{15} \\ &\quad + 2630q^{16} + 3216q^{17} + 4104q^{18} \\ &\quad + 5238q^{19} + 6326q^{20} + 7728q^{21} \\ &\quad + 9628q^{22} + 11324q^{23} + 13512q^{24} \\ &\quad + 16716q^{25} + 19260q^{26} + 22754q^{27} \\ &\quad + 27344q^{28} + 31160q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_2}(q)$$

$$\begin{aligned} &= 1 + 4q + 6q^2 + 10q^3 + 24q^4 + 40q^5 \\ &\quad + 70q^6 + 128q^7 + 184q^8 + 278q^9 \\ &\quad + 436q^{10} + 596q^{11} + 828q^{12} \\ &\quad + 1172q^{13} + 1552q^{14} + 2014q^{15} \\ &\quad + 2602q^{16} + 3260q^{17} + 4086q^{18} \\ &\quad + 5166q^{19} + 6266q^{20} + 7700q^{21} \\ &\quad + 9640q^{22} + 11436q^{23} + 13616q^{24} \\ &\quad + 16418q^{25} + 19180q^{26} + 22566q^{27} \\ &\quad + 26780q^{28} + 31238q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_5}(q)$$

$$\begin{aligned} &= 1 + 2q + 10q^2 + 20q^3 + 42q^4 + 80q^5 \\ &\quad + 100q^6 + 160q^7 + 170q^8 + 192q^9 \\ &\quad + 302q^{10} + 344q^{11} + 670q^{12} + 1000q^{13} \\ &\quad + 1364q^{14} + 1930q^{15} + 2082q^{16} \\ &\quad + 2500q^{17} + 2814q^{18} + 3414q^{19} \\ &\quad + 4984q^{20} + 6340q^{21} + 9180q^{22} \\ &\quad + 11060q^{23} + 12974q^{24} + 14922q^{25} \\ &\quad + 16404q^{26} + 18560q^{27} + 24028q^{28} \\ &\quad + 28360q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_1 \oplus \Psi_5}(q)$$

$$\begin{aligned} &= 1 + 2q + 10q^3 + 22q^4 + 50q^6 + 120q^7 \\ &\quad + 10q^8 + 182q^9 + 430q^{10} + 100q^{11} \\ &\quad + 570q^{12} + 1140q^{13} + 500q^{14} \\ &\quad + 1572q^{15} + 2456q^{16} + 1720q^{17} \\ &\quad + 3684q^{18} + 4744q^{19} + 4634q^{20} \\ &\quad + 7280q^{21} + 8820q^{22} + 10280q^{23} \\ &\quad + 12758q^{24} + 15892q^{25} + 19240q^{26} \\ &\quad + 21490q^{27} + 27040q^{28} \\ &\quad + 31650q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_1 \oplus \Lambda_5}(q)$$

$$\begin{aligned} &= 1 + 2q + 12q^4 + 30q^5 + 30q^6 + 20q^7 \\ &\quad + 60q^8 + 182q^9 + 300q^{10} + 320q^{11} \\ &\quad + 370q^{12} + 680q^{13} + 1220q^{14} + 1710q^{15} \\ &\quad + 2052q^{16} + 2500q^{17} + 3374q^{18} \\ &\quad + 4680q^{19} + 6166q^{20} + 7564q^{21} \\ &\quad + 8800q^{22} + 10120q^{23} + 12398q^{24} \\ &\quad + 15554q^{25} + 18664q^{26} + 21150q^{27} \\ &\quad + 24740q^{28} + 29648q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_4 \oplus \Psi_1}(q)$$

$$\begin{aligned} &= 1 + 2q + 8q^2 + 18q^3 + 30q^4 + 64q^5 \\ &\quad + 82q^6 + 120q^7 + 186q^8 + 190q^9 \\ &\quad + 334q^{10} + 452q^{11} + 578q^{12} \\ &\quad + 996q^{13} + 1172q^{14} + 1612q^{15} \\ &\quad + 2128q^{16} + 2424q^{17} + 3300q^{18} \\ &\quad + 3904q^{19} + 5234q^{20} + 6608q^{21} \\ &\quad + 8100q^{22} + 10696q^{23} + 12086q^{24} \\ &\quad + 14732q^{25} + 18088q^{26} + 20058q^{27} \\ &\quad + 25392q^{28} + 29946q^{29} + \dots, \end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_4 \oplus \Lambda_1}(q)$$

$$= 1 + 2q + 8q^2 + 16q^3 + 28q^4 + 54q^5$$

$$\begin{aligned}
& + 70q^6 + 116q^7 + 172q^8 + 238q^9 \\
& + 380q^{10} + 496q^{11} + 666q^{12} \\
& + 856q^{13} + 1060q^{14} + 1422q^{15} \\
& + 1908q^{16} + 2692q^{17} + 3574q^{18} \\
& + 4544q^{19} + 5638q^{20} + 6364q^{21} \\
& + 7680q^{22} + 9160q^{23} + 11550q^{24} \\
& + 14658q^{25} + 18840q^{26} + 22390q^{27} \\
& + 26980q^{28} + 30016q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_4}(q)$$

$$\begin{aligned}
& = 1 + 2q + 2q^2 + 12q^3 + 18q^4 + 16q^5 \\
& + 68q^6 + 80q^7 + 74q^8 + 264q^9 \\
& + 254q^{10} + 280q^{11} + 790q^{12} \\
& + 680q^{13} + 900q^{14} + 1890q^{15} \\
& + 1674q^{16} + 2404q^{17} + 3806q^{18} \\
& + 3918q^{19} + 5344q^{20} + 6996q^{21} \\
& + 8460q^{22} + 10148q^{23} + 12542q^{24} \\
& + 16162q^{25} + 17540q^{26} + 22008q^{27} \\
& + 27420q^{28} + 29472q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Psi_4 \oplus \Lambda_1}(q)$$

$$\begin{aligned}
& = 1 + 2q + 8q^3 + 20q^4 + 6q^5 + 38q^6 \\
& + 100q^7 + 60q^8 + 150q^9 + 332q^{10} \\
& + 288q^{11} + 514q^{12} + 872q^{13} \\
& + 932q^{14} + 1462q^{15} + 2060q^{16} \\
& + 2388q^{17} + 3414q^{18} + 4520q^{19} \\
& + 5262q^{20} + 6876q^{21} + 8992q^{22} \\
& + 10296q^{23} + 12798q^{24} + 16170q^{25} \\
& + 18216q^{26} + 22478q^{27} \\
& + 27012q^{28} + 30104q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_1 \oplus \Lambda_4}(q)$$

$$= 1 + 2q + 2q^2 + 4q^3 + 10q^4 + 24q^5$$

$$\begin{aligned}
& + 44q^6 + 64q^7 + 90q^8 + 152q^9 \\
& + 270q^{10} + 424q^{11} + 598q^{12} \\
& + 792q^{13} + 1076q^{14} + 1530q^{15} \\
& + 2130q^{16} + 2820q^{17} + 3590q^{18} \\
& + 4398q^{19} + 5352q^{20} + 6676q^{21} \\
& + 8476q^{22} + 10484q^{23} + 12718q^{24} \\
& + 14922q^{25} + 17732q^{26} + 21192q^{27} \\
& + 25820q^{28} + 30680q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Psi_1 \oplus \Lambda_4}(q)$$

$$\begin{aligned}
& = 1 + 2q + 2q^3 + 14q^4 + 24q^5 + 26q^6 \\
& + 40q^7 + 90q^8 + 166q^9 + 238q^{10} \\
& + 324q^{11} + 506q^{12} + 788q^{13} \\
& + 1076q^{14} + 1452q^{15} + 2048q^{16} \\
& + 2760q^{17} + 3516q^{18} + 4440q^{19} \\
& + 5682q^{20} + 7200q^{21} + 8820q^{22} \\
& + 10520q^{23} + 12822q^{24} + 15564q^{25} \\
& + 18280q^{26} + 214748q^{27} + 25824q^{28} \\
& + 30426q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_3 \oplus \Psi_2}(q)$$

$$\begin{aligned}
& = 1 + 2q + 6q^2 + 16q^3 + 22q^4 + 48q^5 \\
& + 76q^6 + 88q^7 + 178q^8 + 212q^9 \\
& + 278q^{10} + 520q^{11} + 554q^{12} \\
& + 856q^{13} + 1276q^{14} + 1382q^{15} \\
& + 2102q^{16} + 2596q^{17} + 3174q^{18} \\
& + 4274q^{19} + 5164q^{20} + 6812q^{21} \\
& + 8004q^{22} + 10220q^{23} + 12782q^{24} \\
& + 14118q^{25} + 18556q^{26} + 21516q^{27} \\
& + 24732q^{28} + 30996q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_3 \oplus \Lambda_2}(q)$$

$$= 1 + 2q + 6q^2 + 12q^3 + 18q^4 + 36q^5$$

$$\begin{aligned}
& + 56q^6 + 96q^7 + 162q^8 + 236q^9 \\
& + 342q^{10} + 464q^{11} + 626q^{12} \\
& + 848q^{13} + 1172q^{14} + 1586q^{15} \\
& + 2042q^{16} + 2612q^{17} + 3226q^{18} \\
& + 4002q^{19} + 5216q^{20} + 6668q^{21} \\
& + 8562q^{22} + 10580q^{23} + 12758q^{24} \\
& + 14714q^{25} + 17500q^{26} + 20356q^{27} \\
& + 24924q^{28} + 29824q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Phi_3 \oplus \Psi_1 \oplus \Lambda_1}(q) \\
= 1 + 2q + 6q^2 + 14q^3 + 20q^4 + 42q^5 \\
+ 64q^6 + 92q^7 + 168q^8 + 224q^9 \\
+ 332q^{10} + 492q^{11} + 610q^{12} \\
+ 852q^{13} + 1124q^{14} + 1472q^{15} \\
+ 1982q^{16} + 2616q^{17} + 3440q^{18} \\
+ 4246q^{19} + 5452q^{20} + 6632q^{21} \\
+ 7960q^{22} + 10028q^{23} + 12138q^{24} \\
+ 14756q^{25} + 18324q^{26} + 21716q^{27} \\
+ 25920q^{28} + 30282q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Phi_2 \oplus \Psi_3}(q) \\
= 1 + 2q + 4q^2 + 14q^3 + 18q^4 + 32q^5 \\
+ 74q^6 + 72q^7 + 138q^8 + 250q^9 \\
+ 222q^{10} + 468q^{11} + 670q^{12} \\
+ 652q^{13} + 1284q^{14} + 1528q^{15} \\
+ 1748q^{16} + 2840q^{17} + 3228q^{18} \\
+ 4116q^{19} + 5446q^{20} + 6704q^{21} \\
+ 8380q^{22} + 9816q^{23} + 13086q^{24} \\
+ 14960q^{25} + 17416q^{26} + 22982q^{27} \\
+ 25208q^{28} + 30014q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Psi_3 \oplus \Lambda_2}(q) \\
= 1 + 2q + 6q^3 + 18q^4 + 12q^5 + 30q^6 \\
+ 80q^7 + 90q^8 + 142q^9 + 262q^{10}
\end{aligned}$$

