Research Article

An Extended Auxiliary Function Method and Its Application in mKdV Equation

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An extended auxiliary function method is presented for constructing exact traveling wave solutions to nonlinear partial differential equations. The main idea of this method is to take full advantage of the solutions to the elliptic equation to construct exact traveling wave solutions for nonlinear partial differential equations. mKdV equation is chosen to illustrate the application of the extended auxiliary function method. Consequently, more new exact traveling wave solutions are derived that are not obtained by the previously known methods.

1. Introduction

The investigation of exact solutions, in particular solitons, for nonlinear mathematical physics equations is an important and interesting subject. Many authors have paid attention to this subject. And many effective methods have been presented, such as Bäcklund transformation, Darboux transformation, variable separation approach, various tanh methods, Painlevé method, generalized hyperbolic-function method, homogeneous balance method, and similarity reduction method [1–17].

With the rapid development of computer algebraic systems like Maple or Mathematical, many powerful algebraic methods have been presented to obtain a great many exact solutions (especially traveling wave solutions) for nonlinear partial differential equation (NLPDE), for example, sine-cosine method [18], Fu's method [19], the hyperbolic tangent function expansion method [8, 20], Riccati equation expansion method [9, 21], the generalized Riccati equation method [22], auxiliary ordinary differential equation method [23], modified extended tanh method [24], Fan's unified algebraic method [25], the extended Fan's subequation method [26], and so on [26–36]. Generally speaking, these methods have a common characteristic: the possible solutions are

constructed from the known functions or the solutions for some simple and solvable nonlinear ordinary differential equations such as sine-cosine function, tanh function, Jacobi elliptic function or the Riccati equation, the coupled Riccati equation, and sinh-Gordon equation. Recently, Xu presents the elliptic equation expansion method to construct new and more general solutions of NLPDE [37]. The main purpose of this paper is to summarize and extend Xu's method to unified basic frame so that it can be used to obtain more exact traveling wave solutions for NLPDE. For illustration, we apply the extended method to the modified Korteweg-de Vries (mKdV) equation and get successfully many new exact solutions.

This paper is organized as follows. In Section 2, we give the main steps of our extended algorithm for computing exact solutions of NLPDE. In Section 3, many families of traveling wave solutions of mKdV equation are obtained. Finally, some conclusions and discussions are provided in Section 4.

2. The Extended Elliptic Equation Method and Its Algorithm

In the following we would like to outline the main steps of our method.

Step 1 (reduce the nonlinear PDE to the nonlinear ODE). For a Given nonlinear partial differential equation with the physical field $u(x, t)$ in two variables x, t

$$\Delta(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0. \quad (1)$$

We consider its traveling wave solution in the form

$$u(x, t) = U(\xi), \quad \xi = k(x - ct), \quad (2)$$

where k and c are constants to be determined later. Substituting (2) into (1) gives rise to a nonlinear ordinary differential equation with constant coefficients

$$\Theta(U, U', U'', \dots) = 0. \quad (3)$$

Step 2 (set the series of formal solutions). To seek the traveling wave solutions to (3), in [37] Xu assumes that (3) has the solutions in the form of

$$U(\xi) = a_0 + \sum_{i=1}^n a_i \varphi^i(\xi), \quad (4)$$

with the new variable $\varphi(\xi)$ satisfying the elliptic equation

$$\varphi'^2 = A\varphi(\varphi - \alpha)(\varphi - \beta)(\varphi - \gamma), \quad (5)$$

where A, α, β , and γ are constants and n is the integer to be determined later.

In order to get more exact solutions for (3), here we assume that (3) has the following formal solutions:

$$U(\xi) = a_0 + \sum_{i=1}^n [a_i \varphi^i(\xi) + b_i \varphi^{i-1}(\xi) \varphi'(\xi) + c_i \varphi^{-i}(\xi) + d_i \varphi^{-i}(\xi) \varphi'(\xi)]. \quad (6)$$

Step 3 (determine the truncation expansion terms in (6)). In order to obtain the value of n in (6). We define the degree of $U(\xi)$ as $D[U(\xi)] = n$ which gives rise to the degrees of other expressions as

$$D\left[\frac{d^q U}{d\xi^q}\right] = n + q, \quad D\left[U^p \left(\frac{d^q U}{d\xi^q}\right)^s\right] = np + s(n + q). \quad (7)$$

Therefore we can get the value of n in (6) by balancing the highest-order contributions from such terms in (1) or (3). If n is a nonnegative integer, we first make the transformation $U = w^n$.

