Research Article

Free Vibration Analysis of an Euler Beam of Variable Width on the Winkler Foundation Using Homotopy Perturbation Method

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Homotopy Perturbation Method (HPM) is employed to investigate the vibration of an Euler beam resting on an elastic foundation. The beam is assumed to have variable stiffness along its length. HPM is an easy-to-use and very efficient technique for the solution of linear or nonlinear problems. HPM produces analytical approximate expression which is continuous in the solution domain. This work assures that HPM is a promising method for the vibration analysis of the variable stiffness Euler beams on elastic foundation. Different case problems have been solved by using the technique, and solutions have been compared with those available in the literature.

1. Introduction

In geotechnical engineering, it is widely seen in application that pipelines, shallow foundations, and piles are modeled as a beam in the analysis procedure. There are also various types of foundation models such as Winkler, Pasternak, and Vlasov that have been used in the analysis of structures on elastic foundations.

The most frequently used foundation model in the analysis of beam on elastic foundation problems is the Winkler foundation model. In the Winkler model, the soil is modeled as uniformly distributed, mutually independent, and linear elastic vertical springs which produce distributed reactions in the direction of the deflection of the beam. The Winkler model requires a single parameter k , representing the soil properties. However, since the model does not take into account either continuity or cohesion of the soil, it may be considered as a rather crude representation of the soil.

As stated before pipelines, shallow foundations and piles can be modeled as a beam. Hence, produced results may find various application areas. There are also different beam types in theory. The well-known is the Euler-Bernoulli beam which is suitable for slender beams. For moderately short and thick

beams, the Timoshenko beam model has to be used in the analysis.

Vibration of a constant stiffness Euler beam on elastic foundation was studied previously by Balkaya et al. [1] and Ozturk and Coskun [2]. Balkaya et al. [1] used Differential Transform Method while Ozturk and Coskun [2] used HPM in their studies. Avramidis and Morfidis [3] studied bending of beams on three-parameter elastic foundation. De Rosa [4] analyzed free vibration of Timoshenko beams on two-parameter elastic foundation. Matsunaga [5] studied vibration and buckling of deep beam columns on two-parameter elastic foundations. El-Mously [6] determined fundamental frequencies of the Timoshenko beams mounted on the Pasternak foundation. Chen [7, 8] analyzed vibration of beam resting on an elastic foundation by the differential quadrature element method (DQEM). Coşkun [9] investigated the response of a finite beam on a tensionless Pasternak foundation subjected to a harmonic load. Chen et al. [10] developed a mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation. Maheshwari et al. [11] studied the response of beams on a tensionless extensible geosynthetic-reinforced earth bed subjected to moving loads. Auciello and De Rosa [12] developed different

FIGURE 1: Representation of a beam on the Winkler foundation.

approaches to the dynamic analysis of beams on soils subjected to subtangential forces.

2. The Equations of Motion and Boundary Conditions

An Euler beam resting on the Winkler foundation shown in Figure 1 is considered in this study. The equation of motion for this problem is given as follows:

$$\frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) + k(x) w + \rho A(x) \left(\frac{\partial^2 w}{\partial t^2} \right) = 0, \quad (1)$$

where k is the spring constant, w is deflection, ρ is the mass density, A is the cross-sectional area, EI is the beam stiffness, and I is the area moment of inertia about the neutral axis. The deflection is a function of both space and time, that is, $w = w(x, t)$ in which space variable x is measured along the length of the beam and t represents any particular instant of time.

Due to the support conditions at both ends of the beam, different conditions have to be imposed to the obtained solution to determine unknowns included in final approximation produced by HPM. These conditions are given as follows.

(a) For clamped-clamped beam the end conditions are

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, L. \quad (2)$$

(b) For cantilever beam (clamped-free) the end conditions are

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \quad (3a)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \quad \text{at } x = L. \quad (3b)$$

(c) For simply supported beam (pinned-pinned) the end conditions are

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, L. \quad (4)$$

(d) For clamped-simply supported (pinned) beam the end conditions are

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \quad (5a)$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = L. \quad (5b)$$

Now, free vibration analysis of the beams with variable stiffness resting on elastic foundations will be formulated.