$$\begin{aligned}
& + 356q^{11} + 530q^{12} + 772q^{13} \\
& + 1028q^{14} + 1480q^{15} + 2028q^{16} \\
& + 2512q^{17} + 3360q^{18} + 4544q^{19} \\
& + 5462q^{20} + 6856q^{21} + 8852q^{22} \\
& + 10528q^{23} + 12934q^{24} + 15776q^{25} \\
& + 18376q^{26} + 22314q^{27} + 26656q^{28} \\
& + 30382q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_3 \oplus \Lambda_1}(q) \\
= 1 + 2q + 2q^2 + 10q^3 + 16q^4 + 18q^5 \\
+ 56q^6 + 76q^7 + 96q^8 + 220q^9 \\
+ 268q^{10} + 356q^{11} + 646q^{12} \\
+ 772q^{13} + 1036q^{14} + 1572q^{15} \\
+ 1946q^{16} + 2528q^{17} + 3432q^{18} \\
+ 4322q^{19} + 5312q^{20} + 6912q^{21} \\
+ 8520q^{22} + 10076q^{23} + 12938q^{24} \\
+ 15360q^{25} + 18036q^{26} + 22416q^{27} \\
+ 26024q^{28} + 30462q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Phi_2 \oplus \Lambda_3}(q) \\
= 1 + 2q + 4q^2 + 8q^3 + 12q^4 + 26q^5 \\
+ 50q^6 + 84q^7 + 132q^8 + 194q^9 \\
+ 284q^{10} + 424q^{11} + 638q^{12} \\
+ 896q^{13} + 1204q^{14} + 1566q^{15} \\
+ 1980q^{16} + 2564q^{17} + 3370q^{18} \\
+ 4340q^{19} + 5502q^{20} + 6724q^{21} \\
+ 8112q^{22} + 9832q^{23} + 12302q^{24} \\
+ 15090q^{25} + 18688q^{26} + 22234q^{27} \\
+ 26292q^{28} + 30072q^{29} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Psi_2 \oplus \Lambda_3}(q) \\
= 1 + 2q + 4q^3 + 16q^4 + 18q^5 \\
+ 26q^6 + 60q^7 + 100q^8 + 150q^9 \\
+ 228q^{10} + 352q^{11} + 546q^{12}
\end{aligned}$$

$$\begin{aligned}
& + 776q^{13} + 1028q^{14} + 1458q^{15} \\
& + 2072q^{16} + 2636q^{17} + 3378q^{18} \\
& + 4488q^{19} + 5666q^{20} + 7012q^{21} \\
& + 8656q^{22} + 10544q^{23} + 12982q^{24} \\
& + 15670q^{25} + 18360q^{26} + 21990q^{27} \\
& + 26468q^{28} + 30500q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_3}(q)$$

$$\begin{aligned}
& = 1 + 2q + 2q^2 + 6q^3 + 12q^4 + 22q^5 \\
& + 44q^6 + 68q^7 + 104q^8 + 172q^9 \\
& + 268q^{10} + 396q^{11} + 582q^{12} \\
& + 828q^{13} + 1124q^{14} + 1528q^{15} \\
& + 2022q^{16} + 2640q^{17} + 3484q^{18} \\
& + 4394q^{19} + 5452q^{20} + 6832q^{21} \\
& + 8432q^{22} + 10228q^{23} + 12570q^{24} \\
& + 15236q^{25} + 18324q^{26} + 21976q^{27} \\
& + 26072q^{28} + 30426q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_2 \oplus \Psi_2 \oplus \Lambda_1}(q)$$

$$\begin{aligned}
& = 1 + 2q + 4q^2 + 12q^3 + 16q^4 + 30q^5 \\
& + 62q^6 + 76q^7 + 140q^8 + 226q^9 \\
& + 276q^{10} + 456q^{11} + 622q^{12} \\
& + 784q^{13} + 1156q^{14} + 1506q^{15} \\
& + 1952q^{16} + 2604q^{17} + 3406q^{18} \\
& + 4244q^{19} + 5322q^{20} + 6860q^{21} \\
& + 8160q^{22} + 10112q^{23} + 12662q^{24} \\
& + 14838q^{25} + 18272q^{26} + 21938q^{27} \\
& + 25732q^{28} + 30492q^{29} + \dots,
\end{aligned}$$

$$\Theta_{F_1 \oplus \Phi_2 \oplus \Psi_1 \oplus \Lambda_2}(q)$$

$$\begin{aligned}
& = 1 + 2q + 4q^2 + 10q^3 + 14q^4 + 28q^5 \\
& + 54q^6 + 80q^7 + 138q^8 + 210q^9
\end{aligned}$$

$$\begin{aligned}
& + 294q^{10} + 436q^{11} + 614q^{12} \\
& + 844q^{13} + 1156q^{14} + 1564q^{15} \\
& + 2008q^{16} + 2576q^{17} + 3352q^{18} \\
& + 4172q^{19} + 5346q^{20} + 6776q^{21} \\
& + 8348q^{22} + 10336q^{23} + 12622q^{24} \\
& + 14988q^{25} + 18040q^{26} + 21518q^{27} \\
& + 25544q^{28} + 30234q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Psi_1 \oplus \Phi_5}(q)$$

$$\begin{aligned}
& = 1 + 10q^2 + 2q^3 + 40q^4 + 20q^5 \\
& + 82q^6 + 80q^7 + 112q^8 + 170q^9 \\
& + 224q^{10} + 260q^{11} + 612q^{12} \\
& + 484q^{13} + 1240q^{14} + 1010q^{15} \\
& + 1906q^{16} + 1780q^{17} + 2774q^{18} \\
& + 2990q^{19} + 4646q^{20} + 4940q^{21} \\
& + 8304q^{22} + 7620q^{23} + 12672q^{24} \\
& + 11742q^{25} + 16924q^{26} + 17762q^{27} \\
& + 24220q^{28} + 26382q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_5 \oplus \Lambda_1}(q)$$

$$\begin{aligned}
& = 1 + 10q^2 + 42q^4 + 2q^5 + 102q^6 + 20q^7 \\
& + 190q^8 + 90q^9 + 362q^{10} + 240q^{11} \\
& + 672q^{12} + 440q^{13} + 1104q^{14} + 756q^{15} \\
& + 1906q^{16} + 1640q^{17} + 3452q^{18} \\
& + 3310q^{19} + 5302q^{20} + 4920q^{21} \\
& + 7164q^{22} + 6860q^{23} + 10952q^{24} \\
& + 12044q^{25} + 18364q^{26} + 20062q^{27} \\
& + 26624q^{28} + 27020q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Psi_2 \oplus \Phi_4}(q)$$

$$\begin{aligned}
& = 1 + 8q^2 + 4q^3 + 24q^4 + 32q^5 + 40q^6 \\
& + 96q^7 + 92q^8 + 144q^9 + 284q^{10}
\end{aligned}$$

$$\begin{aligned}
& + 216q^{11} + 576q^{12} + 584q^{13} \\
& + 888q^{14} + 1264q^{15} + 1512q^{16} \\
& + 2048q^{17} + 2720q^{18} + 3168q^{19} \\
& + 4808q^{20} + 4896q^{21} + 7904q^{22} \\
& + 8152q^{23} + 11200q^{24} + 13300q^{25} \\
& + 16368q^{26} + 18620q^{27} + 25272q^{28} \\
& + 26420q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_4 \oplus \Lambda_2}(q)$$

$$\begin{aligned}
& = 1 + 8q^2 + 28q^4 + 4q^5 + 68q^6 + 32q^7 \\
& + 156q^8 + 112q^9 + 324q^{10} + 248q^{11} \\
& + 576q^{12} + 480q^{13} + 1040q^{14} + 972q^{15} \\
& + 1892q^{16} + 1800q^{17} + 3004q^{18} \\
& + 2944q^{19} + 4684q^{20} + 5040q^{21} \\
& + 7904q^{22} + 8632q^{23} + 12136q^{24} \\
& + 12536q^{25} + 16520q^{26} + 17308q^{27} \\
& + 23824q^{28} + 27000q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_4 \oplus \Psi_1 \oplus \Lambda_1}(q)$$

$$\begin{aligned}
& = 1 + 8q^2 + 2q^3 + 26q^4 + 18q^5 + 52q^6 \\
& + 68q^7 + 110q^8 + 156q^9 + 270q^{10} \\
& + 292q^{11} + 548q^{12} + 532q^{13} \\
& + 968q^{14} + 978q^{15} + 1732q^{16} \\
& + 1876q^{17} + 2990q^{18} + 3324q^{19} \\
& + 4852q^{20} + 5052q^{21} + 7564q^{22} \\
& + 7704q^{23} + 11492q^{24} + 12426q^{25} \\
& + 17488q^{26} + 18940q^{27} + 25428q^{28} \\
& + 27194q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_3 \oplus \Psi_3}(q)$$

$$\begin{aligned}
& = 1 + 6q^2 + 6q^3 + 12q^4 + 36q^5 + 26q^6 \\
& + 72q^7 + 120q^8 + 86q^9 + 288q^{10} \\
& + 276q^{11} + 384q^{12} + 756q^{13} \\
& + 720q^{14} + 1174q^{15} + 1662q^{16} \\
& + 1812q^{17} + 2870q^{18} + 3186q^{19}
\end{aligned}$$

$$\begin{aligned}
& + 4674q^{20} + 5364q^{21} + 7032q^{22} \\
& + 9276q^{23} + 10376q^{24} + 13458q^{25} \\
& + 17364q^{26} + 18230q^{27} + 25356q^{28} \\
& + 28098q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_3 \oplus \Lambda_3}(q)$$

$$\begin{aligned}
& = 1 + 6q^2 + 18q^4 + 6q^5 + 50q^6 + 36q^7 \\
& + 126q^8 + 102q^9 + 258q^{10} + 240q^{11} \\
& + 524q^{12} + 552q^{13} + 1032q^{14} \\
& + 1052q^{15} + 1722q^{16} + 1776q^{17} \\
& + 2832q^{18} + 3186q^{19} + 4902q^{20} \\
& + 5416q^{21} + 7572q^{22} + 8124q^{23} \\
& + 11160q^{24} + 12636q^{25} + 17604q^{26} \\
& + 19874q^{27} + 25728q^{28} + 27612q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_3 \oplus \Psi_2 \oplus \Lambda_1}(q)$$