Step 4 (lead to the set of algebraic equations). We substitute (6) along with (5) into (3) and then set to zero the coefficients of like powers of $\varphi^i \varphi'^j$ ($i = 0, 1, 2, \dots; j = 0, 1$) to obtain a set of nonlinear algebraic equations with respect to the unknowns a_0, a_i, b_i, c_i , and d_i ($i = 1, 2, \dots$).

Step 5 (solve the algebraic equations). By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system Maple, we can get these unknowns a_0, a_i, b_i, c_i and d_i ($i = 1, 2, \dots$).

Step 6 (solve (5)). Motivated by [37], we obtain some special solutions for (5) which are now listed in the following.

Case 1. If $A > 0$ and $\alpha < 0 < \beta < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\alpha \beta s n^2 (\alpha_1 \xi)}{\alpha - \beta c n^2 (\alpha_1 \xi)}, \quad \alpha_1 = \frac{1}{2} \sqrt{A\gamma(\beta - \alpha)}, \quad (8)$$

$$m = \sqrt{\frac{\beta(\gamma - \alpha)}{\gamma(\beta - \alpha)}}.$$

Case 2. If $A > 0$ and $\alpha < \beta < 0 < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\alpha \gamma s n^2 (\alpha_1 \xi)}{\gamma - \alpha c n^2 (\alpha_1 \xi)}, \quad \alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \quad (9)$$

$$m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}}.$$

Case 3. If $A < 0$ and $0 < \alpha < \beta < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\alpha(\gamma - \beta) s n^2 (\alpha_1 \xi) - \beta(\gamma - \alpha)}{(\gamma - \beta) s n^2 (\alpha_1 \xi) - (\gamma - \alpha)}, \quad (10)$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)},$$

$$m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}}.$$

Case 4. If $A < 0$ and $\alpha < 0 < \beta < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\beta \gamma}{\gamma + (\beta - \gamma) s n^2 (\alpha_1 \xi)}, \quad (11)$$

$$\alpha_1 = \frac{1}{2} \sqrt{-A\gamma(\beta - \alpha)},$$

$$m = \sqrt{\frac{\alpha(\beta - \gamma)}{\gamma(\beta - \alpha)}}.$$

Case 5. If $A < 0$ and $\alpha < \beta < 0 < \gamma$, (5) has the solution

$$\varphi(\xi) = \frac{\beta \gamma s n^2 (\alpha_1 \xi)}{\beta - \gamma c n^2 (\alpha_1 \xi)}, \quad (12)$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\alpha(\gamma - \beta)},$$

$$m = \sqrt{\frac{\gamma(\alpha - \beta)}{\alpha(\gamma - \beta)}}.$$

Step 7 (obtain the exact solutions to (3)). By inserting each solution to the above set of algebraic equations into (6) and making use of the solutions (8)–(12), some new types of elliptic function solutions for (3) can be obtained.

Remark 1. The method proposed here is more general than Xu's method for finding exact solutions for NLPDE. When $b_i = 0, c_i = 0$ and $d_i = 0$ ($i = 1, 2, \dots$), our method can degenerate to Xu's method.

3. The Application of the Elliptic Equation Expansion Method

In the following, we will apply the extended elliptic equation method to construct new explicit solutions of mKdV equation [1]:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \quad (13)$$

where α and β are constants.