A solution is assumed as the following form to formulate the analysis of the presented problem by the separation of variables

$$w(x, t) = W(x) e^{i\omega t}, \quad (6)$$

where ω is the circular frequency for the vibration. Substituting (6) into (1), equations of motion become as follows:

$$\frac{d^2}{dx^2} \left(EI(x) \frac{d^2 W}{dx^2} \right) + kW = \rho A \omega^2 W. \quad (7)$$

This equation can be rearranged as

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} + 2 \frac{EI(x)'}{EI(x)} \frac{\partial^3 w}{\partial x^3} + \frac{EI(x)''}{EI(x)} \frac{\partial^2 w}{\partial x^2} + \frac{k(x)}{EI(x)} w \\ + \frac{\rho A(x)}{EI(x)} \left(\frac{\partial^2 w}{\partial t^2} \right) = 0, \end{aligned} \quad (8)$$

where $(\cdot)'$ denotes total derivative with respect to x . The governing equation is now rewritten in a nondimensional form. This procedure is provided from [1] in which a constant stiffness beam was analyzed. The notation is maintained in this study for the comparison purposes. The nondimensional parameters for the Eulerbeam on the Winkler foundation are defined as [1]

$$\bar{x} = \frac{x}{L}, \quad \bar{W} = \frac{W}{L}, \quad \lambda = \frac{kL^4}{EI}, \quad \bar{\omega} = \omega \sqrt{\frac{\rho A}{k}}. \quad (9)$$

Using these parameters, nondimensional form of the equations and formulation procedures are explained in the following sections.

3. Homotopy Perturbation Method

In recent years, efforts towards application of analytical approximate solution techniques for nonlinear problems have increased. In these studies, He's Homotopy Perturbation Method [13–18] can be considered as one of the most promising methods for nonlinear problems. The HPM provides an analytical approximate expression as the solution for the problems which are continuous in the solution domain. The technique is applied to an equation of the form $L(u) + N(u) = f(r)$, $r \in \Omega$ with boundary conditions $B(u, \partial u / \partial n) = 0$, $r \in \Gamma$, where L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω , and $f(r)$ is a known analytic function. HPM defines a homotopy as $v(r, p) = \Omega \times [0, 1] \rightarrow R$ which satisfies the following inequalities:

$$\begin{aligned} H(v, p) = (1 - p) [L(v) - L(u_0)] \\ + p [L(v) + N(v) - f(r)] = 0 \end{aligned} \quad (10)$$

or

$$H(v, p) = L(v) - L(u_0) + p[L(u_0) + p[N(v) - f(r)]] = 0, \quad (11)$$

FIGURE 2: Beam with a linearly varying width.

FIGURE 3: Beam with an exponentially varying width.

where $r \in \Omega$ and $p \in [0, 1]$ are imbedding parameters, u_0 is an initial approximation which satisfies the boundary conditions. From (10)-(11), we have

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0, \\ H(v, 1) &= L(v) + N(v) - f(r) = 0. \end{aligned} \quad (12)$$

The changing process of p from zero to unity is that of $v(r, p)$ from u_0 to $u(r)$. In topology, this deformation $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are called homotopic. The method expresses the solution of (10)-(11) as a power series in p as follows:

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \dots \quad (13)$$

The approximate solution of $L(u) + N(u) = f(r)$, $r \in \Omega$ can be obtained as

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (14)$$

The convergence of the series in (14) has been proven in [13–18].

4. HPM Formulation and Solution Procedure for Presented Problem

In this study, a variable stiffness is assumed for the beam considered. The variation is applied by changing the width of the beam along its length. Two different variations in stiffness are assumed. These are linear and exponential variations.

In the linear variation, beam width varies in a linear manner along the beam's length which is shown in Figure 2.

The variable width is formulated as

$$b(x) = b_0(1 - \alpha x), \quad (15)$$

where the dimension of α is $[1/L]$. For the exponential variation, beam width varies exponentially along the beam's length as shown in Figure 3.

Variable width of the beam in this case is

$$b(x) = b_0 e^{-\alpha x}, \quad (16)$$

where α has the same dimension as in linear case. By the use of variable width, both cross-sectional area and moment of inertia are variable,

$$\begin{aligned} A(x) &= b(x)h, \\ I(x) &= b(x) \frac{h^3}{12}. \end{aligned} \quad (17)$$

In the following sections, formulation procedures are given separately for each case.