$$\begin{aligned}
& = 1 + 6q^2 + 4q^3 + 14q^4 + 26q^5 + 30q^6 \\
& + 68q^7 + 102q^8 + 126q^9 + 258q^{10} \\
& + 280q^{11} + 468q^{12} + 600q^{13} \\
& + 856q^{14} + 1104q^{15} + 1574q^{16} \\
& + 2016q^{17} + 2732q^{18} + 3362q^{19} \\
& + 4706q^{20} + 5232q^{21} + 7420q^{22} \\
& + 8476q^{23} + 11136q^{24} + 13072q^{25} \\
& + 17100q^{26} + 19154q^{27} + 24984q^{28} \\
& + 27784q^{29} + \dots,
\end{aligned}$$

$$\Theta_{\Phi_3 \oplus \Psi_1 \oplus \Lambda_2}(q)$$

$$\begin{aligned}
& = 1 + 6q^2 + 2q^3 + 16q^4 + 16q^5 + 38q^6 \\
& + 56q^7 + 104q^8 + 134q^9 + 240q^{10} \\
& + 284q^{11} + 480q^{12} + 564q^{13} \\
& + 944q^{14} + 1066q^{15} + 1690q^{16} \\
& + 1908q^{17} + 2794q^{18} + 3194q^{19} \\
& + 4686q^{20} + 5276q^{21} + 7632q^{22} \\
& + 8524q^{23} + 11608q^{24} + 12798q^{25} \\
& + 17036q^{26} + 18710q^{27} + 24676q^{28}
\end{aligned}$$

$$\begin{aligned}
& + 27622q^{29} + \dots, \\
\Theta_{\Phi_2 \oplus \Psi_4}(q) &= 1 + 4q^2 + 8q^3 + 4q^4 + 32q^5 + 32q^6 \\
& + 32q^7 + 140q^8 + 84q^9 + 180q^{10} \\
& + 408q^{11} + 252q^{12} + 656q^{13} \\
& + 952q^{14} + 820q^{15} + 1748q^{16} \\
& + 1968q^{17} + 2480q^{18} + 3548q^{19} \\
& + 4356q^{20} + 5776q^{21} + 6696q^{22} \\
& + 9240q^{23} + 11408q^{24} + 12272q^{25} \\
& + 18128q^{26} + 19632q^{27} + 23424q^{28} \\
& + 30192q^{29} + \dots, \\
\Theta_{\Phi_2 \oplus \Lambda_4}(q) &= 1 + 4q^2 + 12q^4 + 8q^5 + 40q^6 \\
& + 32q^7 + 92q^8 + 84q^9 + 212q^{10} \\
& + 248q^{11} + 492q^{12} + 544q^{13} \\
& + 904q^{14} + 1004q^{15} + 1644q^{16} \\
& + 1984q^{17} + 3016q^{18} + 3420q^{19} \\
& + 4684q^{20} + 5168q^{21} + 7208q^{22} \\
& + 8568q^{23} + 11872q^{24} + 13496q^{25} \\
& + 17216q^{26} + 18832q^{27} + 24176q^{28} \\
& + 27960q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_5}(q) &= 1 + 2q^2 + 10q^3 + 20q^5 + 50q^6 + 112q^8 \\
& + 162q^9 + 32q^{10} + 420q^{11} + 412q^{12} \\
& + 260q^{13} + 1160q^{14} + 954q^{15} + 1130q^{16} \\
& + 2484q^{17} + 2342q^{18} + 3222q^{19} \\
& + 4670q^{20} + 5660q^{21} + 7040q^{22} \\
& + 8564q^{23} + 12256q^{24} + 12934q^{25} \\
& + 16300q^{26} + 22314q^{27} + 23100q^{28} \\
& + 29110q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Lambda_5}(q) &= 1 + 2q^2 + 10q^4 + 10q^5 + 30q^6 + 20q^7 \\
& + 62q^8 + 82q^9 + 202q^{10} + 240q^{11} \\
& + 392q^{12} + 440q^{13} + 800q^{14} \\
& + 1092q^{15} + 1730q^{16} + 2040q^{17} \\
& + 2772q^{18} + 3222q^{19} + 4662q^{20} \\
& + 5880q^{21} + 8044q^{22} + 9100q^{23} \\
& + 11192q^{24} + 12492q^{25} + 16620q^{26} \\
& + 20134q^{27} + 25984q^{28} + 29004q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_4 \oplus \Lambda_1}(q) &= 1 + 2q^2 + 8q^3 + 2q^4 + 18q^5 + 38q^6 \\
& + 20q^7 + 94q^8 + 130q^9 + 122q^{10} \\
& + 320q^{11} + 392q^{12} + 472q^{13} \\
& + 864q^{14} + 1068q^{15} + 1402q^{16} \\
& + 2008q^{17} + 2700q^{18} + 3286q^{19} \\
& + 4462q^{20} + 5864q^{21} + 6908q^{22} \\
& + 9068q^{23} + 11560q^{24} + 13204q^{25} \\
& + 17276q^{26} + 20486q^{27} + 24368q^{28} \\
& + 29380q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_1 \oplus \Lambda_4}(q) &= 1 + 2q^2 + 2q^3 + 8q^4 + 12q^5 + 26q^6 \\
& + 32q^7 + 64q^8 + 98q^9 + 176q^{10} \\
& + 244q^{11} + 396q^{12} + 516q^{13} \\
& + 808q^{14} + 1058q^{15} + 1570q^{16} \\
& + 2004q^{17} + 2830q^{18} + 3398q^{19} \\
& + 4630q^{20} + 5612q^{21} + 7504q^{22} \\
& + 8916q^{23} + 11584q^{24} + 13406q^{25} \\
& + 17084q^{26} + 19786q^{27} + 24844q^{28} \\
& + 28638q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_3 \oplus \Lambda_2}(q) &= 1 + 2q^2 + 6q^3 + 4q^4 + 16q^5 + 30q^6 \\
& + 32q^7 + 80q^8 + 114q^9 + 160q^{10} \\
& + 268q^{11} + 396q^{12} + 532q^{13} \\
& + 792q^{14} + 1102q^{15} + 1462q^{16}
\end{aligned}$$

$$\begin{aligned}
& + 1972q^{17} + 2714q^{18} + 3326q^{19} \\
& + 4554q^{20} + 5716q^{21} + 7160q^{22} \\
& + 9044q^{23} + 11440q^{24} + 13506q^{25} \\
& + 17108q^{26} + 20314q^{27} + 24644q^{28} \\
& + 29002q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_2 \oplus \Lambda_3}(q) \\
& = 1 + 2q^2 + 4q^3 + 6q^4 + 14q^5 + 26q^6 \\
& + 36q^7 + 70q^8 + 106q^9 + 170q^{10} \\
& + 248q^{11} + 400q^{12} + 536q^{13} \\
& + 800q^{14} + 1080q^{15} + 1502q^{16} \\
& + 1992q^{17} + 2736q^{18} + 3398q^{19} \\
& + 4594q^{20} + 5680q^{21} + 7316q^{22} \\
& + 8892q^{23} + 11488q^{24} + 13520q^{25} \\
& + 17188q^{26} + 20134q^{27} + 24904q^{28} \\
& + 28744q^{29} + \dots, \\
\Theta_{\Psi_5 \oplus \Lambda_1}(q) \\
& = 1 + 10q^3 + 2q^4 + 2q^5 + 52q^6 + 20q^7 \\
& + 30q^8 + 180q^9 + 110q^{10} + 180q^{11} \\
& + 492q^{12} + 420q^{13} + 680q^{14} \\
& + 1194q^{15} + 1292q^{16} + 1860q^{17} \\
& + 2782q^{18} + 3284q^{19} + 4236q^{20} \\
& + 6044q^{21} + 7180q^{22} + 8680q^{23} \\
& + 12052q^{24} + 13730q^{25} + 16800q^{26} \\
& + 21636q^{27} + 24500q^{28} + 29570q^{29} + \dots, \\
\Theta_{\Psi_4 \oplus \Lambda_2}(q) \\
& = 1 + 8q^3 + 4q^4 + 4q^5 + 36q^6 + 32q^7 \\
& + 44q^8 + 120q^9 + 148q^{10} + 216q^{11} \\
& + 376q^{12} + 496q^{13} + 720q^{14} \\
& + 1076q^{15} + 1404q^{16} + 1864q^{17} \\
& + 2732q^{18} + 3384q^{19} + 4324q^{20} \\
& + 5936q^{21} + 7280q^{22} + 8984q^{23}
\end{aligned}$$

$$\begin{aligned}
& + 11736q^{24} + 13872q^{25} + 17128q^{26} \\
& + 21092q^{27} + 24928q^{28} + 29296q^{29} + \dots, \\
\Theta_{\Psi_3 \oplus \Lambda_3}(q) \\
& = 1 + 6q^3 + 6q^4 + 6q^5 + 24q^6 + 36q^7 \\
& + 54q^8 + 92q^9 + 150q^{10} + 228q^{11} \\
& + 356q^{12} + 492q^{13} + 720q^{14} \\
& + 1070q^{15} + 1440q^{16} + 1884q^{17} \\
& + 2698q^{18} + 3468q^{19} + 4488q^{20} \\
& + 5788q^{21} + 7404q^{22} + 9144q^{23} \\
& + 11652q^{24} + 13926q^{25} + 17208q^{26} \\
& + 20884q^{27} + 25116q^{28} + 29142q^{29} + \dots, \\
\Theta_{\Psi_2 \oplus \Lambda_4}(q) \\
& = 1 + 4q^3 + 8q^4 + 18q^5 + 16q^6 + 32q^7 \\
& + 60q^8 + 88q^9 + 140q^{10} + 216q^{11} \\
& + 360q^{12} + 488q^{13} + 712q^{14} \\
& + 1040q^{15} + 1496q^{16} + 1936q^{17} \\
& + 2664q^{18} + 3480q^{19} + 4648q^{20} \\
& + 5824q^{21} + 7472q^{22} + 9176q^{23} \\
& + 11776q^{24} + 13940q^{25} + 17216q^{26} \\
& + 20516q^{27} + 25208q^{28} + 28980q^{29} + \dots, \\
\Theta_{\Psi_1 \oplus \Lambda_5}(q) \\
& = 1 + 2q^3 + 10q^4 + 10q^5 + 12q^6 + 20q^7 \\
& + 62q^8 + 100q^9 + 142q^{10} + 180q^{11} \\
& + 332q^{12} + 500q^{13} + 760q^{14} \\
& + 1010q^{15} + 1492q^{16} + 1940q^{17} \\
& + 2710q^{18} + 3460q^{19} + 4804q^{20} \\
& + 5980q^{21} + 7660q^{22} + 9064q^{23} \\
& + 11732q^{24} + 13994q^{25} + 17424q^{26} \\
& + 20292q^{27} + 25028q^{28}
\end{aligned}$$

$$\begin{aligned}
& + 28682q^{29} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_4 \oplus \Lambda_1}(q) \\
& = 1 + 2q^2 + 8q^3 + 2q^4 + 18q^5 + 38q^6 \\
& + 20q^7 + 94q^8 + 130q^9 + 122q^{10} \\
& + 320q^{11} + 392q^{12} + 472q^{13} \\
& + 864q^{14} + 1068q^{15} + 1402q^{16} \\
& + 2008q^{17} + 2700q^{18} + 3286q^{19} \\
& + 4462q^{20} + 5864q^{21} + 6908q^{22} \\
& + 9068q^{23} + 11560q^{24} + 13204q^{25} \\
& + 17276q^{26} + 20486q^{27} + 24368q^{28} \\
& + 29380q^{29} + \dots.
\end{aligned} \tag{33}$$

Now, we will easily determine some bases of $S_4(\Gamma_0(71))$, $S_6(\Gamma_0(71))$.

