Research Article

Sharp Upper Bounds for the Laplacian Spectral Radius of Graphs

Houqing Zhou¹ and Youzhuan Xu²

¹ Department of Mathematics, Shaoyang University, Hunan 422000, China

² Shaoyang Radio & TV University, Hunan 422000, China

Correspondence should be addressed to Houqing Zhou; zhouhq2004@163.com

Received 15 August 2013; Accepted 10 October 2013

Academic Editor: Miguel A. F. Sanjuán

Copyright © 2013 H. Zhou and Y. Xu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The spectrum of the Laplacian matrix of a network plays a key role in a wide range of dynamical problems associated with the network, from transient stability analysis of power network to distributed control of formations. Let $G = (V, E)$ be a simple connected graph on n vertices and let $\mu(G)$ be the largest Laplacian eigenvalue (i.e., the spectral radius) of G . In this paper, by using the Cauchy-Schwarz inequality, we show that the upper bounds for the Laplacian spectral radius of G .

1. Introduction

The eigenvalue spectrum of the Laplacian matrix of a network provides valuable information regarding the behavior of many dynamical processes taking place on the network. In [1], Pecora and Carroll related the problem of synchronization in a network of coupled oscillators to the largest and second-smallest Laplacian eigenvalues (usually denoted by Laplacian spectral radius and spectral gap, resp.) of the network. More recently, Dorfler and Bullo (see [2]) derived conditions for transient stability in power networks in terms of the spectral gap of the Laplacian matrix. Apart from their applicability to the problems of synchronization and transient stability analysis, the Laplacian eigenvalues are also relevant in the analysis of many distributed estimation and control problems (see [3]).

Understanding the relationship between the structure of a complex network and the behavior of dynamical processes taking place in it is a central question in the research field of network science. Since the behavior of many networked dynamical processes is closely related to the Laplacian eigenvalues, it is of interest to study the relationship between structural features of the network and its Laplacian eigenvalues. In this paper, we mainly study the spectral radius of the Laplacian matrix.

2. Preliminaries

Let $G = (V, E)$ be a simple undirected graph on n vertices. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency and $D(G)$ is the diagonal matrix of vertex degrees. It is well known that $L(G)$ is a positive semidefinite matrix and that $(0, \mathbf{e})$ is an eigenpair of $L(G)$ where $\mathbf{e}$ is the all-ones vector. In [4], some of the many results known for Laplacian matrices are given. The spectrum of $L(G)$ is $\{\mu_1(G), \mu_2(G), \dots, \mu_n(G)\}$, where $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) = 0$. The largest eigenvalue $\mu_1(G)$ is called the Laplacian spectral radius of the graph G , denoted by $\mu(G)$. For a star graph of order n , the spectrum is $\{n, \underbrace{1, 1, \dots, 1}_{n-2}, 0\}$.

We recall that upper bounds of the spectral radius of $L(G)$. It is a well-known fact that $\mu(G) \leq 2\Delta$ with equality if and only if G is bipartite regular. Shi [5] gave an upper bound for the Laplacian spectral radius of irregular graphs as follows.

Let G be a connected irregular graph of order n with maximum degree Δ and diameter D . Then $\mu(G) < 2\Delta - 2/(2D + 1)n$.

Li et al. [6] improve Shi's upper bound for the Laplacian spectral radius of irregular graphs. They show the following result:

$$\mu(G) < 2\Delta - \frac{1}{nD}. \quad (1)$$

Dyilek Maden and Buyukkose [7] proved the following. Let G be a simple graph. Then,

$$\sqrt{m + \frac{s}{\sqrt{n-1}}} \leq \mu_1(G) \leq \sqrt{m + s\sqrt{n-1}}, \quad (2)$$

where $m = (\sum_{i=1}^n d_i(d_i + 1))/n$, and

$$\begin{aligned} s^2 = & \frac{\sum_{i=1}^n (d_i^2 + d_i)^2}{n} \\ & + \frac{2 \sum_{i < j, i \sim j} (d_i + d_j) (d_i + d_j - 2 |N_i \cap N_j|)}{n} \\ & + \frac{2 \sum_{i < j} |N_i \cap N_j|^2}{n} - m^2. \end{aligned} \quad (3)$$

In this paper, we continue to consider the upper bounds for the Laplacian spectral radius of graphs. The rest of the paper is organized as follows. Section 3 contains some lemmas which play a fundamental role. Section 4 contains two theorems on the upper bounds of $\mu(G)$.

