Research Article

Wild Fluctuations of Random Functions with the Pareto Distribution

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This paper provides the fluctuation analysis of random functions with the Pareto distribution. By the introduced concept of wild fluctuations, we give an alternative way to classify the fluctuations from those with light-tailed distributions. Moreover, the suggested term wildest fluctuation may be used to classify random functions with infinite variance from those with finite variances.

1. Introduction

The Pareto distribution is typically a type of heavy-tailed distributions gaining research interests and applications in many fields of sciences and technologies, ranging from financial engineering to geosciences; see, for example, [1–9]. By heavy tail of a probability distribution density (PDF) function, we mean that the PDF is in the form of a certain power function instead of exponential functions, such as the Poisson distribution and Gaussian distribution. The subject of heavy-tailed PDFs, including the Pareto one, may be in the field of power laws, which has been attracted by researchers in various fields; see, for example, [10–16].

Though Pareto reported his distribution in 1895 [17, 18] from a view of economics, its applications to the other fields are widely reported from a view of fractals in particular. Due to the fact that the Pareto distribution plays a role in so many areas, we give an introductory description about it from a view of its fluctuations in comparison with some common PDFs, such as Gaussian distribution.

The remainder of this paper is organized as follows. Section 2 briefs the study background. The wild fluctuations of random functions with the Pareto distribution are discussed in Section 3. Conclusions are given in Section 4.

2. Background

Denote by $p(x)$ the PDF of a second-order random function (random function for short) $x(t)$. Then, its mean, that is, its first moment, and its variance, that is, its second central moment, play a role in characterizing $x(t)$.

Without generality losing in the discussions, we assume that $x(t)$ is stationary. Let μ and σ^2 be the mean and the variance of $x(t)$. Then, μ and σ^2 are expressed by (1) and (2), respectively;

$$\mu = E[x(t)] = \int_{-\infty}^{\infty} xp(x)dx, \quad (1)$$

$$\sigma^2 = E\{[x(t) - \mu]^2\} = \int_{-\infty}^{\infty} [x(t) - \mu]^2 p(x)dx. \quad (2)$$

Note that μ represents the average value around which $x(t)$ fluctuates. On the other side, σ^2 is a parameter for measuring the dispersion or fluctuation of $x(t)$ around μ . Two parameters are essential. For instance, in the field of measurements, an accurate measurement implies that the variance of that measurement should be small [19–21].

Now, we consider two random functions $x(t)$ and $y(t)$. Suppose that their means are equal. Denote by σ_x^2 and σ_y^2 the

variances of $x(t)$ and $y(t)$, respectively. Then, in engineering, one may say that $x(t)$ is more random than $y(t)$ if

$$\sigma_x^2 > \sigma_y^2. \quad (3)$$

In other words, one may also say that $x(t)$ is more diverse than $y(t)$ if (3) holds [21, 22]. Using the term fluctuation, one may say that the fluctuation range of $x(t)$ is larger than that of $y(t)$ when (3) holds. As a matter of fact, variance analysis plays a role in statistics [23, 24].

Note that if $x(t)$ is Gaussian, its PDF is uniquely determined by its μ and σ^2 because

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty. \quad (4)$$

The particularly useful result by using σ^2 can be explained as follows. Though $-\infty < x < \infty$ in general, the fluctuation of a Gaussian random function $x(t)$ can be simply determined with a certain probability. For example, it is well known that, with probability 95%, the fluctuation interval of $x(t)$ is given by

$$-(\mu - 3\sigma) < x < \mu + 3\sigma. \quad (5)$$

It is obvious that the tool of variance analysis may work if variances to be studied exist. In fact, one may use (3) to identify whether the fluctuation of $x(t)$ is more severe than that of $y(t)$ if both variances of $x(t)$ and $y(t)$ exist.

3. Fluctuation Analysis of Random Function with the Pareto PDF

Recall that the necessary and sufficient condition for a function $p(x)$ to be a PDF is $p(x) \geq 0$ and

$$\int_{-\infty}^{\infty} p(x) dx = 1. \quad (6)$$

Denote by $p_{\text{Pareto}}(x)$ the PDF of the Pareto distribution. Then,

$$p_{\text{Pareto}}(x) = \begin{cases} \frac{ab^a}{x^{a+1}}, & x \geq b, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

In the above, a and b are positive parameters (<http://math-world.wolfram.com/ParetoDistribution.html>).

Note 1 (heavy tail). The function $p_{\text{Pareto}}(x)$ decays hyperbolically. Hence, heavy tail is compared to PDFs that are exponentially decayed.

The mean and variance of $x(t)$ that follows $p_{\text{Pareto}}(x)$ are, respectively, given by

$$\mu_{\text{Pareto}} = \int_b^{\infty} x p_{\text{Pareto}}(x) dx = \frac{ab}{a-1}, \quad (8)$$

$$\text{Var}(x)_{\text{Pareto}} = \frac{ab^2}{(a-1)^2(a-2)}. \quad (9)$$

Note 2 (infinite variance). If $a \rightarrow 1$ or $a \rightarrow 2$, $\text{Var}(x)_{\text{Pareto}} \rightarrow \infty$ as can be seen from (9).

Since the heavy tails of random functions imply their larger fluctuation ranges than those with light tails, that is, exponential type distributions, such as the Gaussian or Poisson distribution, we specifically, though informal, introduce a term “wild fluctuation,” in comparison with those with light tails. In addition, because infinite invariance implies that the fluctuation range of a random function is infinite, we, though informal again, introduce another term “wildest fluctuation,” in comparison with those with finite variances.

Remark 1 (wildest fluctuation). The fluctuation of a random function $x(t)$ that follows $p_{\text{Pareto}}(x)$ may be wildest if $a \rightarrow 1$ or $a \rightarrow 2$.

Case Study 1. Suppose that there are two different random functions $x(t)$ and $y(t)$. Both obey the Pareto distribution. When $x(t)$ is with $a = 1$ while $y(t)$ is with $a = 2$, we have $\text{Var}(x) \rightarrow \infty$ and $\text{Var}(y) \rightarrow \infty$. In this case, one may fail to identify whether the fluctuation of $x(t)$ is more severe than that of $y(t)$ based on the tool of variance analysis. More precisely, variance analysis that plays a role in conventional statistics fails to be used for the fluctuation analysis of random functions with infinite variance.

4. Conclusions

We have explained our introduction of the term wild fluctuation and wildest one by using the Pareto distributions. Though the present analysis is based on the Pareto distribution, it may yet be an alternative material to shortly describe the fact that caution should be paid to variance analysis of a random function with a heavy-tailed distribution unless its variance exists.

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