The mKdV equation is of important significance in many branches of nonlinear science field. When $\alpha = 6$, (13) is called the positive mKdV equation, while $\alpha = -6$, (13) is called the negative mKdV equation. The mKdV equation appears in many fields such as acoustic waves in certain anharmonic lattices, Alfvén waves in a collisionless plasma, transmission lines in Schottky barrier, models of traffic congestion, ion acoustic solitons, elastic media, and so forth. It possesses many remarkable properties such as miura transformation, conservation laws, inverse scattering transformation, bilinear transformation, N-solitons, breather solutions, Bäcklund transformation, Painlevé integrability, Darboux transformation, and doublyperiodic solutions.

According to Step 1 in Section 2, in order to look for the traveling wave solutions for (13), we make the following traveling wave transformation:

$$u(x, t) = U(\xi), \quad \xi = k(x - ct), \quad (14)$$

and substitute (14) into (13). Then (13) changes into the form

$$-cU' + \alpha U^2 U' + \beta k^2 U''' = 0. \quad (15)$$

According to Steps 2 and 3 in Section 2, by balancing the highest-order partial derivative term U''' with the highest nonlinear term $U^2 U'$ in (15), we can obtain $n = 1$. So we have

$$U = a_0 + a_1 \varphi + b_1 \varphi' + \frac{c_1}{\varphi} + d_1 \frac{\varphi'}{\varphi}, \quad (16)$$

where a_0, a_1, b_1, c_1 and d_1 are constants to be determined later and φ satisfies (5).

According to Step 4 in Section 2, with the aid of Maple, substituting (16) into (15) yields a set of algebraic equations for $\varphi^i \varphi'^j (i = -1, 0, 1; j = 0, 1)$. Setting the coefficients of these terms $\varphi^i \varphi'^j$ of the resulting system's numerator to zero yields

a set of overdetermined algebraic equations with respect to the unknowns a_0, a_1, b_1, c_1 , and d_1 :

$$\begin{aligned} \frac{\alpha}{3\beta k^2} b_1^3 A &= 0, & \frac{\alpha}{\beta k^2} c_1^2 d_1 &= 0, \\ \frac{\alpha}{\beta k^2} a_1 b_1^2 A &= 0, & \frac{\alpha}{3\beta k^2} c_1^3 &= 0, \\ -\frac{\alpha}{3\beta k^2} (-3b_1^2 A d_1 + b_1^3 A \beta + b_1^3 A \alpha + b_1^3 A \gamma) &= 0, \\ 6b_1 A - \frac{\alpha}{3\beta k^2} (-3a_1^2 b_1 - b_1^3 A \alpha \gamma - b_1^3 A \alpha \beta & \\ + 3b_1^2 A d_1 \beta + 3b_1^2 A d_1 \alpha - b_1^3 A \beta \gamma & \\ - 3b_1 A d_1^2 + 3b_1^2 A d_1 \gamma) &= 0, \\ 2d_1 A - 3h_3 a_1^2 d_1 - 3h_3 b_1^2 A d_1 \alpha \gamma - 3h_3 b_1^2 A d_1 \alpha \beta & \\ - 3h_3 b_1^2 A d_1 \beta \gamma - 3b_1 A \gamma - 3b_1 A \beta & \\ + h_3 b_1^3 A \alpha \beta \gamma + 3h_3 b_1 A d_1^2 \gamma - 3b_1 A \alpha & \\ + 3h_3 b_1 A d_1^2 \alpha - h_3 d_1^3 A & \\ - 6h_3 a_0 a_1 b_1 + 3h_3 b_1 A d_1^2 \beta &= 0, \\ -\frac{\alpha}{\beta k^2} (-a_0 b_1^2 A - 2a_1 b_1 A d_1 + a_1 b_1^2 A \alpha & \\ + a_1 b_1^2 A \gamma + a_1 b_1^2 A \beta) &= 0, \\ 2a_1 A - \frac{\alpha}{3\beta k^2} (6a_1 b_1 A d_1 \beta - 3a_1 b_1^2 A \beta \gamma - 3a_1 b_1^2 A \alpha \gamma & \\ - 3a_1 b_1^2 A \alpha \beta - 3b_1^2 A c_1 + 3a_0 b_1^2 A \alpha & \\ + 6a_1 b_1 A d_1 \alpha - a_1^3 - 3a_1 d_1^2 A + 3a_0 b_1^2 A \gamma & \\ - 6a_0 b_1 A d_1 + 6a_1 b_1 A d_1 \gamma + 3a_0 b_1^2 A \beta) &= 0, \\ -\frac{3}{2} a_1 A \alpha - \frac{3}{2} a_1 A \beta - \frac{3}{2} a_1 A \gamma & \\ -\frac{\alpha}{\beta k^2} (2a_0 b_1 A d_1 \alpha - 2a_1 b_1 A d_1 \alpha \beta - 2a_1 b_1 A d_1 \alpha \gamma & \\ - a_0 b_1^2 A \alpha \beta - 2b_1 A c_1 d_1 + b_1^2 A c_1 \alpha & \\ + 2a_0 b_1 A d_1 \beta - a_0 a_1^2 + b_1^2 A c_1 \beta & \\ - a_0 b_1^2 A \alpha \gamma - a_0 b_1^2 A \beta \gamma + 2a_0 b_1 A d_1 - a_0 d_1^2 A & \\ + b_1^2 A c_1 \gamma + a_1 d_1^2 A \beta + a_1 b_1^2 A \alpha \beta \gamma & \\ + a_1 d_1^2 A \gamma - 2a_1 b_1 A d_1 \beta \gamma + a_1 d_1^2 A \alpha) &= 0, \\ a_1 A (\alpha \beta + \beta \gamma + \alpha \gamma) - \frac{c}{\beta k^2} a_1 & \\ -\frac{\alpha}{\beta k^2} (2b_1 A c_1 d_1 \gamma + 2b_1 A c_1 d_1 \beta - a_0^2 a_1 + 2b_1 A c_1 d_1 \alpha & \end{aligned}$$