Theorem 4. *The following generalized 17 theta series*

$$\begin{aligned}
\Theta_{F_2, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{F_2=n} \left(x_1^2 - \frac{18}{71} F_2 \right) q^n, \\
\Theta_{\Phi_2, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_1^2 - \frac{9}{142} \Phi_2 \right) q^n, \\
\Theta_{\Phi_2, \varphi_{12}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_1 x_2 - \frac{1}{142} \Phi_2 \right) q^n, \\
\Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_1^2 - \frac{18}{71} F_1 \oplus \Phi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) \\
&= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} \left(x_1 x_2 + \frac{1}{142} F_1 \oplus \Phi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_3^2 - \frac{9}{71} F_1 \oplus \Phi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\
&= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} \left(x_3 x_4 - \frac{1}{142} F_1 \oplus \Phi_1 \right) q^n,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Psi_2, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{\Psi_2=n} \left(x_1^2 - \frac{6}{71} \Psi_2 \right) q^n, \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q) \\
&= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} \left(x_1^2 - \frac{18}{71} F_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) &= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} \left(x_1 x_2 + \frac{1}{142} F_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) &= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} \left(x_3^2 - \frac{6}{71} F_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{44}}(q) &= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} \left(x_4^2 - \frac{3}{71} F_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{\Psi_2=n} \left(x_1^2 - \frac{9}{71} \Phi_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) &= \sum_{n=1}^{\infty} \sum_{\Psi_2=n} \left(x_3 x_4 - \frac{1}{142} \Phi_1 \oplus \Psi_1 \right) q^n, \\
\Theta_{\Lambda_2, \varphi_{22}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Lambda_2=n} (71x_2^2 - 4\Lambda_2) q^n, \\
\Theta_{F_1 \oplus \Lambda_1, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Lambda_1=n} \left(x_1^2 - \frac{18}{71} (F_1 \oplus \Lambda_1) \right) q^n, \\
\Theta_{\Psi_1 \oplus \Lambda_1, \varphi_{34}}(q) \\
&= \sum_{n=1}^{\infty} \sum_{\Psi_1 \oplus \Lambda_1=n} \left(x_3 x_4 - \frac{3}{142} (\Psi_1 \oplus \Lambda_1) \right) q^n,
\end{aligned} \tag{34}$$

are a basis of $S_4(\Gamma_0(71))$, and the following generalized theta series

$$\begin{aligned}
\Theta_{F_4, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{F_4=n} \left(x_1^2 - \frac{9}{71} F_4 \right) q^n, \\
\Theta_{\Phi_4, \varphi_{11}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_4=n} \left(x_1^2 - \frac{9}{142} \Phi_4 \right) q^n, \\
\Theta_{\Phi_4, \varphi_{22}}(q) &= \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_2^2 - \frac{1}{71} \Phi_4 \right) q^n, \\
\Theta_{F_3 \oplus \Phi_1, \varphi_{11}}(q) \\
&= \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Phi_1 \right) q^n, \\
\Theta_{F_3 \oplus \Phi_1, \varphi_{22}}(q) &= \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_2^2 - \frac{1}{142} F_3 \oplus \Phi_1 \right) q^n,
\end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_1, \varphi_{77}}(q) = \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1 = n} \left(x_7^2 - \frac{9}{142} F_3 \oplus \Phi_1 \right) q^n,$$

$$\Theta_{F_3 \oplus \Phi_1, \varphi_{88}}(q) = \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1 = n} \left(x_8^2 - \frac{1}{71} F_3 \oplus \Phi_1 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_2, \varphi_{11}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_4 = n} \left(x_1^2 - \frac{9}{71} F_2 \oplus \Phi_2 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_2, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_4 = n} \left(x_2^2 - \frac{1}{142} F_2 \oplus \Phi_2 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_2, \varphi_{55}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_4 = n} \left(x_5^2 - \frac{9}{142} F_2 \oplus \Phi_2 \right) q^n,$$

$$\Theta_{\Psi_4, \varphi_{11}}(q) = \sum_{n=1}^{\infty} \sum_{\Psi_4 = n} \left(x_1^2 - \frac{3}{71} \Psi_4 \right) q^n,$$

$$\Theta_{\Psi_4, \varphi_{12}}(q) = \sum_{n=1}^{\infty} \sum_{\Psi_4 = n} \left(x_1 x_2 - \frac{1}{284} \Psi_4 \right) q^n,$$

$$\Theta_{\Psi_4, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Psi_4 = n} \left(x_2^2 - \frac{3}{142} \Psi_4 \right) q^n,$$

$$\Theta_{F_3 \oplus \Psi_1, \varphi_{11}}(q) = \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Psi_1 = n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Psi_1 \right) q^n,$$

$$\Theta_{F_2 \oplus \Psi_2, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Psi_2 = n} \left(x_2^2 - \frac{1}{142} F_2 \oplus \Psi_2 \right) q^n,$$

$$\Theta_{F_2 \oplus \Psi_2, \varphi_{66}}(q) = \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Psi_2 = n} \left(x_6^2 - \frac{3}{142} F_2 \oplus \Psi_2 \right) q^n,$$

$$\Theta_{\Phi_3 \oplus \Psi_1, \varphi_{11}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_3 \oplus \Psi_1 = n} \left(x_1^2 - \frac{9}{71} \Phi_3 \oplus \Psi_1 \right) q^n,$$

$$\Theta_{\Phi_3 \oplus \Psi_1, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_3 \oplus \Psi_1 = n} \left(x_2^2 - \frac{1}{71} \Phi_3 \oplus \Psi_1 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Psi_3, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_3 = n} \left(x_2^2 - \frac{1}{71} \Phi_1 \oplus \Psi_3 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Psi_3, \varphi_{44}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_3 = n} \left(x_4^2 - \frac{3}{142} \Phi_1 \oplus \Psi_3 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Psi_3, \varphi_{44}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_3 = n} \left(x_4^2 - \frac{3}{142} \Phi_1 \oplus \Psi_3 \right) q^n,$$

$$\Theta_{F_3 \oplus \Lambda_1, \varphi_{11}}(q) = \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Lambda_1 = n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Lambda_1 \right) q^n,$$

$$\Theta_{F_3 \oplus \Lambda_1, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Lambda_1 = n} \left(x_2^2 - \frac{1}{142} F_3 \oplus \Lambda_1 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{12}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3 = n} \left(x_1 x_2 - \frac{1}{284} \Phi_1 \oplus \Lambda_3 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3 = n} \left(x_2^2 - \frac{1}{71} \Phi_1 \oplus \Lambda_3 \right) q^n,$$

$$\Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{44}}(q) = \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3 = n} \left(x_4^2 - \frac{2}{71} \Phi_1 \oplus \Lambda_3 \right) q^n,$$

$$\Theta_{\Psi_3 \oplus \Lambda_1, \varphi_{22}}(q) = \sum_{n=1}^{\infty} \sum_{\Psi_3 \oplus \Lambda_1 = n} \left(x_2^2 - \frac{3}{142} \Psi_3 \oplus \Lambda_1 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1, \varphi_{55}}(q)$$

$$= \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1 = n} \left(x_5^2 - \frac{9}{142} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1, \varphi_{56}}(q)$$

$$= \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1 = n} \left(x_5 x_6 - \frac{1}{284} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n,$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1, \varphi_{78}}(q)$$

$$= \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1 = n} \left(x_7 x_8 - \frac{1}{284} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n$$

(35)

are a basis of $S_6(\Gamma_0(71))$.

Proof.

$$F_2 = x_1^2 + x_1 x_2 + 18x_2^2 + x_3^2 + x_3 x_4 + 18x_4^2 = n \quad (36)$$

has the following solutions for

$$n = 1 \implies x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = 0;$$

$$x_1 = 0, x_2 = 0, x_3 = \pm 1, x_4 = 0,$$

$$n = 2 \implies x_1 = \pm 1, x_2 = 0, x_3 = \pm 1, x_4 = 0,$$

$$n = 4 \implies x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = 0;$$

$$x_1 = 0, x_2 = 0, x_3 = \pm 2, x_4 = 0,$$

$$n = 5 \implies x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 = 0;$$

$$x_1 = \pm 2, x_2 = 0, x_3 = \pm 1, x_4 = 0,$$

$$n = 8 \implies x_1 = \pm 2, x_2 = 0, x_3 = \pm 2, x_4 = 0,$$

$$n = 9 \implies x_1 = \pm 3, x_2 = 0, x_3 = 0, x_4 = 0;$$

$$x_1 = 0, x_2 = 0, x_3 = \pm 3, x_4 = 0,$$

$$n = 10 \implies x_1 = \pm 3, x_2 = 0, x_3 = \pm 1, x_4 = 0;$$

$$x_1 = \pm 1, x_2 = 0, x_3 = \pm 3, x_4 = 0,$$

$$\begin{aligned}
n = 13 &\implies x_1 = \pm 2, x_2 = 0, x_3 = \pm 3, x_4 = 0; \\
&\quad x_1 = \pm 3, x_2 = 0, x_3 = \pm 2, x_4 = 0, \\
n = 16 &\implies x_1 = \pm 4, x_2 = 0, x_3 = 0, x_4 = 0; \\
&\quad x_1 = 0, x_2 = 0, x_3 = \pm 4, x_4 = 0, \\
n = 17 &\implies x_1 = \pm 1, x_2 = 0, x_3 = \pm 4, x_4 = 0; \\
&\quad x_1 = \pm 4, x_2 = 0, x_3 = \pm 1, x_4 = 0,
\end{aligned} \tag{37}$$

and there is no solutions for $n = 3, 6, 7, 11, 12, 14, 15$. Therefore,

$$\begin{aligned}
\Theta_{F_2, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{F_2=n} (71x_1^2 - 9F_2) q^n \\
&= \frac{1}{71} \left((71 \cdot 2 - 9 \cdot 4) q + (71 \cdot 4 - 9 \cdot 2 \cdot 4) q^2 \right. \\
&\quad + (71 \cdot 4 \cdot 2 - 9 \cdot 4 \cdot 4) q^4 \\
&\quad + (71 \cdot 1 \cdot 4 + 71 \cdot 4 \cdot 4 - 9 \cdot 5 \cdot 8) q^5 \\
&\quad + (71 \cdot 4 \cdot 4 - 9 \cdot 8 \cdot 4) q^8 \\
&\quad + (71 \cdot 9 \cdot 2 - 9 \cdot 9 \cdot 4) q^9 \\
&\quad + (71 \cdot 9 \cdot 4 + 71 \cdot 1 \cdot 4 - 9 \cdot 10 \cdot 8) q^{10} \\
&\quad + (71 \cdot 4 \cdot 4 + 71 \cdot 9 \cdot 4 - 9 \cdot 13 \cdot 8) q^{13} \\
&\quad + (71 \cdot 16 \cdot 2 - 9 \cdot 16 \cdot 4) q^{16} \\
&\quad \left. + (71 \cdot 1 \cdot 4 + 71 \cdot 16 \cdot 4 - 9 \cdot 17 \cdot 8) q^{17} + \dots \right) \\
&= \frac{1}{71} \left(106q + 212q^2 + 424q^4 + 1060q^5 + 848q^8 \right. \\
&\quad + 954q^9 + 2120q^{10} + 2756q^{13} + 1696q^{16} \\
&\quad \left. + 3604q^{17} + \dots \right).
\end{aligned} \tag{38}$$