3. Some Useful Lemmas

In the proof of several theorems we will use the following lemmas.

Lemma 1 (see [8]). *Let G be a connected graph with n vertices and m edges; then*

$$\sum_{i=1}^n d_i^2 \leq \frac{2m(2m + (n-1)(\Delta - \delta))}{n + \Delta - \delta}. \quad (4)$$

Equality holds if and only if $G = B_{n,t}$ for some $1 \leq t \leq n$ or G is regular, where $B_{n,t}$ denotes the graph on n vertices with exactly t vertices of degree $n-1$ and the remaining of $n-t$ vertices forming an independent set. Notice that $B_{n,1} = K_{1,n-1}$ and $B_{n,n} = K_n$.

Let M be an $m \times n$ matrix. Then $s_i(M)$ will denote the i th row sum of M ; that is, $s_i(M) = \sum_{j=1}^n M_{ij}$, where $1 \leq i \leq m$.

Lemma 2 (see [9]). *Let G be a connected n -vertex graph and A its adjacency matrix, with spectral radius $\rho(A)$. Let P be any polynomial. Then*

$$\min_{i \in V(G)} s_i(P(A)) \leq P(\rho(A)) \leq \max_{i \in V(G)} s_i(P(A)). \quad (5)$$

Lemma 3 (see [10]). *Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Let ρ denote the spectral radius of the line graph $l(G)$ of G . Then the inequality*

$$\mu(G) \leq 2 + \rho \quad (6)$$

holds, and the equality occurs if and only if G is a bipartite graph.

4. Main Results

In this section, we consider simple connected graph with n vertices. The main result of the paper is the following theorem.

Theorem 4. *Let G be a graph with n vertices and m edges; then*

$$\begin{aligned} \mu(G) \leq & \frac{1}{n-1} \left(2m + \left((n-2)m(n(n-1) - 2m) \right. \right. \\ & \left. \left. \times \frac{2(\Delta - \delta + 1)}{n + \Delta - \delta} \right)^{1/2} \right), \end{aligned} \quad (7)$$

with equality if and only if G is the star $K_{1,n-1}$ or the complete graph K_n .

Proof. Since

$$\sum_{i=1}^n \mu_i(G) = \text{tr}(L(G)) = 2m, \quad (8)$$

where $\text{tr}(L(G))$ denotes the trace of $L(G)$. Notice that $\mu_1^2(G), \mu_2^2(G), \dots, \mu_n^2(G)$ are eigenvalues of $L^2(G)$; hence we have

$$\sum_{i=1}^n \mu_i^2(G) = \text{tr}(L^2(G)) = 2m + \sum_{i=1}^n d_i^2. \quad (9)$$

According to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (n-2) (\mu_2^2(G) + \mu_3^2(G) + \dots + \mu_{n-1}^2(G)) \\ \geq (\mu_2(G) + \mu_3(G) + \dots + \mu_{n-1}(G))^2. \end{aligned} \quad (10)$$

That is,

$$(n-2) \left(2m - \mu^2(G) + \sum_{i=1}^n d_i^2 \right) \geq (2m - \mu(G))^2. \quad (11)$$

By means of Lemma 1, we obtain

$$\begin{aligned} (2m - \mu(G))^2 \leq (n-2) (2m - \mu(G)^2) \\ + (n-2) \cdot \frac{2m(2m + (n-1)(\Delta - \delta))}{n + \Delta - \delta}. \end{aligned} \quad (12)$$

Suppose that $t = (n-2) \cdot (2m(2m + (n-1)(\Delta - \delta)))/(n + \Delta - \delta)$; simplifying the inequality above, we get

$$(n-1)\mu(G)^2 - 4m\mu(G) + 4m^2 - 2m(n-2) - t \leq 0. \quad (13)$$

Hence, we have

$$\frac{2m - w}{n-1} \leq \mu(G) \leq \frac{2m + w}{n-1}, \quad (14)$$

where $w = \sqrt{(n-1)t + 2m(n-1)(n-2) - 4m^2(n-2)}$.