$$\begin{aligned}
& -a_1 d_1^2 A \beta \gamma - a_1 d_1^2 A \alpha \gamma - a_1 d_1^2 A \alpha \beta \\
& -b_1^2 A c_1 \beta \gamma - a_1^2 c_1 - b_1^2 A c_1 \alpha \gamma \\
& -b_1^2 A c_1 \alpha \beta + a_0 d_1^2 A \gamma + a_0 d_1^2 A \beta - c_1 d_1^2 A \\
& + a_0 d_1^2 A \alpha + 2a_1 b_1 A d_1 \alpha \beta \gamma + a_0 b_1^2 A \alpha \beta \gamma \\
& -2a_0 b_1 A d_1 \alpha \gamma - 2a_0 b_1 A d_1 \beta \gamma - 2a_0 b_1 A d_1 \alpha \beta = 0.
\end{aligned} \quad (17)$$

According to Step 5 in Section 2, by use of the Maple software package “Charsets” by Dongming Wang, which is based on the Wu-elimination method [38], solving the previous overdetermined algebraic equations, we can get the following three sets of explicit expressions for a_0 , a_1 , b_1 , c_1 , and d_1 .

Family 1. Consider

$$\begin{aligned}
a_0 &= \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma), \\
a_1 &= \pm \sqrt{-\frac{6\beta k^2 A}{\alpha}}, \\
b_1 &= 0, \quad c_1 = 0, \quad d_1 = 0, \\
\frac{c}{\beta k^2} &= -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2).
\end{aligned} \quad (18)$$

Family 2. Consider

$$\begin{aligned}
a_0 &= 0, \quad a_1 = 0, \\
b_1 &= 0, \quad c_1 = 0, \\
d_1 &= \pm \sqrt{-\frac{6\beta k^2}{\alpha}}
\end{aligned} \quad (19)$$

Family 3. Consider

$$\begin{aligned}
a_0 &= 0, \quad a_1 = \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha}}, \\
b_1 &= 0, \quad c_1 = 0, \quad d_1 = \pm \sqrt{-\frac{3\beta k^2}{2\alpha}}, \\
\frac{c}{\beta k^2} &= -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma).
\end{aligned} \quad (20)$$

According to Step 7 in Section 2, substituting (18)–(20) into (16), respectively, and combining Step 6 in Section 2, we get the following traveling wave solutions to (15).

