Research Article

Exact Solutions of the Kudryashov-Sinelshchikov Equation Using the Multiple G'/G -Expansion Method

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Exact traveling wave solutions of the Kudryashov-Sinelshchikov equation are studied by the G'/G -expansion method and its variants. The solutions obtained include the form of Jacobi elliptic functions, hyperbolic functions, and trigonometric and rational functions. Many new exact traveling wave solutions can easily be derived from the general results under certain conditions. These methods are effective, simple, and many types of solutions can be obtained at the same time.

1. Introduction

The investigation of the traveling wave solutions to nonlinear evolution equations (NLEEs) plays an important role in mathematical physics. A lot of physical models have supported a wide variety of solitary wave solutions. Here, we study the Kudryashov-Sinelshchikov equation. In 2010, Kudryashov and Sinelshchikov [1] obtained a more common nonlinear partial differential equation for describing the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer, that is,

$$u_t + \alpha u u_x + u_{xxx} - (u u_{xx})_x - \beta u_x u_{xx} = 0, \quad (1)$$

where α, β are real parameters. In [2], they derived partial cases of nonlinear evolution equations of the fourth order for describing nonlinear pressure waves in a mixture liquid and gas bubbles. Some exact solutions are found and properties of nonlinear waves in a liquid with gas bubbles are discussed. Equation (1) is called Kudryashov-Sinelshchikov equation; it is generalization of the KdV and the BKdV equations and similar but not identical to the Camassa-Holm (CH) equation; it has been studied by some authors [1, 3–5]. Undistorted waves are governed by a corresponding ordinary differential equation which, for special values of some integration constant, is solved analytically in [1]. Solutions are derived in a more straightforward manner and cast into a simpler form, and some new types of solutions which contain

solitary wave and periodic wave solutions are presented in [4]. Ryabov [5] obtained some exact solutions for $\beta = -3$ and $\beta = -4$ using a modification of the truncated expansion method [6, 7]. Li and He discussed the equation by the bifurcation method of dynamical systems and the method of phase portraits analysis [8–10]. In [11], the equation is studied by the Lie symmetry method.

The G'/G -expansion method proposed by Wang et al. [12] is one of the most effective direct methods to obtain travelling wave solutions of a large number of nonlinear evolution equations, such as the KdV equation, the mKdV equation, the variant Boussinesq equations, and the Hirota-Satsuma equations. Later, the further developed methods named the generalized G'/G -expansion method, the modified G'/G -expansion method, the extended G'/G -expansion method, and the improved G'/G -expansion method have been proposed in [13–15], respectively. The aim of this paper is to derive more and new traveling wave solutions of the Kudryashov-Sinelshchikov equation by the G'/G -expansion method and its variants.

The organization of the paper is as follows: in Section 2, a brief account of the G'/G -expansion and its variants that is, the generalized, improved, and extended versions, for finding the traveling wave solutions of nonlinear equations, is given. In Section 3, we will study the Kudryashov-Sinelshchikov equation by these methods. Finally conclusions are given in Section 4.

2. Description of Methods

2.1. The G'/G -Expansion Method

Step 1. Consider a general nonlinear PDE in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0. \quad (2)$$

Using $u(x, t) = U(\xi)$, $\xi = x - ct$, we can rewrite (2) as the following nonlinear ODE:

$$F(U, U', U'', \dots) = 0, \quad (3)$$

where the prime denotes differentiation with respect to ξ .

Step 2. Suppose that the solution of ODE (3) can be written as follows:

$$U(\xi) = \sum_{i=-n}^n a_i \left(\frac{G'}{G} \right)^i, \quad (4)$$

where σ, a_i ($i = -n, -n+1, \dots$) are constants to be determined later, n is a positive integer, and $G = G(\xi)$ satisfies the following second-order linear ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \quad (5)$$

where λ, μ are real constants. The general solutions of (5) can be listed as follows.