Let $G = K_{1,n-1}$ and then $m = n-1$, $t = n(n-1)(n-2)$; $w = (n-1)(n-2)$; we have

$$\frac{2m + w}{n-1} = \frac{2(n-1) + (n-1)(n-2)}{n-1} = n = \mu(G). \quad (15)$$

FIGURE 1: The graph G on 6 vertices.

Next we examine the complete graph K_n . The complete graph has n vertices and $n(n-1)/2$ edges, $\Delta = n-1 = \delta$.

Thus, $t = n(n-1)^2(n-2)$; hence, $w = 0$; we get

$$\frac{2m+w}{n-1} = n = \mu(G). \quad (16)$$

Equality holds on the right in (14) if and only if G is the star $K_{1,n-1}$ or the complete graph K_n .

This completes the proof of the theorem. $\square$

Example 5. In this example we illustrate the technique of Theorem 4. Consider the graph G on 6 vertices and 8 edges in Figure 1; this graph has the largest degree $\Delta = 3$ and the smallest degree $\delta = 2$.

Now we estimate the largest eigenvalue $\mu(G)$ with Theorem 4. Applying this upper bound on $\mu(G)$, it follows that

$$0 \leq \mu(G) \leq 6.4. \quad (17)$$

By a straightforward calculation, we show that the Laplacian eigenvalues of G are

$$\begin{aligned} \mu_1(G) &= 3 + \sqrt{3}, & \mu_2(G) &= 4, & \mu_3(G) &= 4, \\ \mu_4(G) &= 2, & \mu_5(G) &= 3 - \sqrt{3}, & \mu_6(G) &= 0. \end{aligned} \quad (18)$$

Clearly, $\mu_1(G) = 3 + \sqrt{3} < 6.4$ holds.

It is easy to see that we can use the method to estimate the upper bound of the largest Laplacian eigenvalue.

The following Theorem 6 is associated with edge and degree of graph G , that is, associated with the largest and the second largest degree Δ' , the smallest degree of G , respectively.

Theorem 6. Let $G = (V, E)$ be a simple graph with n vertices and m edges; then

$$\begin{aligned} \mu(G) &\leq \left(\Delta' + \delta - 1 \right. \\ &\quad \left. + \left((\Delta' + \delta - 1) \right. \right. \\ &\quad \left. \left. + 4 \left(\Delta^2 - \Delta \cdot \Delta' + 4m - 2(n-1)\delta \right) \right)^{1/2} \right) \\ &\quad \times (2)^{-1} \end{aligned} \quad (19)$$

with equality if and only if G is a regular bipartite graph.

Proof. Let $A = (a_{ij})$ be the adjacency matrix of a graph G with vertices $1, 2, \dots, n$; let $A^k = (a_{ij}^{(k)})$; further, let $N_i(A^k)$ denote the number of walks of length k starting at vertex i . Hence $N_i(A^2) = \sum_{j \sim i} d_j$.

Similarly, for the adjacency matrix B of line graph $l(G)$ of G , let $N'_u(B^k)$ denote the number of walks of length k starting at vertex u and let $d_{l(G)}(ij)$ denote the degree of the vertex ij in line graph $l(G)$; then we have

$$N'_{ij}(B) = d_{l(G)}(ij) = d_i + d_j - 2. \quad (20)$$

It can easily be seen that

$$N'_{ij}(B^2) = \sum_{pq \sim ij} d_{l(G)}(pq) = \sum_{pq \sim ij} (d_p + d_q - 2); \quad (21)$$

that is,

$$\begin{aligned} &\sum_{q \sim i} (d_q + d_i - 2) + \sum_{p \sim j} (d_p + d_j - 2) - 2(d_i + d_j - 2) \\ &= d_i^2 - 2d_i + \sum_{q \sim i} d_q + d_j^2 - 2d_j \\ &\quad + \sum_{p \sim j} d_p - 2(d_i + d_j - 2) \\ &\leq \Delta d_i + \Delta' d_j + 4m - 3(d_i + d_j) \\ &\quad - \sum_{q \sim i} d_q - \sum_{p \sim j} d_p - 2(d_i + d_j - 2) \\ &\leq \Delta' (d_i + d_j) + (\Delta - \Delta') \Delta + 4m - 3(d_i + d_j) \\ &\quad - (n - d_i - 1)\delta - (n - d_j - 1)\delta - 2(d_i + d_j - 2) \\ &= (\Delta' + \delta - 5)(d_i + d_j - 2) + \Delta^2 \\ &\quad + (2 - \Delta)\Delta' - 2(n - 2)\delta + 4m - 6. \end{aligned} \quad (22)$$