4.1. Formulation for Linear Variation. Employing (15) into (17), variable cross-sectional areas and variable stiffness become

$$\begin{aligned} A(x) &= b_0 h (1 - \alpha x) = A_0 (1 - \alpha x), \\ EI(x) &= E b_0 \frac{h^3}{12} (1 - \alpha x) = EI_0 (1 - \alpha x), \end{aligned} \quad (18)$$

where A_0 and I_0 are the cross-sectional area and moment of inertia of the section at the origin, respectively. Inserting (18) into (8)

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} - \frac{2\alpha EI_0}{EI_0(1 - \alpha x)} \frac{\partial^3 w}{\partial x^3} + \frac{k(x)}{EI_0(1 - \alpha x)} w \\ + \frac{\rho A_0}{EI_0} \left(\frac{\partial^2 w}{\partial t^2} \right) = 0. \end{aligned} \quad (19)$$

This equation can be rewritten as

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} - 2\alpha \left(\frac{1}{1 - \alpha x} \right) \frac{\partial^3 w}{\partial x^3} + \frac{k(x)}{EI_0} \left(\frac{1}{1 - \alpha x} \right) w \\ + \frac{\rho A_0}{EI_0} \left(\frac{\partial^2 w}{\partial t^2} \right) = 0. \end{aligned} \quad (20)$$

Employing (6)

$$W^{iv} - 2\alpha \xi(x) W''' + \frac{k(x)}{EI_0} \xi(x) W - \frac{\rho A_0}{EI_0} \omega^2 W = 0. \quad (21)$$

Equation (21) can be made nondimensional in view of (9) as follows:

$$\bar{W}^{iv} - 2\bar{\alpha}\xi(\bar{x})\bar{W}''' + \lambda(\xi(\bar{x}) - \bar{\omega}^2)W = 0, \quad (22)$$

where

$$\begin{aligned} \bar{x} &= \frac{x}{L}, & \bar{W} &= \frac{W}{L}, & \lambda &= \frac{kL^4}{EI_0}, & \bar{\omega} &= \omega\sqrt{\frac{\rho A_0}{k}}, \\ \xi(\bar{x}) &= \frac{1}{1 - \bar{\alpha}\bar{x}}, & \bar{\alpha} &= \alpha L. \end{aligned} \quad (23)$$

By the application of HPM, the following iteration algorithm is obtained:

$$\bar{W}_0^{iv} - u_0^{iv} = 0, \quad (24a)$$

$$\bar{W}_1^{iv} + u_0^{iv} - 2\bar{\alpha}\xi(\bar{x})\bar{W}_0''' + \lambda(\xi(\bar{x}) - \bar{\omega}^2)\bar{W}_0 = 0, \quad (24b)$$

$$\bar{W}_n^{iv} - 2\bar{\alpha}\xi(\bar{x})\bar{W}_{n-1}''' + \lambda(\xi(\bar{x}) - \bar{\omega}^2)\bar{W}_{n-1} = 0, \quad n \geq 2. \quad (24c)$$

4.2. Formulation for Exponential Variation. Employing (16) into (17), variable cross-sectional areas and variable stiffness become

$$\begin{aligned} A(x) &= b_0 h e^{-\alpha x} = A_0 e^{-\alpha x}, \\ EI(x) &= E \frac{h^3}{12} b_0 e^{-\alpha x} = EI_0 e^{-\alpha x}. \end{aligned} \quad (25)$$

Inserting (25) into (8)

$$\frac{\partial^4 w}{\partial x^4} - 2\alpha \frac{\partial^3 w}{\partial x^3} + \alpha^2 \frac{\partial^2 w}{\partial x^2} + \frac{k(x)}{EI_0 e^{-\alpha x}} w + \frac{\rho A_0}{EI_0} \left(\frac{\partial^2 w}{\partial t^2} \right) = 0. \quad (26)$$

Employing (6)

$$W^{iv} - 2\alpha W''' + \alpha^2 W'' + \frac{k(x)}{EI_0} e^{\alpha x} W - \frac{\rho A_0}{EI_0} \omega^2 W = 0. \quad (27)$$

Equation (21) can be made nondimensional in view of (9) as follows:

$$\bar{W}^{iv} - 2\bar{\alpha}\bar{W}''' + \bar{\alpha}^2 \bar{W}'' + \lambda(e^{\bar{\alpha}\bar{x}} - \bar{\omega}^2)W = 0. \quad (28)$$