Similarly, we have found the remaining 16 linearly independent theta series as follows:

$$\begin{aligned}
&\Theta_{\Phi_2, \varphi_{11}}(q) \\
&= \frac{1}{2 \cdot 71} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (2 \cdot 71x_1^2 - 9\Phi_2) q^n \\
&= \frac{1}{71} \left(70q^2 + 140q^4 + 280q^8 - 324q^9 + 482q^{10} \right. \\
&\quad - 508q^{11} - 586q^{12} - 440q^{14} + 28q^{15} \\
&\quad \left. + 560q^{16} + 108q^{17} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{\Phi_2, \varphi_{12}}(q) \\
&= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (142x_1x_2 - \Phi_2) q^n \\
&= \frac{1}{71} \left(-4q^2 - 8q^4 - 16q^8 - 18q^9 + 82q^{10} \right. \\
&\quad - 44q^{11} + 70q^{12} - 340q^{14} \\
&\quad \left. + 254q^{15} - 32q^{16} + 432q^{17} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) \\
&= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (71x_1^2 - 9F_1 \oplus \Phi_1) q^n \\
&= \frac{1}{71} \left(106q - 72q^2 + 68q^3 + 424q^4 + 704q^6 \right. \\
&\quad - 288q^8 + 266q^9 - 796q^{10} + 1256q^{11} \\
&\quad - 160q^{12} - 452q^{13} + 128q^{14} - 540q^{15} \\
&\quad \left. + 812q^{16} + 1332q^{17} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) \\
&= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (142x_1x_2 + F_1 \oplus \Phi_1) q^n \\
&= \frac{1}{71} \left(q + 2q^2 + 6q^3 + 4q^4 + 12q^6 \right. \\
&\quad + 8q^8 + 36q^9 + 30q^{10} + 44q^{11} + 36q^{12} \\
&\quad \left. + 52q^{13} + 28q^{14} + 15q^{15} + 80q^{16} + 34q^{17} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) \\
&= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (71x_3^2 - 9F_1 \oplus \Phi_1) q^n \\
&= \frac{1}{71} \left(-18q + 106q^2 + 176q^3 - 72q^4 + 68q^6 \right. \\
&\quad + 424q^8 + 488q^9 - 398q^{10} - 224q^{11} \\
&\quad + 630q^{12} - 652q^{13} - 220q^{14} + 298q^{15} \\
&\quad \left. - 20q^{16} + 524q^{17} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\
&= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} (142x_3x_4 - F_1 \oplus \Phi_1) q^n \\
&= \frac{1}{71} \left(-q - 2q^2 - 6q^3 - 4q^4 - 12q^6 - 8q^8 \right.
\end{aligned}$$

$$\begin{aligned}
& -36q^9 + 112q^{10} + 240q^{11} - 178q^{12} \\
& -336q^{13} + 256q^{14} + 269q^{15} + 204q^{16} \\
& -34q^{17} + \dots), \\
\Theta_{\Psi_2, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (71x_1^2 - 6\Psi_2) q^n \\
&= \frac{1}{71} (70q^3 - 4q^6 - 50q^8 - 148q^9 - 98q^{10} \\
&\quad + 40q^{11} - 8q^{12} - 56q^{13} \\
&\quad - 388q^{14} + 700q^{15} - 400q^{16} + \dots), \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} (71x_1^2 - 18F_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (106q - 108q^3 + 420q^4 - 216q^6 + 412q^7 \\
&\quad - 288q^8 + 590q^9 + 56q^{10} - 508q^{11} + 1532q^{12} \\
&\quad - 652q^{13} + 128q^{14} + 1476q^{15} + 1104q^{16} \\
&\quad + 392q^{17} + \dots), \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) &= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} (142x_1x_2 + F_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (q + 3q^3 + 12q^4 + 6q^6 + 28q^7 + 8q^8 \\
&\quad + 27q^9 + 30q^{10} + 22q^{11} \\
&\quad + 60q^{12} + 26q^{13} + 28q^{14} + 30q^{15} \\
&\quad + 64q^{16} + 68q^{17} + \dots), \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} (71x_1x_2 - 6F_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (-12q + 106q^3 + 140q^4 - 72q^6 - 52q^7 \\
&\quad + 46q^8 - 40q^9 - 218q^{10} + 20q^{11} \\
&\quad + 416q^{12} + 824q^{13} - 52q^{14} - 360q^{15}
\end{aligned}$$

$$\begin{aligned}
& + 936q^{16} + 604q^{17} + \dots), \\
\Theta_{F_1 \oplus \Psi_1, \varphi_{44}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} (71x_4^2 - 3F_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (-6q - 18q^3 - 72q^4 + 106q^6 + 116q^7 \\
&\quad + 94q^8 + 122q^9 + 246q^{10} + 152q^{11} \\
&\quad - 76q^{12} - 156q^{13} + 116q^{14} + 104q^{15} \\
&\quad - 242q^{16} + 160q^{17} + \dots), \\
\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (71x_1^2 - 9\Phi_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (106q^2 - 54q^3 + 104q^5 - 108q^6 + 276q^8 \\
&\quad - 162q^9 - 294q^{10} + 740q^{11} - 870q^{12} \\
&\quad - 184q^{13} + 412q^{14} - 498q^{15} - 20q^{16} \\
&\quad - 612q^{17} + \dots), \\
\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) &= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (142x_3x_4 - \Phi_1 \oplus \Psi_1) q^n \\
&= \frac{1}{71} (-2q^2 - 3q^3 - 10q^5 - 6q^6 + 110q^8 \\
&\quad - 9q^9 + 102q^{10} - 22q^{11} - 356q^{12} \\
&\quad - 26q^{13} - 56q^{14} - 75q^{15} + 488q^{16} \\
&\quad + 250q^{17} + \dots), \\
\Theta_{\Lambda_2, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{\Lambda_2=n} (71x_2^2 - 4\Lambda_2) q^n \\
&= \frac{1}{71} (-64q^4 + 62q^5 + 46q^6 - 128q^8 \\
&\quad - 4q^9 + 88q^{10} + 216q^{11} + 42q^{12} \\
&\quad - 98q^{15} - 484q^{16} + 24q^{17} + \dots), \\
\Theta_{F_1 \oplus \Lambda_1, \varphi_{11}}(q) &= \frac{1}{71} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Lambda_1=n} (71x_1^2 - 18(F_1 \oplus \Lambda_1)) q^n
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{71} \left(134q + 504q^4 + 164q^5 + 140q^6 \right. \\
&\quad + 172q^7 + 1008q^8 + 2198q^9 + 976q^{10} \\
&\quad - 96q^{12} + 2424q^{13} + 2332q^{14} + 2196q^{15} \\
&\quad \left. + 2924q^{16} + 12q^{17} + \dots \right), \\
\Theta_{\Psi_1 \oplus \Lambda_1, \varphi_{34}}(q) \\
&= \frac{1}{142} \sum_{n=1}^{\infty} \sum_{\Psi_1 \oplus \Lambda_1 = n} \left(142x_1^2 - 3(\Psi_1 \oplus \Lambda_1) \right) q^n \\
&= \frac{1}{71} \left(-24q^3 - 32q^4 - 40q^5 + 46q^6 - 112q^7 \right. \\
&\quad - 192q^8 + 140q^9 - 240q^{10} - 176q^{11} \\
&\quad - 434q^{12} - 208q^{13} - 164q^{14} - 600q^{15} \\
&\quad \left. - 484q^{16} - 272q^{17} + \dots \right). \tag{39}
\end{aligned}$$

Since the 17th determinant of the coefficients of the theta series is

$$\frac{22370813204653670400000}{9095120158391}, \tag{40}$$

the first 17 generalized theta series in the theorem are a basis of $S_4(\Gamma_0(71))$. Similarly, the second 29 generalized theta series in the theorem are the following, and they are a basis of $S_6(\Gamma_0(71))$ since the 29th determinant of the coefficients is

$$-\frac{6041160948656511821438109137218074776490167579085845299200}{231122292121701565271}. \tag{41}$$