According to (20), we have

$$\begin{aligned} N'_{ij}((B^2 - (\Delta' + \delta - 5)B)) \\ \leq \Delta^2 + (2 - \Delta)\Delta' - 2(n - 2)\delta + 4m - 6. \end{aligned} \quad (23)$$

Using Lemma 2, we obtain

$$\begin{aligned} \rho^2 - (\Delta' + \delta - 5)\rho \\ \leq \Delta^2 + (2 - \Delta)\Delta' - 2(n - 2)\delta + 4m - 6. \end{aligned} \quad (24)$$

From the inequality above, we have

$$\begin{aligned}
 \rho &\leq \left(\Delta' + \delta - 5 \right. \\
 &\quad \left. + \left((\Delta' + \delta - 5)^2 \right. \right. \\
 &\quad \left. \left. + 4(\Delta^2 + (2 - \Delta)\Delta' - 2(n - 2)\delta + 4m - 6) \right)^{1/2} \right) \\
 &\quad \times (2)^{-1} \\
 &= \left(\Delta' + \delta - 5 \right. \\
 &\quad \left. + \left((\Delta' + \delta - 1)^2 \right. \right. \\
 &\quad \left. \left. + 4(\Delta^2 - \Delta \cdot \Delta' - 2(n - 1)\delta + 4m) \right)^{1/2} \right) \\
 &\quad \times (2)^{-1}.
 \end{aligned} \tag{25}$$

Using Lemma 3, we have

$$\begin{aligned}
 \mu(G) &\leq \left(\Delta' + \delta - 1 \right. \\
 &\quad \left. + \left((\Delta' + \delta - 1)^2 \right. \right. \\
 &\quad \left. \left. + 4(\Delta^2 - \Delta \cdot \Delta' + 4m - 2(n - 1)\delta) \right)^{1/2} \right) \\
 &\quad \times (2)^{-1}.
 \end{aligned} \tag{26}$$

If equality holds in (19), applying Lemma 3, hence we deduce that G is a bipartite graph. And by Lemma 2, for any $ij \in E(G)$, the equality in (25) occurs. That is, for any ij , the equality in (22) holds. Then, we have

$$\Delta = \Delta' = \delta; \tag{27}$$

that is, G is a regular graph; hence G is a regular bipartite.

More generally, if G is a regular bipartite graph, then by verifying straightforward (19), the equality holds. $\square$

Acknowledgments

The authors thank Professor Miguel A. F. Sanjuán for his useful comments and suggestions. Project is supported by Hunan Provincial Natural Science Foundation of China no. 13JJ3118 and by Scientific Research Fund of Shaoyang Science & Technology no. M230.

References

- [1] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109–2112, 1998.
- [2] F. Dorfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform kuramoto oscillators," *SIAM Journal on Control and Optimization*, vol. 50, pp. 1616–1642, 2012.

- [3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [4] R. Merris, "Laplacian matrices of graphs: a survey," *Linear Algebra and Its Applications*, vol. 197–198, pp. 143–176, 1994.
- [5] L. Shi, "Bounds on the (Laplacian) spectral radius of graphs," *Linear Algebra and Its Applications*, vol. 422, no. 2–3, pp. 755–770, 2007.
- [6] J. Li, W. C. Shiu, and A. Chang, "The Laplacian spectral radius of graphs," *Czechoslovak Mathematical Journal*, vol. 60, no. 3, pp. 835–847, 2010.
- [7] A. Dyilek Maden and S. Buyukkose, "Bounds for Laplacian graph eigenvalues," *Mathematical Inequalities & Applications*, vol. 12, pp. 529–536, 2012.
- [8] S. M. Cioabă, "Sums of powers of the degrees of a graph," *Discrete Mathematics*, vol. 306, no. 16, pp. 1959–1964, 2006.
- [9] M. N. Ellingham and X. Zha, "The spectral radius of graphs on surfaces," *Journal of Combinatorial Theory B*, vol. 78, no. 1, pp. 45–56, 2000.
- [10] J. Shu, Y. Hong, and R. Wen, "A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph," *Linear Algebra and Its Applications*, vol. 347, no. 1–3, pp. 123–129, 2002.