When $\Delta = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution of (5)

$$G(\xi) = e^{-(\lambda/2)\xi} \left(A_1 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right). \quad (6)$$

When $\Delta = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution of (5)

$$G(\xi) = e^{-(\lambda/2)\xi} \left(A_1 \cos \left(\frac{\sqrt{-\Delta}}{2} \xi \right) + A_2 \sin \left(\frac{\sqrt{-\Delta}}{2} \xi \right) \right). \quad (7)$$

When $\Delta = \lambda^2 - 4\mu = 0$, we obtain the solution of (5)

$$G(\xi) = e^{-(\lambda/2)\xi} (A_1 + A_2 \xi), \quad (8)$$

where A_1 and A_2 are arbitrary constants.

Step 3. Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in (3).

Step 4. Substituting (4) along with (5) into (3) and then setting all the coefficients of $(G'/G)^k$ ($k = 1, 2, \dots$) of the resulting system's numerator to zero yields a set of over-determined nonlinear algebraic equations for c and a_i ($i = -n, -n+1, \dots$).

Step 5. Assuming that the constants c and a_i ($i = -n, -n+1, \dots$) can be obtained by solving the algebraic equations in Step 4 and then substituting these constants and the known general solutions of (5) into (4), we can obtain the explicit solutions of (2) immediately.

TABLE 1: Relations between values of r, p, q and corresponding $G(\xi)$ in (10).

r	p	q	$G(\xi)$
1	$-(m^2 + 1)$	m^2	$sn(\xi), cd(\xi)$
$1 - m^2$	$m^2 - 1$	$-m^2$	$cn(\xi)$
$m^2 - 1$	$2 - m^2$	-1	$dn(\xi)$
m^2	$-(m^2 + 1)$	1	$ns(\xi), dc(\xi)$
$-m^2$	$2m^2 - 1$	$1 - m^2$	$nc(\xi)$
-1	$2 - m^2$	$m^2 - 1$	$nd(\xi)$
1	$2 - m^2$	$1 - m^2$	$sc(\xi)$
1	$2m^2 - 1$	$-m^2(1 - m^2)$	$sd(\xi)$
$1 - m^2$	$2 - m^2$	1	$cs(\xi)$
$-m^2(1 - m^2)$	$2m^2 - 1$	1	$sd(\xi)$
1/4	$(1 - 2m^2)/2$	1/4	$ns(\xi) \pm cs(\xi)$
$(1 - m^2)/4$	$(1 + m^2)/2$	$(1 - m^2)/4$	$nc(\xi) \pm sc(\xi)$
$m^2/4$	$(m^2 - 2)/2$	1/4	$ns(\xi) \pm ds(\xi)$
$m^2/4$	$(m^2 - 2)/2$	$m^2/4$	$sn(\xi) \pm icn(\xi)$

2.2. The Generalized G'/G -Expansion Method. In generalized version [13], one makes an ansatz for the solution $U(\xi)$ as

$$U(\xi) = \sum_{i=0}^n a_i \left(\frac{G'}{G} \right)^i, \quad (9)$$

where $G = G(\xi)$ satisfies the following Jacobi elliptic equation:

$$(G')^2 = qG^4 + pG^2 + r, \quad (10)$$

where q, p , and r are the arbitrary constants to be determined later and $a_n \neq 0$. Substituting (9) into (3) and using (10), we obtain a polynomial in $G^i, G'G^i$ ($i = 1, 2, \dots$). Equating each coefficient of the resulted polynomials to zero yields a set of algebraic equations for a_i, q, p , and r . Now, substituting a_i and the general solutions of (10) (see Table 1) into (9), we obtain many new traveling wave solutions in terms of Jacobi elliptic functions of the nonlinear PDE (2).

2.3. The Extended G'/G -Expansion Method. In the extended form of this method [15], the solution $U(\xi)$ of (3) can be expressed as

$$U(\xi) = a_0 + \sum_{i=1}^n \left(a_i \left(\frac{G'}{G} \right)^i + b_i \left(\frac{G'}{G} \right)^{i-1} \sqrt{\sigma \left(1 + \frac{1}{\mu} \left(\frac{G'}{G} \right)^2 \right)} \right), \quad (11)$$

where a_0, a_i, b_i ($i = 1, 2, \dots, n$) are constants to be determined later, $\sigma = \pm 1$, n is a positive integer, and $G = G(\xi)$ satisfies the following second order linear ODE:

$$G'' + \mu G = 0, \quad (12)$$

where μ is a constant. Substituting (11) into (3), using (12), collecting all terms with the same order of $(G'/G)^k$ and $(G'/G)^k \sqrt{\sigma(1 + (1/\mu)(G'/G)^2)}$ together, and then equating

each coefficient of the resulting polynomial to zero yield a set of algebraic equations for μ, a_0, a_i, b_i ($i = 1, \dots, n$). On solving these algebraic equations, we obtain the values of the constants μ, a_0, a_i, b_i ($i = 1, \dots, n$), and then substituting these constants and the known general solutions of (12) into (11), we obtain the explicit solutions of nonlinear differential equation (2).