Therefore,

$$\begin{aligned}
\Theta_{F_4, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(70q + 420q^2 + 840q^3 + 840q^4 + 2100q^5 \right. \\
&\quad + 5040q^6 + 3920q^7 + 1680q^8 + 8190q^9 \\
&\quad + 12600q^{10} + 9240q^{11} + 10080q^{12} \\
&\quad + 12740q^{13} + 23520q^{14} + 25200q^{15} \\
&\quad + 3360q^{16} + 21420q^{17} + 46690q^{18} \\
&\quad + 12740q^{19} - 3010q^{20} + 18620q^{21} \\
&\quad \left. + 21420q^{22} - 17360q^{23} - 29750q^{24} \right.
\end{aligned}$$

$$\begin{aligned}
&\quad + 10150q^{25} + 12600q^{26} + 30380q^{27} \\
&\quad \left. + 2940q^{28} - 10080q^{29} + \dots \right), \\
\Theta_{\Phi_4, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(70q^2 + 420q^4 + 840q^6 + 840q^8 - 324q^9 \right. \\
&\quad + 1882q^{10} - 1524q^{11} + 3862q^{12} \\
&\quad - 2208q^{13} + 1664q^{14} - 884q^{15} \\
&\quad - 184q^{16} + 324q^{17} + 5978q^{18} \\
&\quad - 728q^{19} + 7464q^{20} - 11080q^{21} \\
&\quad + 1992q^{22} - 20568q^{23} - 3966q^{24} \\
&\quad + 140q^{25} - 5044q^{26} + 6288q^{27} \\
&\quad \left. + 14688q^{28} - 26740q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_4, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-16q^2 - 96q^4 - 192q^6 - 192q^8 + 70q^9 \right. \\
&\quad - 418q^{10} + 324q^{11} - 830q^{12} \\
&\quad + 456q^{13} - 356q^{14} + 198q^{15} \\
&\quad - 104q^{16} + 72q^{17} - 1480q^{18} \\
&\quad + 422q^{19} - 1560q^{20} + 2492q^{21} \\
&\quad - 504q^{22} + 4044q^{23} + 586q^{24} \\
&\quad - 32q^{25} + 1372q^{26} + 72q^{27} \\
&\quad \left. - 2700q^{28} + 6112q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_3 \oplus \Phi_1, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(88q + 316q^2 + 312q^3 + 624q^4 + 2176q^5 \right. \\
&\quad + 2600q^6 + 2656q^7 + 4624q^8 + 3110q^9 \\
&\quad + 3904q^{10} + 8120q^{11} + 5160q^{12} \\
&\quad + 9396q^{13} + 10520q^{14} + 6530q^{15} \\
&\quad + 18348q^{16} + 12240q^{17} + 14862q^{18} \\
&\quad + 29326q^{19} + 13910q^{20} + 12000q^{21} \\
&\quad + 13540q^{22} + 21844q^{23} + 3578q^{24} \\
&\quad + 2948q^{25} + 62160q^{26} + 21596q^{27} \\
&\quad \left. + 24068q^{28} + 44156q^{29} + \dots \right),
\end{aligned}$$

$$\Theta_{F_3 \oplus \Phi_1, \varphi_{22}}(q)$$

$$= \frac{1}{71} \left(-3q - 14q^2 - 30q^3 - 60q^4 - 100q^5 \right. \\ \left. - 108q^6 - 168q^7 - 248q^8 - 198q^9 \right. \\ \left. - 430q^{10} - 748q^{11} - 660q^{12} \right. \\ \left. - 1040q^{13} - 1372q^{14} - 1215q^{15} \right. \\ \left. - 1584q^{16} - 1734q^{17} - 2038q^{18} \right. \\ \left. - 2189q^{19} - 2012q^{20} - 1842q^{21} \right. \\ \left. - 652q^{22} - 1942q^{23} - 1308q^{24} \right. \\ \left. + 77q^{25} - 3836q^{26} - 1088q^{27} \right. \\ \left. - 856q^{28} - 4568q^{29} + \dots \right),$$

$$\Theta_{F_3 \oplus \Phi_1, \varphi_{77}}(q)$$

$$= \frac{1}{71} \left(-27q + 16q^2 + 582q^3 + 1164q^4 + 236q^5 \right. \\ \left. - 120q^6 + 1896q^7 + 1744q^8 + 1626q^9 \right. \\ \left. + 4792q^{10} + 2924q^{11} + 2722q^{12} \right. \\ \left. + 9668q^{13} + 4976q^{14} - 2415q^{15} \right. \\ \left. + 7044q^{16} + 11658q^{17} + 4520q^{18} \right. \\ \left. + 6427q^{19} + 12848q^{20} + 14946q^{21} \right. \\ \left. + 4072q^{22} + 2970q^{23} + 25574q^{24} \right. \\ \left. + 8361q^{25} + 5520q^{26} + 25424q^{27} \right. \\ \left. + 7632q^{28} + 26480q^{29} + \dots \right),$$

$$\Theta_{F_3 \oplus \Phi_1, \varphi_{88}}(q)$$

$$= \frac{1}{71} \left(-6q - 28q^2 - 60q^3 - 120q^4 - 200q^5 \right. \\ \left. - 216q^6 - 336q^7 - 496q^8 - 254q^9 \right. \\ \left. + 134q^{10} + 1060q^{11} + 1662q^{12} \right. \\ \left. + 760q^{13} + 3220q^{14} + 5664q^{15} \right. \\ \left. + 1944q^{16} + 3348q^{17} + 5864q^{18} \right. \\ \left. + 2012q^{19} + 4780q^{20} + 3416q^{21} \right. \\ \left. - 2440q^{22} + 3216q^{23} + 1218q^{24} \right. \\ \left. - 3822q^{25} + 564q^{26} - 6152q^{27} \right. \\ \left. - 4836q^{28} + 7336q^{29} + \dots \right),$$

$$\Theta_{F_2 \oplus \Phi_2, F_4, \varphi_{11}}(q)$$

$$= \frac{1}{71} \left(106q + 140q^2 + 136q^3 + 840q^4 + 908q^5 \right. \\ \left. + 1680q^6 + 3664q^7 + 1680q^8 + 2990q^9 \right. \\ \left. + 2968q^{10} - 760q^{11} + 5872q^{12} \right. \\ \left. + 2948q^{13} + 6096q^{14} + 11028q^{15} \right. \\ \left. + 1336q^{16} + 14948q^{17} + 9018q^{18} \right. \\ \left. + 11416q^{19} + 19926q^{20} + 4804q^{21} \right. \\ \left. + 11468q^{22} - 8616q^{23} + 17986q^{24} \right. \\ \left. + 20146q^{25} + 9704q^{26} + 69068q^{27} \right. \\ \left. + 29308q^{28} + 5752q^{29} + \dots \right),$$

$$\Theta_{F_2 \oplus \Phi_2, \varphi_{22}}(q)$$

$$= \frac{1}{71} \left(-2q - 8q^2 - 24q^3 - 48q^4 - 60q^5 \right. \\ \left. - 96q^6 - 112q^7 - 96q^8 - 252q^9 \right. \\ \left. - 340q^{10} - 484q^{11} - 936q^{12} \right. \\ \left. - 988q^{13} - 1176q^{14} - 1470q^{15} \right. \\ \left. - 1472q^{16} - 1836q^{17} - 1984q^{18} \right. \\ \left. - 3042q^{19} - 2940q^{20} - 2368q^{21} \right. \\ \left. - 4112q^{22} - 1804q^{23} - 2748q^{24} \right. \\ \left. - 5610q^{25} - 2648q^{26} - 5740q^{27} \right. \\ \left. - 4888q^{28} - 1416q^{29} + \dots \right),$$

$$\Theta_{F_2 \oplus \Phi_2, \varphi_{55}}(q)$$

$$= \frac{1}{71} \left(-18q + 70q^2 + 352q^3 + 420q^4 + 596q^5 \right. \\ \left. + 840q^6 + 128q^7 + 840q^8 + 2276q^9 \right. \\ \left. + 1342q^{10} + 2744q^{11} + 4214q^{12} \right. \\ \left. + 764q^{13} + 1344q^{14} + 4378q^{15} \right. \\ \left. - 184q^{16} + 3072q^{17} + 10686q^{18} \right. \\ \left. + 4714q^{19} + 7904q^{20} + 9928q^{21} \right. \\ \left. + 7864q^{22} + 3076q^{23} - 1302q^{24} \right. \\ \left. + 22498q^{25} + 4852q^{26} + 9684q^{27} \right. \\ \left. + 31552q^{28} - 7348q^{29} + \dots \right),$$