By the application of HPM, the following iteration algorithm is obtained:

$$\bar{W}_0^{iv} - u_0^{iv} = 0, \quad (29a)$$

$$\bar{W}_1^{iv} + u_0^{iv} - 2\bar{\alpha}\bar{W}_0''' + \bar{\alpha}^2 \bar{W}_0'' + \lambda(e^{\bar{\alpha}\bar{x}} - \bar{\omega}^2)\bar{W}_0 = 0, \quad (29b)$$

$$\begin{aligned} \bar{W}_n^{iv} - 2\bar{\alpha}\bar{W}_{n-1}''' + \bar{\alpha}^2 \bar{W}_{n-1}'' + \lambda(e^{\bar{\alpha}\bar{x}} - \bar{\omega}^2)\bar{W}_{n-1} &= 0, \\ n &\geq 2. \end{aligned} \quad (29c)$$

4.3. Solution Procedure. An initial approximation may be chosen as a cubic polynomial with four unknown coefficients. There exist four boundary conditions, that is, two at each end of the column, in the presented problem. Hence, an initial approximation of the following choice may be employed:

$$\bar{W}_0 = A\bar{x}^3 + B\bar{x}^2 + C\bar{x} + D. \quad (30)$$

In the computations, twenty iterations are conducted and four boundary conditions for each case are rewritten by using the final approximation of iteration. Each boundary condition produces an equation containing four unknowns spread from the initial approximation. These nondimensional boundary conditions are as follows.

Clamped-clamped beam:

$$\bar{W} = \frac{d\bar{W}}{d\bar{x}} = 0 \quad \text{at } \bar{x} = 0, 1. \quad (31)$$

Clamped-free (cantilever) beam:

$$\bar{W} = \frac{d\bar{W}}{d\bar{x}} = 0 \quad \text{at } \bar{x} = 0, \quad (32a)$$

$$\frac{d^2\bar{W}}{d\bar{x}^2} = \frac{d^3\bar{W}}{d\bar{x}^3} = 0 \quad \text{at } \bar{x} = 1. \quad (32b)$$

Pinned-pinned (simply supported) beam:

$$\bar{W} = \frac{d^2\bar{W}}{d\bar{x}^2} = 0 \quad \text{at } \bar{x} = 0, 1. \quad (33)$$

Clamped-pinned beam:

$$\bar{W} = \frac{d\bar{W}}{d\bar{x}} = 0 \quad \text{at } \bar{x} = 0, \quad (34a)$$

$$\bar{W} = \frac{d^2\bar{W}}{d\bar{x}^2} = 0 \quad \text{at } \bar{x} = 1. \quad (34b)$$

Hence, four equations in four unknowns may be written with respect to the boundary conditions of the problem. These equations can be represented in matrix form as follows:

$$[M(\bar{\omega})]\{A\} = \{0\}, \quad (35)$$

where $\{A\} = \langle A \ B \ C \ D \rangle^T$. For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of coefficient matrix yields a characteristic equation in terms of $\bar{\omega}$. Positive real roots of this equation are the normalized free vibration frequencies for the case considered.

5. Numerical Results

5.1. Constant Stiffness Case. As the first example, Euler beam of constant stiffness (i.e., EI is constant) with different boundary conditions is investigated. For the sake of comparison, all the values are set to unity such as $I = E = A = \rho = 1$; hence, $\lambda = 1$, according to previous studies [7]. Both algorithms given for linear and exponential variations lead to constant stiffness when $\alpha = 0$.

TABLE 1: Normalized free vibration frequencies of simply supported beam resting on the Winkler foundation.

Method	ω_1	ω_2	ω_3	ω_4	ω_5
HPM	9.92014	39.4911	88.8321	157.9168	246.7421
DTM [1]	9.92014	39.4911	88.8321	—	—
DQEM [7]	9.92014	39.4913	89.4002	—	—
Exact solution [1]	9.92014	39.4911	88.8321	—	—

FIGURE 4: Variation of normalized first mode frequency with respect to normalized variation coefficient (linear variation in stiffness).

FIGURE 5: Variation of normalized second mode frequency with respect to normalized variation coefficient (linear variation in stiffness).

In Table 1, first three normalized free vibration frequencies for simply supported (pinned-pinned) beam are compared with previous available results in the literature and the exact solution. Excellent agreement is observed with the exact solution.

The first five natural frequencies for clamped-clamped beam and cantilever (clamped-free) beam are presented in Tables 2 and 3, respectively. They are compared with the available solutions.

Excellent agreement is observed with previous available results for both cantilever and clamped-clamped beams. Clamped-pinned beam was not included in the studies used for comparison. Hence, only the result obtained from HPM is tabulated for this case in Table 4.

As one can see, perfect agreement is obtained for constant stiffness case. This issue is mainly due to constant coefficient governing equation. In the following sections, variable stiffness cases are investigated.