After the brief description of the methods, we now apply these for solving the Kudryashov-Sinelshchikov equation.

3. The Exact Solutions of the Kudryashov-Sinelshchikov Equation

3.1. Using G'/G -Expansion Method. Let $u(x, t) = 1 - \varphi(\xi)$, with $\xi = x - ct$, $c \neq 0$, that is, $u(x, t) = 1 - \varphi(x - ct)$, where c is the wave speed. Under this transformation, (1) can be reduced to the following ordinary differential equation (ODE):

$$c\varphi' - \alpha(1 - \varphi)\varphi' - \varphi''' + ((1 - \varphi)\varphi'')' - \beta\varphi'\varphi'' = 0. \quad (13)$$

Integrating (13) once with respect to φ and setting the constant of integration to zero, we have

$$\frac{1}{2}\alpha\varphi^2 + (c - \alpha)\varphi + \frac{1}{2}\alpha - \varphi\varphi'' - \frac{1}{2}\beta(\varphi')^2 = 0. \quad (14)$$

Balancing $\varphi\varphi''$ with $(\varphi')^2$ in (10) we find that $n + n + 2 = 2(n + 1)$, so n is an arbitrary positive integer. For simplify, we take $n = 1$. Suppose that (14) owns the solutions in the form

$$\varphi(\xi) = a_0 + a_1 \frac{G'}{G} + b_1 \left(\frac{G'}{G} \right)^{-1}. \quad (15)$$

Substituting (15) along with (5) into (14) and then setting all the coefficients of G'/G ($k = 0, 1, \dots$) of the resulting system's numerator to zero yield a set of overdetermined nonlinear algebraic equations about $a_0, a_1, b_1, c, \alpha, \beta$. Solving the overdetermined algebraic equations, we can obtain the following results.

Case 1. We have

$$\begin{aligned} a_0 &= \pm \sqrt{(-\lambda^2 + 4\mu)^{-1}}\lambda, & a_1 &= 0, \\ b_1 &= \pm 2\sqrt{(-\lambda^2 + 4\mu)^{-1}}\mu, \\ c &= \lambda^2 - 4\mu, & \alpha &= \lambda^2 - 4\mu, & \beta &= -4, \end{aligned} \quad (16)$$

where λ, μ are arbitrary constants and $\lambda^2 \neq 4\mu$.

Case 2. We have

$$\begin{aligned} a_0 &= \pm \sqrt{(-\lambda^2 + 4\mu)^{-1}}\lambda, & a_1 &= \pm 2\sqrt{(-\lambda^2 + 4\mu)^{-1}}, \\ b_1 &= 0, & c &= \lambda^2 - 4\mu, \\ \alpha &= \lambda^2 - 4\mu, & \beta &= -4, \end{aligned} \quad (17)$$

where λ, μ are arbitrary constants and $\lambda^2 \neq 4\mu$.

Case 3. We have

$$\begin{aligned} a_0 &= \frac{W}{2}, & a_1 &= \frac{W}{\lambda}, & b_1 &= \frac{\mu W}{\lambda}, \\ c &= 6W\mu + 8\mu + \lambda^2, & \alpha &= 8\mu + \lambda^2, & \beta &= -4, \end{aligned} \quad (18)$$

where λ, μ are arbitrary constants, $\lambda \neq 0$, $W = \text{Root Of } (-32\mu - 4\lambda^2 + (16\mu - \lambda^2)Z^2)$.

Using Case 3, (15) and the general solutions of (5), we can find the following travelling wave solutions of Kudryashov-Sinelshchikov equation (1).