$$\begin{aligned}
& \Theta_{\Psi_4, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(130q + 236q^2 + 424q^3 + 1416q^4 + 1628q^5 \right. \\
&\quad + 2832q^6 + 5008q^7 + 2832q^8 + 6014q^9 \\
&\quad + 7048q^{10} + 5048q^{11} + 17104q^{12} \\
&\quad + 14804q^{13} + 20208q^{14} + 28668q^{15} \\
&\quad + 19000q^{16} + 36980q^{17} + 36234q^{18} \\
&\quad + 54736q^{19} + 72246q^{20} + 67300q^{21} \\
&\quad + 94892q^{22} + 67560q^{23} + 102082q^{24} \\
&\quad + 118296q^{25} + 116512q^{26} + 200020q^{27} \\
&\quad \left. + 190620q^{28} + 213692q^{29} \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Psi_4, \varphi_{12}}(q) \\
&= \frac{1}{71} \left(-q - 4q^2 - 12q^3 - 24q^4 - 30q^5 \right. \\
&\quad - 48q^6 - 56q^7 - 48q^8 - 126q^9 \\
&\quad - 170q^{10} - 242q^{11} - 468q^{12} \\
&\quad - 494q^{13} - 588q^{14} - 735q^{15} \\
&\quad - 736q^{16} - 918q^{17} - 1276q^{18} \\
&\quad - 2089q^{19} - 2890q^{20} - 4024q^{21} \\
&\quad - 4896q^{22} - 5446q^{23} - 5634q^{24} \\
&\quad - 5620q^{25} - 8088q^{26} - 9010q^{27} \\
&\quad \left. - 13012q^{28} - 18142q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Psi_4, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-6q - 24q^2 - 72q^3 - 144q^4 - 180q^5 \right. \\
&\quad - 288q^6 - 336q^7 - 288q^8 - 756q^9 \\
&\quad - 1020q^{10} - 1452q^{11} - 2808q^{12} \\
&\quad - 2964q^{13} - 3528q^{14} - 4410q^{15} \\
&\quad - 4416q^{16} - 5508q^{17} - 6520q^{18} \\
&\quad - 10262q^{19} - 11660q^{20} - 12784q^{21} \\
&\quad - 18016q^{22} - 14500q^{23} - 16764q^{24} \\
&\quad - 22360q^{25} - 21264q^{26} - 29068q^{27} \\
&\quad \left. - 34904q^{28} - 38420q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_3 \oplus \Psi_1, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(88q + 352q^2 + 298q^3 + 204q^4 + 1816q^5 \right. \\
&\quad + 2276q^6 - 92q^7 + 2896q^8 + 6040q^9 \\
&\quad + 2372q^{10} + 5616q^{11} + 7508q^{12} \\
&\quad + 8528q^{13} + 16328q^{14} + 5240q^{15} \\
&\quad + 13824q^{16} + 34624q^{17} + 10182q^{18} \\
&\quad + 14444q^{19} + 38886q^{20} + 27684q^{21} \\
&\quad + 7336q^{22} + 11636q^{23} + 40714q^{24} \\
&\quad + 478q^{25} + 3156q^{26} + 23902q^{27} \\
&\quad \left. - 29536q^{28} + 38042q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_2 \oplus \Psi_2, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-2q - 4q^2 - 6q^3 - 40q^4 - 60q^5 \right. \\
&\quad - 24q^6 - 168q^7 - 288q^8 - 126q^9 \\
&\quad - 460q^{10} - 748q^{11} - 528q^{12} \\
&\quad - 1144q^{13} - 1288q^{14} - 1260q^{15} \\
&\quad - 2336q^{16} - 2108q^{17} - 2128q^{18} \\
&\quad - 3004q^{19} - 3556q^{20} - 3000q^{21} \\
&\quad - 2088q^{22} - 4844q^{23} - 2572q^{24} \\
&\quad - 1416q^{25} - 5432q^{26} + 1238q^{27} \\
&\quad \left. - 1816q^{28} - 5598q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_2 \oplus \Psi_2, \varphi_{66}}(q) \\
&= \frac{1}{71} \left(-6q - 12q^2 - 18q^3 - 120q^4 - 180q^5 \right. \\
&\quad + 70q^6 + 64q^7 - 154q^8 + 474q^9 \\
&\quad + 1034q^{10} + 880q^{11} + 972q^{12} \\
&\quad + 1396q^{13} + 2384q^{14} + 2468q^{15} \\
&\quad + 1370q^{16} + 2764q^{17} + 4976q^{18} \\
&\quad + 2632q^{19} + 1118q^{20} + 3496q^{21} \\
&\quad + 836q^{22} + 520q^{23} - 48q^{24} \\
&\quad - 3680q^{25} + 5288q^{26} - 262q^{27} \\
&\quad \left. - 5164q^{28} + 7630q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_3 \oplus \Psi_1, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(106q + 212q^2 - 108q^3 + 416q^4 + 1476q^5 \right. \\
&\quad - 432q^6 + 384q^7 + 3904q^8 - 422q^9 \\
&\quad + 1376q^{10} + 5280q^{11} + 152q^{12} \\
&\quad + 8092q^{13} + 2944q^{14} + 1176q^{15} \\
&\quad + 21568q^{16} + 1532q^{17} + 5858q^{18} \\
&\quad + 30276q^{19} + 13382q^{20} + 12172q^{21} \\
&\quad + 20636q^{22} + 42880q^{23} + 7522q^{24} \\
&\quad + 11006q^{25} + 65240q^{26} - 17192q^{27} \\
&\quad \left. + 27804q^{28} + 44644q^{29} \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_3 \oplus \Psi_1, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-4q - 8q^2 - 12q^3 - 80q^4 - 120q^5 \right. \\
&\quad - 48q^6 - 336q^7 - 576q^8 - 252q^9 \\
&\quad - 920q^{10} - 1496q^{11} - 1056q^{12} \\
&\quad - 2288q^{13} - 2576q^{14} - 2520q^{15} \\
&\quad - 4672q^{16} - 4216q^{17} - 4540q^{18} \\
&\quad - 6576q^{19} - 7396q^{20} - 7704q^{21} \\
&\quad - 7016q^{22} - 10824q^{23} - 10540q^{24} \\
&\quad - 10216q^{25} - 14272q^{26} - 10588q^{27} \\
&\quad \left. - 17832q^{28} - 20852q^{29} \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_1 \oplus \Psi_3, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-4q^2 - 18q^3 - 60q^5 - 108q^6 - 352q^8 \right. \\
&\quad - 164q^9 - 58q^{10} - 1100q^{11} + 178q^{12} \\
&\quad - 240q^{13} - 2352q^{14} + 1360q^{15} \\
&\quad - 804q^{16} - 2888q^{17} + 2504q^{18} \\
&\quad - 504q^{19} - 2528q^{20} + 4292q^{21} \\
&\quad - 12q^{22} + 960q^{23} + 3354q^{24} \\
&\quad + 5154q^{25} + 1556q^{26} + 5066q^{27} \\
&\quad \left. + 6812q^{28} + 6342q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_1 \oplus \Psi_3, \varphi_{44}}(q) \\
&= \frac{1}{71} \left(-6q^2 - 27q^3 - 90q^5 - 20q^6 - 102q^8 \right. \\
&\quad + 109q^9 + 126q^{10} + 54q^{11} + 196q^{12} \\
&\quad + 66q^{13} + 732q^{14} - 729q^{15} \\
&\quad + 1350q^{16} + 354q^{17} - 362q^{18} \\
&\quad + 2013q^{19} + 42q^{20} - 946q^{21} \\
&\quad + 2112q^{22} - 2394q^{23} + 984q^{24} \\
&\quad - 3771q^{25} + 3612q^{26} - 5891q^{27} \\
&\quad \left. + 420q^{28} - 2415q^{29} \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_3 \oplus \Lambda_1, \varphi_{11}}(q) \\
&= \frac{1}{71} \left(88q + 352q^2 + 352q^3 + 280q^4 + 1414q^5 \right. \\
&\quad + 1480q^6 - 720q^7 + 1072q^8 + 5756q^9 \\
&\quad + 7432q^{10} + 6864q^{11} + 4776q^{12} \\
&\quad + 10952q^{13} + 20020q^{14} + 14282q^{15} \\
&\quad + 11356q^{16} + 20888q^{17} + 27410q^{18} \\
&\quad + 17852q^{19} + 9398q^{20} + 24364q^{21} \\
&\quad + 27004q^{22} + 13348q^{23} - 2814q^{24} \\
&\quad - 2492q^{25} + 15596q^{26} - 17286q^{27} \\
&\quad \left. - 37880q^{28} + 12452q^{29} \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_3 \oplus \Lambda_1, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-3q - 12q^2 - 12q^3 - 16q^4 - 95q^5 \right. \\
&\quad - 186q^6 - 182q^7 - 256q^8 - 477q^9 \\
&\quad - 660q^{10} - 660q^{11} - 492q^{12} \\
&\quad - 780q^{13} - 1428q^{14} - 1305q^{15} \\
&\quad - 1152q^{16} - 2074q^{17} - 2398q^{18} \\
&\quad - 1296q^{19} - 1600q^{20} - 2938q^{21} \\
&\quad - 2176q^{22} - 1420q^{23} - 748q^{24} \\
&\quad - 383q^{25} - 1800q^{26} + 1075q^{27} \\
&\quad \left. + 2976q^{28} - 1880q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{12}}(q) \\
&= \frac{1}{142} \left(-2q^2 - 12q^4 - 15q^5 - 54q^6 - 42q^7 \right. \\
&\quad - 104q^8 - 117q^9 - 26q^{10} - 396q^{11} \\
&\quad - 956q^{12} - 624q^{13} + 752q^{14} + 1252q^{15} \\
&\quad - 1776q^{16} - 3880q^{17} - 1464q^{18} \\
&\quad + 6787q^{19} + 6604q^{20} - 288q^{21} \\
&\quad - 8956q^{22} - 4090q^{23} + 6476q^{24} \\
&\quad + 17282q^{25} + 9308q^{26} - 8255q^{27} \\
&\quad \left. - 22264q^{28} - 11766q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-4q^2 - 24q^4 - 30q^5 - 108q^6 \right. \\
&\quad - 84q^7 - 208q^8 - 92q^9 - 478q^{10} \\
&\quad - 792q^{11} - 1202q^{12} - 396q^{13} \\
&\quad - 200q^{14} - 194q^{15} - 1848q^{16} \\
&\quad - 1796q^{17} - 372q^{18} + 3776q^{19} \\
&\quad + 3836q^{20} + 3968q^{21} - 20q^{22} \\
&\quad + 1192q^{23} + 2018q^{24} + 8152q^{25} \\
&\quad + 2428q^{26} + 1950q^{27} - 4484q^{28} \\
&\quad \left. + 324q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Phi_1 \oplus \Lambda_3, \varphi_{44}}(q) \\
&= \frac{1}{71} \left(-8q^2 - 48q^4 + 82q^5 - 74q^6 \right. \\
&\quad + 116q^7 - 132q^8 + 100q^9 \\
&\quad - 104q^{10} + 688q^{11} + 294q^{12} \\
&\quad + 628q^{13} - 116q^{14} + 606q^{15} \\
&\quad - 4q^{16} + 1520q^{17} + 108q^{18} \\
&\quad + 452q^{19} - 1416q^{20} - 3424q^{21} \\
&\quad - 4868q^{22} - 2160q^{23} - 224q^{24} \\
&\quad + 1536q^{25} + 2584q^{26} - 2206q^{27} \\
&\quad \left. - 4424q^{28} - 1056q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{\Psi_3 \oplus \Lambda_1, \varphi_{22}}(q) \\
&= \frac{1}{71} \left(-27q^3 - 12q^4 - 15q^5 - 38q^6 - 126q^7 \right. \\
&\quad - 74q^8 - 26q^9 - 204q^{10} - 138q^{11} \\
&\quad + 48q^{12} + 38q^{13} + 552q^{14} - 493q^{15} \\
&\quad + 1018q^{16} + 1002q^{17} - 294q^{18} \\
&\quad + 874q^{19} + 1654q^{20} - 286q^{21} \\
&\quad + 1816q^{22} + 48q^{23} + 308q^{24} - 561q^{25} \\
&\quad + 1552q^{26} - 6580q^{27} + 1716q^{28} \\
&\quad \left. - 3853q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_2 \oplus \Phi_1 \oplus \Lambda_1, \varphi_{55}}(q) \\
&= \frac{1}{71} \left(-18q + 88q^2 + 433q^3 + 208q^4 - 166q^5 \right. \\
&\quad + 1002q^6 + 760q^7 - 444q^8 + 2357q^9 \\
&\quad + 1994q^{10} - 354q^{11} + 4954q^{12} \\
&\quad + 3954q^{13} - 1972q^{14} + 5933q^{15} \\
&\quad + 8148q^{16} - 1626q^{17} + 5256q^{18} \\
&\quad + 11771q^{19} + 6740q^{20} + 5090q^{21} \\
&\quad + 4468q^{22} + 18450q^{23} + 5798q^{24} \\
&\quad + 3171q^{25} + 27768q^{26} - 3535q^{27} \\
&\quad \left. + 20296q^{28} + 21929q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_2 \oplus \Phi_1 \oplus \Lambda_1, \varphi_{56}}(q) \\
&= \frac{1}{142} \left(-2q - 6q^2 - 15q^3 - 40q^4 - 50q^5 \right. \\
&\quad - 78q^6 - 168q^7 - 144q^8 - 243q^9 \\
&\quad - 236q^{10} + 718q^{11} + 156q^{12} \\
&\quad - 1738q^{13} + 1264q^{14} + 3089q^{15} \\
&\quad - 2408q^{16} + 198q^{17} + 6406q^{18} \\
&\quad + 1939q^{19} + 1380q^{20} - 34q^{21} \\
&\quad + 2232q^{22} + 7730q^{23} - 8996q^{24} \\
&\quad + 1299q^{25} + 10564q^{26} - 8471q^{27} \\
&\quad \left. + 13552q^{28} - 2139q^{29} + \dots \right),
\end{aligned}$$

$$\begin{aligned}
& \Theta_{F_2 \oplus \Phi_1 \oplus \Lambda_1, \varphi_{78}}(q) \\
&= \frac{1}{71} \left(-2q - 6q^2 - 15q^3 - 40q^4 - 50q^5 \right. \\
&\quad - 78q^6 - 168q^7 + 140q^8 + 893q^9 \\
&\quad + 900q^{10} + 718q^{11} + 1008q^{12} \\
&\quad - 1170q^{13} - 2144q^{14} + 817q^{15} \\
&\quad - 1840q^{16} - 370q^{17} + 8678q^{18} \\
&\quad + 4211q^{19} + 1948q^{20} + 2238q^{21} \\
&\quad - 5152q^{22} - 4198q^{23} - 11552q^{24} \\
&\quad - 923q^{25} + 14076q^{26} + 2997q^{27} \\
&\quad \left. + 18432q^{28} + 6161q^{29} + \dots \right). \tag{42}
\end{aligned}$$