5.2. Linearly Varying Stiffness. Linearly varying beam width results in a linearly varying flexural stiffness. The variation is based on parameter α . A number of case studies are conducted with respect to parameter α , and the results are given in Tables 5, 6, 7, and 8.

Variation of normalized free vibration frequencies ($\bar{\omega}$) with respect to nondimensional variation parameter $\bar{\alpha}$ for each beam are also given in Figures 4, 5, 6, 7, and 8.

5.3. Exponentially Varying Stiffness. Exponentially varying beam width results in an exponentially varying flexural

FIGURE 6: Variation of normalized third mode frequency with respect to normalized variation coefficient (linear variation in stiffness).

stiffness. The variation is again based on parameter α , and a number of case studies are also conducted with respect to parameter α . Results are given in Tables 9, 10, 11, and 12.

Variation of normalized free vibration frequencies ($\bar{\omega}$) with respect to nondimensional variation parameter $\bar{\alpha}$ for each beam are also given in Figures 9, 10, 11, 12, and 13.

TABLE 2: Normalized free vibration frequencies of clamped-clamped beam resting on the Winkler foundation.

Method	ω_1	ω_2	ω_3	ω_4	ω_5
HPM	22.3956	61.6809	120.908	199.862	298.557
DTM [1]	22.3733	61.6728	120.903	199.859	298.556
DQEM [7]	22.3956	61.6811	120.910	199.885	298.675

TABLE 3: Normalized free vibration frequencies of cantilever beam resting on the Winkler foundation.

Method	ω_1	ω_2	ω_3	ω_4	ω_5
HPM	3.65546	22.0572	61.7053	120.906	199.862
DTM [1]	3.65546	22.0572	61.7053	120.906	199.862
DQEM [7]	3.65544	22.0572	61.7057	120.911	199.894

TABLE 4: Normalized free vibration frequencies of clamped-pinned beam resting on the Winkler foundation.

Method	ω_1	ω_2	ω_3	ω_4	ω_5
HPM	15.451	49.975	104.253	178.273	272.033

FIGURE 7: Variation of normalized fourth mode frequency with respect to normalized variation coefficient (linear variation in stiffness).

6. Conclusions

In this study, HPM is introduced for the free vibration analysis of variable stiffness Euler beams on elastic foundations. As a demonstration of application of the method, firstly the case studies for which previous results were available are chosen. In these studies, constant stiffness Euler beams were considered, and first analyses are conducted for constant stiffness case for comparison with the available results. HPM produced results which are in excellent agreement with the previously available solutions that encourage the application of the method for variable stiffness beams. To represent a variation in stiffness, a rectangular beam with varying width is considered, and two types of variation are taken into consideration. These are, namely, linear and exponential changes. Such varying widths lead to linearly and exponentially varying stiffnesses. The analyses are expanded

FIGURE 8: Variation of normalized fifth mode frequency with respect to normalized variation coefficient (linear variation in stiffness).

FIGURE 9: Variation of normalized first mode frequency with respect to normalized variation coefficient (exponential variation in stiffness).

for variable stiffness cases. HPM also produced reasonable results for the vibration of variable stiffness beams which show the efficiency of the method. For the variable stiffness problems, the governing equation is a differential equation with variable coefficients, and it is not easy to obtain analytical

TABLE 5: Normalized free vibration frequencies of cantilever beam resting on Winkler foundation with linearly varying flexural stiffness.

α	0.00	0.10	0.20	0.30	0.40	0.50
ω_1	3.65546	3.77785	3.91814	4.08113	4.27363	4.50571
ω_2	22.0572	22.2779	22.5271	22.8129	23.1475	23.5506
ω_3	61.7053	61.9182	62.1616	62.4458	62.7867	63.2104
ω_4	120.9061	121.1194	121.3645	121.6528	122.0024	122.4434
ω_5	199.8620	200.0755	200.3213	200.6117	200.9661	201.4172

TABLE 6: Normalized free vibration frequencies of clamped-pinned beam resting on the Winkler foundation with linearly varying flexural stiffness.

α	0.00	0.10	0.20	0.30	0.40	0.50
ω_1	15.4506	15.5615	15.6801	15.8069	15.9427	16.0879
ω_2	49.9749	50.0781	50.1878	50.3052	50.4316	50.5685
ω_3	104.2525	104.3556	104.4655	104.5836	104.7121	104.8544
ω_4	178.2725	178.3756	178.4853	178.6036	178.7333	178.8785
ω_5	272.0328	272.1358	272.2455	272.3640	272.4944	272.6418

TABLE 7: Normalized free vibration frequencies of clamped-clamped beam resting on the Winkler foundation with linearly varying flexural stiffness.