Subcase 3.1. When $\Delta = \lambda^2 - 4\mu > 0$, $\lambda \neq 0$, we obtain the hyperbolic function solutions of (1) as follows:

$$\begin{aligned} u(\xi) &= 1 - \varphi(\xi) = 1 - \left(a_0 + a_1 \frac{G'}{G} + b_1 \left(\frac{G'}{G} \right)^{-1} \right) \\ &= 1 - \frac{W}{\lambda} \left(\frac{\lambda}{2} + \frac{G'}{G} + \mu \left(\frac{G'}{G} \right)^{-1} \right) \\ &= 1 - \frac{W}{\lambda} \left(\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2} \left(\left(A_1 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) \right. \right. \\ &\quad \times \left(A_1 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right)^{-1} \left. \right) \frac{\lambda}{2} \Big)^{-1} \\ &\quad - \frac{\lambda}{2} + \mu \left(\frac{\sqrt{\Delta}}{2} \left(\left(A_1 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) \right. \right. \\ &\quad \times \left(A_1 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right)^{-1} \left. \right) \frac{\lambda}{2} \Big)^{-1} \Big)^{-1} \\ &= 1 - \frac{W}{\lambda} \left(\frac{\sqrt{\Delta}}{2} \left(\left(A_1 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) \right. \right. \\ &\quad \times \left(A_1 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right)^{-1} \left. \right) \frac{\lambda}{2} \Big)^{-1} \Big)^{-1} \\ &\quad + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \Big)^{-1} \Big)^{-1} \end{aligned}$$

$$\begin{aligned}
& + \mu \left(\frac{\sqrt{\Delta}}{2} \left(\left(A_1 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right. \right. \right. \\
& \quad \left. \left. + A_2 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) \right. \\
& \quad \left. \times \left(A_1 \cosh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right. \right. \\
& \quad \left. \left. + A_2 \sinh \left(\frac{\sqrt{\Delta}}{2} \xi \right) \right) \right)^{-1} \\
& \left. - \frac{\lambda}{2} \right)^{-1} \Bigg), \tag{19}
\end{aligned}$$

where $\xi = x - (6W\mu + 8\mu + \lambda^2)t$, $\Delta = \lambda^2 - 4\mu$, A_1, A_2 are arbitrary constants.

It is easy to see that the hyperbolic function solution can be rewritten at $A_1^2 < A_2^2$ and $A_1^2 > A_2^2$ as follows:

$$\begin{aligned}
u(x, t) &= 1 - \varphi(\xi) \\
&= 1 - \frac{W}{\lambda} \left(\frac{\sqrt{\Delta}}{2} \tanh \left(\frac{\sqrt{\Delta}}{2} \xi + \xi_0 \right) \right. \\
& \quad \left. + \frac{\mu}{(\sqrt{\Delta}/2) \tanh((\sqrt{\Delta}/2) \xi + \xi_0) - \lambda/2} \right), \\
u(x, t) &= 1 - \varphi(\xi) \\
&= 1 - \frac{W}{\lambda} \left(\frac{\sqrt{\Delta}}{2} \coth \left(\frac{\sqrt{\Delta}}{2} \xi + \xi_0 \right) \right. \\
& \quad \left. + \frac{\mu}{(\sqrt{\Delta}/2) \coth((\sqrt{\Delta}/2) \xi + \xi_0) - \lambda/2} \right), \tag{20}
\end{aligned}$$

where, $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Subcase 3.2. When $\Delta = \lambda^2 - 4\mu < 0$, $\lambda \neq 0$, the trigonometric function solution of (1) can be rewritten at $A_1^2 < A_2^2$ and $A_1^2 > A_2^2$ as follows:

$$\begin{aligned}
u(x, t) &= 1 - \varphi(\xi) \\
&= 1 - \frac{W}{\lambda} \left(\frac{\sqrt{\Delta}}{2} \tan \left(\frac{\sqrt{\Delta}}{2} \xi + \xi_0 \right) \right. \\
& \quad \left. + \frac{\mu}{(\sqrt{\Delta}/2) \tan((\sqrt{\Delta}/2) \xi + \xi_0) - \lambda/2} \right), \tag{21}
\end{aligned}$$

where, $A_1^2 < A_2^2$,

$$\begin{aligned}
u(x, t) &= 1 - \varphi(\xi) \\
&= 1 - \frac{W}{\lambda} \left(\frac{\sqrt{\Delta}}{2} \cot \left(\frac{\sqrt{\Delta}}{2} \xi + \xi_0 \right) \right. \\
& \quad \left. + \frac{\mu}{(\sqrt{\Delta}/2) \cot((\sqrt{\Delta}/2) \xi + \xi_0) - \lambda/2} \right), \tag{22}
\end{aligned}$$

where, $A_1^2 > A_2^2$.