□

5. Representation Numbers of N

Proposition 5. *The differences between the following theta series of the quadratic forms (4) and the Eisenstein series (22) are the linear combinations of the generalized theta series in the preceding theorem. The coefficients are given in the table [12]. Similarly, the differences between the following theta series of the quadratic forms (5) and the Eisenstein series (23) are the linear combinations of the generalized theta series in the preceding theorem. The coefficients are given in the table [12].*

Proof. Let us see the situation in the following case:

$$\begin{aligned}
& \Theta_{F_4}(q) - E_{F_4}(q) \\
&= \frac{20048}{2521}q + \frac{59424}{2521}q^2 + \frac{77312}{2521}q^3 + \frac{51744}{2521}q^4 \\
&\quad + \frac{105888}{2521}q^5 + \frac{211776}{2521}q^6 + \frac{120064}{2521}q^7 \\
&\quad - \frac{9696}{2521}q^8 + \frac{171344}{2521}q^9 + \frac{226944}{2521}q^{10} \\
&\quad + \frac{82176}{2521}q^{11} - \frac{3264}{2521}q^{12} + \frac{18592}{2521}q^{13} \\
&\quad + \frac{112512}{2521}q^{14} + \frac{60672}{2521}q^{15} - \frac{501216}{2521}q^{16} \\
&\quad - \frac{226656}{2521}q^{17} + \dots \\
&= c_1 \Theta_{F_2, \varphi_{11}}(q) + c_2 \Theta_{\Phi_2, \varphi_{11}}(q) + c_3 \Theta_{\Phi_2, \varphi_{12}}(q) \\
&\quad + c_4 \Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) + c_5 \Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) \\
&\quad + c_6 \Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) + c_7 \Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\
&\quad + c_8 \Theta_{\Psi_2, \varphi_{11}}(q) + c_9 \Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q)
\end{aligned}$$

$$\begin{aligned}
& + c_{10} \Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) + c_{11} \Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) \\
& + c_{12} \Theta_{F_1 \oplus \Psi_1, \varphi_{44}}(q) + c_{13} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) \\
& + c_{14} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) + c_{15} \Theta_{\Lambda_2, \varphi_{22}}(q) \\
& + c_{16} \Theta_{F_1 \oplus \Lambda_1, \varphi_{11}}(q) + c_{17} \Theta_{\Phi_1 \oplus \Lambda_1, \varphi_{34}}(q). \tag{43}
\end{aligned}$$

By equating the coefficients of q^n in both sides for $n = 1, 2, 3, \dots, 17$, we get an equation in coefficients. We repeat the same procedure for the other cases. At the end, by solving 17 linear equations in 17 variables, we get the coefficients in table [12]. Here, the coefficients c_{16} and c_{17} are always zero. Similarly, we can do for the difference between the theta series of the quadratic forms (5) and the Eisenstein series (23). □

Corollary 6. *The representation numbers for the quadratic forms (4) (in these direct sums any form can be replaced by its inverse) are*

$$\begin{aligned}
& r(n; H) \\
&= \frac{120}{2521} \sigma_3^*(n) + c_1 \sum_{n=1}^{\infty} \sum_{F_2=n} \left(x_1^2 - \frac{18}{71} F_2 \right) q^n \\
&\quad + c_2 \sum_{\Phi_2=n} \left(x_1^2 - \frac{9}{71} n \right) + c_3 \sum_{\Phi_2=n} \left(x_1 x_2 - \frac{1}{142} n \right) \\
&\quad + c_4 \sum_{F_1 \oplus \Phi_1=n} \left(x_1^2 - \frac{18}{71} n \right) + c_5 \sum_{F_1 \oplus \Phi_1=n} \left(x_1 x_2 + \frac{1}{142} n \right) \\
&\quad + c_6 \sum_{F_1 \oplus \Phi_1=n} \left(x_3^2 - \frac{9}{71} n \right) + c_7 \sum_{F_1 \oplus \Phi_1=n} \left(x_3 x_4 - \frac{1}{142} n \right) \\
&\quad + c_8 \sum_{\Psi_2=n} \left(x_1^2 - \frac{6}{71} n \right) + c_9 \sum_{F_1 \oplus \Psi_1=n} \left(x_1^2 - \frac{18}{71} n \right) \\
&\quad + c_{10} \sum_{F_1 \oplus \Psi_1=n} \left(x_1 x_2 + \frac{1}{142} n \right) + c_{11} \sum_{F_1 \oplus \Psi_1=n} \left(x_3^2 - \frac{6}{71} n \right) \\
&\quad + c_{12} \sum_{F_1 \oplus \Psi_1=n} \left(x_3^2 - \frac{3}{71} n \right) + c_{13} \sum_{\Phi_1 \oplus \Psi_1=n} \left(x_1^2 - \frac{9}{71} n \right) \\
&\quad + c_{14} \sum_{\Phi_1 \oplus \Psi_1=n} \left(x_3 x_4 - \frac{1}{142} n \right) + c_{15} \sum_{\Lambda_2=n} \left(x_2^2 - \frac{4}{71} n \right). \tag{44}
\end{aligned}$$

Similarly, the representation numbers for the quadratic forms (5) are

$$\begin{aligned}
& r(n; H) \\
&= \frac{36}{25 \cdot 565} \sigma_5^*(n) + d_1 \sum_{n=1}^{\infty} \sum_{F_4=n} \left(x_1^2 - \frac{9}{71} F_4 \right) q^n \\
&\quad + d_2 \sum_{n=1}^{\infty} \sum_{\Phi_4=n} \left(x_1^2 - \frac{9}{142} \Phi_4 \right) q^n
\end{aligned}$$

$$\begin{aligned}
& + d_3 \sum_{n=1}^{\infty} \sum_{\Phi_2=n} \left(x_2^2 - \frac{1}{71} \Phi_4 \right) q^n \\
& + d_4 \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Phi_1 \right) q^n \\
& + d_5 \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_2^2 - \frac{1}{142} F_3 \oplus \Phi_1 \right) q^n \\
& + d_6 \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_7^2 - \frac{9}{142} F_3 \oplus \Phi_1 \right) q^n \\
& + d_7 \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Phi_1=n} \left(x_8^2 - \frac{1}{71} F_3 \oplus \Phi_1 \right) q^n \\
& + d_8 \sum_{n=1}^{\infty} \sum_{\Phi_4=n} \left(x_1^2 - \frac{9}{71} F_2 \oplus \Phi_2 \right) q^n \\
& + d_9 \sum_{n=1}^{\infty} \sum_{\Phi_4=n} \left(x_2^2 - \frac{1}{142} F_2 \oplus \Phi_2 \right) q^n \\
& + d_{10} \sum_{n=1}^{\infty} \sum_{\Phi_4=n} \left(x_5^2 - \frac{9}{142} F_2 \oplus \Phi_2 \right) q^n \\
& + d_{11} \sum_{n=1}^{\infty} \sum_{\Psi_4=n} \left(x_1^2 - \frac{3}{71} \Psi_4 \right) q^n \\
& + d_{12} \sum_{n=1}^{\infty} \sum_{\Psi_4=n} \left(x_1 x_2 - \frac{1}{284} \Psi_4 \right) q^n \\
& + d_{13} \sum_{n=1}^{\infty} \sum_{\Psi_4=n} \left(x_2^2 - \frac{3}{142} \Psi_4 \right) q^n \\
& + d_{14} \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Psi_1=n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Psi_1 \right) q^n \\
& + d_{15} \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Psi_2=n} \left(x_2^2 - \frac{1}{142} F_2 \oplus \Psi_2 \right) q^n \\
& + d_{16} \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Psi_2=n} \left(x_6^2 - \frac{3}{142} F_2 \oplus \Psi_2 \right) q^n \\
& + d_{17} \sum_{n=1}^{\infty} \sum_{\Phi_3 \oplus \Psi_1=n} \left(x_1^2 - \frac{9}{71} \Phi_3 \oplus \Psi_1 \right) q^n \\
& + d_{18} \sum_{n=1}^{\infty} \sum_{\Phi_3 \oplus \Psi_1=n} \left(x_2^2 - \frac{1}{71} \Phi_3 \oplus \Psi_1 \right) q^n \\
& + d_{19} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_3=n} \left(x_2^2 - \frac{1}{71} \Phi_1 \oplus \Psi_3 \right) q^n \\
& + d_{20} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_3=n} \left(x_4^2 - \frac{3}{142} \Phi_1 \oplus \Psi_3 \right) q^n \\
& + d_{21} \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Lambda_1=n} \left(x_1^2 - \frac{9}{71} F_3 \oplus \Lambda_1 \right) q^n \\
& + d_{22} \sum_{n=1}^{\infty} \sum_{F_3 \oplus \Lambda_1=n} \left(x_2^2 - \frac{1}{142} F_3 \oplus \Lambda_1 \right) q^n \\
& + d_{23} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3=n} \left(x_1 x_2 - \frac{1}{284} \Phi_1 \oplus \Lambda_3 \right) q^n \\
& + d_{24} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3=n} \left(x_2^2 - \frac{1}{71} \Phi_1 \oplus \Lambda_3 \right) q^n \\
& + d_{25} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Lambda_3=n} \left(x_4^2 - \frac{2}{71} \Phi_1 \oplus \Lambda_3 \right) q^n \\
& + d_{26} \sum_{n=1}^{\infty} \sum_{\Psi_3 \oplus \Lambda_1=n} \left(x_2^2 - \frac{3}{142} \Psi_3 \oplus \Lambda_1 \right) q^n \\
& + d_{27} \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1=n} \left(x_5^2 - \frac{9}{142} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n \\
& + d_{28} \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1=n} \left(x_5 x_6 - \frac{1}{284} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n \\
& + d_{29} \sum_{n=1}^{\infty} \sum_{F_2 \oplus \Phi_1 \oplus \Psi_1=n} \left(x_7 x_8 - \frac{1}{284} F_2 \oplus \Phi_1 \oplus \Psi_1 \right) q^n.
\end{aligned} \tag{45}$$

Proof. The coefficients are the same coefficients of the preceding theorem. It follows from the preceding theorem. $\square$

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