α	0.00	0.10	0.20	0.30	0.40	0.50
ω_1	22.3956	22.3922	22.3777	22.3477	22.2958	22.2120
ω_2	61.6809	61.6752	61.6541	61.6114	61.5380	61.4183
ω_3	120.9075	120.9009	120.8775	120.8302	120.7485	120.6144
ω_4	199.8620	199.8549	199.8302	199.7803	199.6939	199.5512
ω_5	298.5572	298.5499	298.5243	298.4729	298.3834	298.2351

TABLE 8: Normalized free vibration frequencies of simply supported beam resting on the Winkler foundation with linearly varying flexural stiffness.

α	0.00	0.10	0.20	0.30	0.40	0.50
ω_1	9.9201	9.9217	9.9210	9.9170	9.9084	9.8932
ω_2	39.4911	39.4928	39.4970	39.5047	39.5166	39.5340
ω_3	88.8321	88.8340	88.8398	88.8511	88.8697	88.8986
ω_4	157.9168	157.9189	157.9257	157.9389	157.9612	157.9966
ω_5	246.7421	246.7443	246.7516	246.7661	246.7907	246.8302

TABLE 9: Normalized free vibration frequencies of cantilever beam resting on the Winkler foundation with exponentially varying flexural stiffness.

α	0.00	0.25	0.50	0.75	1.00
ω_1	3.65546	3.99564	4.46148	5.08383	5.89198
ω_2	22.0572	22.6410	23.4038	24.4544	25.9407
ω_3	61.7053	62.2689	63.0286	64.1348	65.8187
ω_4	120.9061	121.4706	122.2407	123.3890	125.1956
ω_5	199.8620	200.4266	201.2020	202.3738	204.2517

TABLE 10: Normalized free vibration frequencies of clamped-pinned beam resting on the Winkler foundation with exponentially varying flexural stiffness.

α	0.00	0.25	0.50	0.75	1.00
ω_1	15.4506	15.7228	16.0355	16.4684	17.1543
ω_2	49.9749	50.2233	50.5095	50.9401	51.7103
ω_3	104.2525	104.4999	104.7874	105.2357	106.0716
ω_4	178.2725	178.5190	178.8070	179.2648	180.1380
ω_5	272.0328	272.2787	272.5671	273.0313	273.9289

TABLE 11: Normalized free vibration frequencies of clamped-clamped beam resting on the Winkler foundation with exponentially varying flexural stiffness.

α	0.00	0.25	0.50	0.75	1.00
ω_1	22.3956	22.3562	22.2409	22.0853	21.9787
ω_2	61.6809	61.6245	61.4646	61.2498	61.1004
ω_3	120.9075	120.8450	120.6698	120.4372	120.2857
ω_4	199.8620	199.7961	199.6124	199.3705	199.2211
ω_5	298.5572	298.4893	298.3002	298.0528	297.9060

TABLE 12: Normalized free vibration frequencies of simply supported beam resting on the Winkler foundation with exponentially varying flexural stiffness.

α	0.00	0.25	0.50	0.75	1.00
ω_1	9.9201	9.8928	9.8179	9.7493	9.7849
ω_2	39.4911	39.4736	39.4584	39.5456	39.9202
ω_3	88.8321	88.8182	88.8204	88.9564	89.4478
ω_4	157.9168	157.9048	157.9164	158.0782	158.6327
ω_5	246.7421	246.7312	246.7485	246.9262	247.5200

FIGURE 10: Variation of normalized second mode frequency with respect to normalized variation coefficient (exponential variation in stiffness).

FIGURE 12: Variation of normalized fourth mode frequency with respect to normalized variation coefficient (exponential variation in stiffness).

FIGURE 11: Variation of normalized third mode frequency with respect to normalized variation coefficient (exponential variation in stiffness).

FIGURE 13: Variation of normalized fifth mode frequency with respect to normalized variation coefficient (exponential variation in stiffness).

solutions for these types of problems. However, it is easy to put those variable parameters into the iteration algorithm of HPM, and the results can be obtained after performing some iterations with the method. The results obtained in this study point out that the proposed method is a powerful and reliable method in the analysis of the presented problem.

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