Subcase 3.3. When $\Delta = \lambda^2 - 4\mu = 0$, $\lambda \neq 0$, we obtain the rational function solutions of (1) as follows:

$$\begin{aligned}
u(x, t) &= 1 - \varphi(\xi) \\
&= 1 - \frac{W}{\lambda} \left(\frac{A_2}{A_1 + A_2 \xi} + \mu \left(\frac{A_2}{A_1 + A_2 \xi} - \frac{\lambda}{2} \right)^{-1} \right). \tag{23}
\end{aligned}$$

Using other two cases, (15), and the general solutions of (5), we could obtain other exact solutions of (1), and here we do not list all of them.

3.2. Using Generalized G'/G -Expansion Method. Suppose that (13) owns the solutions in the form

$$u(\xi) = a_0 + a_1 \frac{G'}{G} + b_1 \left(\frac{G'}{G} \right)^2, \tag{24}$$

in this case, $G = G(\xi)$ satisfies the Jacobi elliptic equation (10).

Substituting (24) along with (10) into (14) and then setting all the coefficients of G^i , $G'G^i$ ($i = 1, 2, \dots$) of the resulting system's numerator to zero yield a set of overdetermined nonlinear algebraic equations about $a_0, a_1, b_1, c, \alpha, \beta$. Solving the overdetermined algebraic equations, we can obtain the following results.

Case 1. We have

$$\begin{aligned}
a_0 &= 0, \quad a_1 = \pm \sqrt{-\frac{p}{p^2 - 4qr}}, \\
a_2 &= 0, \quad c = 4p, \quad \alpha = 4p, \quad \beta = -4. \tag{25}
\end{aligned}$$

Case 2. We have

$$\begin{aligned}
a_0 &= -n(-\alpha + 8p), \quad a_1 = 0, \quad a_2 = 12n, \\
c &= 16np^2 + 192nrq - \alpha^2 n + \alpha, \\
\alpha &= \alpha, \quad \beta = -3, \tag{26}
\end{aligned}$$

where, $n = \text{Root Of } ((48\alpha p^2 + 576qra - \alpha^3 + 128p^3 - 4608qpr)Z^2 + \alpha)$.

Thus using (24) and (26), the following solutions of (1) are obtained:

$$u(\xi) = \pm \sqrt{-\frac{p}{p^2 - 4qr}} \left(\frac{G'}{G} \right), \tag{27}$$

where, $\xi = x - 4pt$. Now, with the aid of Table 1, we get the following set of exact solutions of (1).

Using Case 1, (24), and the general solutions of (10), we can find the following travelling wave solutions of Kudryashov-Sinelshchikov equation (1).

Set 1.1, if $r = 1$, $p = -(m^2 + 1)$, $q = m^2$, $G(\xi) = sn(\xi)$, or $G(\xi) = cd(\xi)$, then we obtain

$$u_1(\xi) = 1 - \varphi(\xi) \\ = 1 \pm \sqrt{-\frac{m^2}{(m^2 + 2)^2}} cs(\xi) dn(\xi), \quad (28)$$

$$u_2(\xi) = 1 - \varphi(\xi) \\ = 1 \pm \sqrt{-\frac{m^2}{(m^2 + 2)^2}} (m^2 - 1) sd(\xi) nc(\xi), \quad (29)$$

where, $\xi = x + 4(m^2 + 1)t$.

When $m \rightarrow 1$, $sn(\xi) \rightarrow \tanh(\xi)$, solution (28) becomes

$$u_1(\xi) = 1 \pm \frac{i}{3} \operatorname{sech}(\xi) \operatorname{csch}(\xi), \quad (30)$$

where, $\xi = x + 8t$.

Set 1.2, if $r = 1 - m^2$, $p = 2m^2 - 1$, $q = -m^2$, $G = cn(\xi)$, then we obtain

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \sqrt{\frac{m^2}{(3m^2 - 2)^2}} sc(\xi) dn(\xi), \quad (31)$$

where, $\xi = x - 4(2m^2 - 1)t$.

When $m \rightarrow 1$, $cn(\xi) \rightarrow \operatorname{sech}(\xi)$, solution (31) becomes

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \tanh(\xi), \quad (32)$$

where, $\xi = x - 4t$.

Set 1.3, if $r = m^2 - 1$, $p = 2 - m^2$, $q = -1$, $G(\xi) = dn(\xi)$, then we obtain

$$u(\xi) = 1 - \varphi(\xi) \\ = 1 \pm \sqrt{\frac{1}{(2m^2 - 3)^2}} m^2 sn(\xi) cd(\xi), \quad (33)$$

where, $\xi = x - 4(2 - m^2)t$.

When $m \rightarrow 1$, $dn(\xi) \rightarrow \operatorname{sech}(\xi)$, solution (33) becomes

$$u(\xi) = 1 \pm \tanh(\xi), \quad (34)$$

where, $\xi = x - 4t$. It is the same with the solution (34).

Set 1.4, if $r = m^2$, $p = -(m^2 + 1)$, $q = 1$, $G(\xi) = ns(\xi)$, or $G(\xi) = dc(\xi)$, then we obtain

$$u_1(\xi) = 1 - \varphi(\xi) = 1 \pm \sqrt{-\frac{1}{(2m^2 + 1)^2}} ds(\xi) cn(\xi), \\ u_2(\xi) = 1 - \varphi(\xi) \\ = 1 \pm \sqrt{-\frac{1}{(2m^2 + 1)^2}} (m^2 - 1) sc(\xi) nd(\xi), \quad (35)$$

where, $\xi = x + 4(m^2 + 1)t$.

When $m \rightarrow 0$, $ns(\xi) \rightarrow csc(\xi)$, $dc(\xi) \rightarrow \sec(\xi)$, solution (35) becomes

$$u_1(\xi) = 1 \pm i \cot(\xi), \quad (36)$$

$$u_2(\xi) = 1 \pm i \tan(\xi), \quad (37)$$

where, $\xi = x + 4t$.

When $m \rightarrow 1$, $ns(\xi) \rightarrow \coth(\xi)$, solution (34) becomes

$$u(\xi) = 1 \pm \frac{i}{3} \operatorname{sech}(\xi) \operatorname{csch}(\xi). \quad (38)$$

It is the same with solution (30). where, $\xi = x + 8t$.

Set 1.5, if $r = -m^2$, $p = 2m^2 - 1$, $q = 1 - m^2$, $G(\xi) = nc(\xi)$, then we obtain

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \sqrt{-2m^2 + 1} dn(\xi) sc(\xi), \quad (39)$$

where, $\xi = x + 4(m^2 + 1)t$.

Set 1.6 if $r = 1/4$, $p = (1 - 2m^2)/2$, $q = 1/4$, $G(\xi) = ns(\xi) + cs(\xi)$, then we obtain

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \frac{\sqrt{2}}{2} \sqrt{\frac{2m^2 - 1}{m^2(m^2 - 1)}} ds(\xi), \quad (40)$$

where, $\xi = x - 2(1 - 2m^2)t$.

Set 1.7 if $r = (1 - m^2)/4$, $p = (1 + m^2)/2$, $q = (1 - m^2)/4$, $G(\xi) = nc(\xi) + sc(\xi)$, solution (39) becomes

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \sqrt{-\frac{m^2 + 1}{2m^2}} dc(\xi), \quad (41)$$

where, $\xi = x - 2(1 + m^2)t$.

Similarly, we can write down the other sets of exact solutions of (1) with the help of Table 1 and the Case 2, which are omitted for convenience. Thus using the generalized form of the G'/G -expansion method, we can obtain families of the exact traveling wave solutions of (1) in terms of Jacobi elliptic functions. Under some conditions, these solutions change into hyperbolic and trigonometric functional forms.

3.3. Using Extended G'/G -Expansion Method. Suppose that (14) owns the solutions in the form

$$\varphi(\xi) = a_0 + a_1 \frac{G'}{G} + a_2 \left(\frac{G'}{G} \right)^2 + b_1 \sqrt{\sigma \left(1 + \frac{1}{\mu} \left(\frac{G'}{G} \right)^2 \right)} \\ + b_2 \frac{G'}{G} \sqrt{\sigma \left(1 + \frac{1}{\mu} \left(\frac{G'}{G} \right)^2 \right)}, \quad (42)$$

where a_0, a_1, a_2, b_1, b_2 are constants to be determined later, $\sigma = \pm 1$, n is a positive integer, and $G = G(\xi)$ satisfies the second-order linear ODE (12).

Substituting (42) along with (12) into (14) and then setting all the coefficients of $(G'/G)^k$ and

$(G'/G)^k \sqrt{\sigma(1 + (1/\mu)(G'/G)^2)}$ ($k = 0, 1, \dots$) of the resulting system to zero yield a set of overdetermined nonlinear algebraic equations about $a_0, a_1, a_2, b_1, b_2, c, \alpha, \beta$. Solving the overdetermined algebraic equations, we can obtain the following results.

Case 1. We have

$$a_0 = 0, \quad a_1 = \pm \sqrt{\frac{1}{\mu}}, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad (43)$$

$$c = -4\mu, \quad \alpha = -4\mu, \quad \beta = -4.$$

Case 2. We have

$$a_0 = 0, \quad a_1 = \pm \sqrt{\frac{1}{\mu}}, \quad a_2 = 0, \quad b_1 = \pm \sqrt{\frac{1}{\delta}}, \quad b_2 = 0, \quad (44)$$

$$c = -\mu, \quad \alpha = -\mu, \quad \beta = -4.$$

Case 3. We have

$$a_0 = \frac{K(\alpha + 8\mu)}{-\alpha + 4\mu}, \quad a_1 = 0, \quad a_2 = 12 \frac{K}{-\alpha + 4\mu}, \quad (45)$$

$$b_1 = 0, \quad b_2 = 0, \quad c = 4K\mu + (K + 1)\alpha,$$

$$\alpha = \alpha, \quad \beta = -3,$$

where, $K = \pm \sqrt{\alpha/(\alpha + 8\mu)}$.

Case 4. We have

$$a_0 = \frac{\alpha(5\mu + \alpha)}{\text{root of } ((\alpha + 2\mu)Z^2 - \alpha)(2\mu^2 - \alpha\mu - \alpha^2)},$$

$$a_1 = 0, \quad a_2 = 6 \frac{\text{Root Of } ((\alpha + 2\mu)Z^2 - \alpha)}{\mu - \alpha},$$

$$b_1 = 0, \quad b_2 = 6 \frac{\text{Root Of } ((\alpha\sigma + 2\mu\sigma)Z^2 - \alpha\mu)}{\mu - \alpha}, \quad (46)$$

$$\alpha = \alpha, \quad \beta = -3,$$

$$c = \left(\alpha(\mu + 2\mu \text{ Root Of } ((\alpha + 2\mu)Z^2 - \alpha)) \right. \\ \left. + \alpha \text{ Root Of } ((\alpha + 2\mu)Z^2 - \alpha) + \alpha \right) \\ \times ((\alpha + 2\mu) \text{ Root Of } ((\alpha + 2\mu)Z^2 - \alpha))^{-1}.$$

Using Case 1, (42), and the general solutions of (12), we can find the following travelling wave solutions of Kudryashov-Sinelshchikov equation (1).

Subcase 1.1. When $\mu < 0$, we have the hyperbolic function solution as

$$u(\xi) = 1 - \varphi(\xi) \\ = 1 \pm i \frac{A_1 \sinh(\sqrt{-\mu}\xi) + A_2 \cosh(\sqrt{-\mu}\xi)}{A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi)}, \quad (47)$$

where, $\xi = x + 4\mu t$.

In particular, setting $A_1 = 0, A_2 \neq 0$, then (47) can be written as

$$u(\xi) = 1 \pm i \coth(\sqrt{-\mu}\xi). \quad (48)$$

Setting $A_1 \neq 0, A_2 = 0$, then (47) can be written as

$$u(\xi) = 1 \pm i \tanh(\sqrt{-\mu}\xi). \quad (49)$$

Subcase 1.2. When $\mu > 0$, we have the trigonometric function solution as

$$u(\xi) = 1 - \varphi(\xi) = 1 \pm \frac{-A_1 \sin(\sqrt{\mu}\xi) + A_2 \cos(\sqrt{\mu}\xi)}{A_1 \cos(\sqrt{\mu}\xi) + A_2 \sin(\sqrt{\mu}\xi)}. \quad (50)$$

In particular, setting $A_1 = 0, A_2 \neq 0$, then (50) can be written as

$$u(\xi) = 1 \pm \cot(\sqrt{\mu}\xi). \quad (51)$$

setting $A_1 \neq 0, A_2 = 0$, then (50) can be written as

$$u(\xi) = 1 \pm \tan(\sqrt{\mu}\xi), \quad (52)$$

where, $\xi = x + 4\mu t$.

Using Case 2, (42), and the general solutions of (12), we can find the following travelling wave solutions of Kudryashov-Sinelshchikov equation (1).

Subcase 2.1. When $\mu < 0$, we have the hyperbolic function solution as

$$u(\xi) = 1 - \varphi(\xi) \\ = 1 \pm i \frac{A_1 \sinh(\sqrt{-\mu}\xi) + A_2 \cosh(\sqrt{-\mu}\xi)}{A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi)} \\ \pm \sqrt{1 - \left(\frac{A_1 \sinh(\sqrt{-\mu}\xi) + A_2 \cosh(\sqrt{-\mu}\xi)}{A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi)} \right)^2}. \quad (53)$$

In particular, setting $A_1 = 0, A_2 \neq 0$, then (53) can be written as

$$u(\xi) = 1 \pm i \coth(\sqrt{-\mu}\xi) \pm \text{csch}(\sqrt{-\mu}\xi). \quad (54)$$

Setting $A_1 \neq 0, A_2 = 0$, then (53) can be written as

$$u(\xi) = 1 \pm i \tanh(\sqrt{-\mu}\xi) \pm \text{sech}(\sqrt{-\mu}\xi), \quad (55)$$

where, $\xi = x + \mu t$.

Subcase 2.2. When $\mu > 0$, we have the trigonometric function solution as

$$u(\xi) = 1 - \varphi(\xi) \\ = 1 \pm \frac{-A_1 \sin(\sqrt{\mu}\xi) + A_2 \cos(\sqrt{\mu}\xi)}{A_1 \cos(\sqrt{\mu}\xi) + A_2 \sin(\sqrt{\mu}\xi)} \\ \pm \sqrt{1 + \left(\frac{-A_1 \sin(\sqrt{\mu}\xi) + A_2 \cos(\sqrt{\mu}\xi)}{A_1 \cos(\sqrt{\mu}\xi) + A_2 \sin(\sqrt{\mu}\xi)} \right)^2}. \quad (56)$$

In particular, setting $A_1 = 0$, $A_2 \neq 0$, then (56) can be written as

$$u(\xi) = 1 \pm \cot(\sqrt{\mu}\xi) \pm \csc(\sqrt{\mu}\xi). \quad (57)$$

Setting $A_1 \neq 0$, $A_2 = 0$, then (56) can be written as

$$u(\xi) = 1 \pm \tan(\sqrt{\mu}\xi) \pm \sec(\sqrt{\mu}\xi), \quad (58)$$

where, $\xi = x + \mu t$.

Similarly, we can get the other exact solutions of (1) in Cases 3 and 4, which are omitted for convenience.

Remark 1. The validity of the solutions we obtained is verified.

Remark 2. The solutions expressed by Jacobi elliptic functions are not given in the related literature. So, the solutions we obtained are new.

Remark 3. The solutions we got are general involving various arbitrary parameters. If we set the parameters to special values, some results in the literature can be obtained.

4. Conclusions

In the present work, we successfully obtained exact traveling wave solutions of the Kudryashov-Sinelshchikov equation using the G'/G -expansion method and its variants. Some obtained new exact and explicit analytic solutions are in general forms involving various arbitrary parameters. These solutions are expressed by the hyperbolic functions, the trigonometric functions, the rational functions, and the Jacobi elliptic functions. The results of [1–11] have been enriched.

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