Family 1. We obtained the following exact solutions:

$$\begin{aligned}
u_{11} &= \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma) \\
&\pm \sqrt{-\frac{6\beta k^2 A}{\alpha}} \frac{\alpha \beta \operatorname{sn}^2(\alpha_1 \xi)}{\alpha - \beta \operatorname{cn}^2(\alpha_1 \xi)}, \\
m &= \sqrt{\frac{\beta(\gamma - \alpha)}{\gamma(\beta - \alpha)}}, \\
\alpha_1 &= \frac{1}{2} \sqrt{A\gamma(\beta - \alpha)},
\end{aligned}$$

$$\frac{c}{\beta k^2} = -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2),$$

$$u_{12} = \pm \sqrt{-\frac{6\beta k^2}{\alpha}} \frac{2\alpha_1 (\alpha - \beta) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) [\alpha - \beta \operatorname{cn}^2(\alpha_1 \xi)]}, \quad (21)$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\gamma(\beta - \alpha)}, \quad m = \sqrt{\frac{\beta(\gamma - \alpha)}{\gamma(\beta - \alpha)}},$$

$$\begin{aligned}
u_{13} &= \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha}} \frac{\alpha \beta \operatorname{sn}^2(\alpha_1 \xi)}{\alpha - \beta \operatorname{cn}^2(\alpha_1 \xi)} \\
&\pm \sqrt{-\frac{3\beta k^2}{2\alpha}} \frac{\alpha_1 (\alpha - \beta) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) (\alpha - \beta \operatorname{cn}^2(\alpha_1 \xi))},
\end{aligned}$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\gamma(\beta - \alpha)}, \quad m = \sqrt{\frac{\beta(\gamma - \alpha)}{\gamma(\beta - \alpha)}},$$

$$\frac{c}{\beta k^2} = -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma),$$

where $\xi = k(x - ct)$, $A > 0$, and $\alpha < 0 < \beta < \gamma$.

Family 2. We obtained the following exact solutions:

$$\begin{aligned}
u_{21} &= \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma) \\
&\pm \sqrt{-\frac{6\beta k^2 A}{\alpha}} \frac{\alpha \gamma \operatorname{sn}^2(\alpha_1 \xi)}{\gamma - \alpha \operatorname{cn}^2(\alpha_1 \xi)}, \\
m &= \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}}, \\
\alpha_1 &= \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)},
\end{aligned}$$

$$\frac{c}{\beta k^2} = -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2),$$

$$u_{22} = \pm \sqrt{-\frac{6\beta k^2}{\alpha} \frac{2\alpha_1 (\gamma - \alpha) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) [\gamma - \alpha \operatorname{cn}^2(\alpha_1 \xi)]}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \quad m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}},$$

$$u_{23} = \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha} \frac{\alpha \gamma \operatorname{sn}^2(\alpha_1 \xi)}{\gamma - \alpha \operatorname{cn}^2(\alpha_1 \xi)}} \\ \pm \sqrt{-\frac{3\beta k^2}{2\alpha} \frac{2\alpha_1 (\gamma - \alpha) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) (\gamma - \alpha \operatorname{cn}^2(\alpha_1 \xi))}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \quad m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}},$$

$$\frac{c}{\beta k^2} = -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma), \quad (22)$$

where $\xi = k(x - ct)$, $A > 0$, and $\alpha < \beta < 0 < \gamma$.

Family 3. We obtained the following exact solutions:

$$u_{31} = \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma) \\ \pm \sqrt{-\frac{6\beta k^2}{\alpha} \frac{\alpha(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - \beta(\gamma - \alpha)}{(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - (\gamma - \alpha)}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \quad m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}},$$

$$\frac{c}{\beta k^2} = -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2),$$

u_{32}

$$= \pm \sqrt{-\frac{6\beta k^2}{\alpha}} \\ \times \frac{2\alpha_1 (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) \operatorname{sn}(\alpha_1 \xi) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{[\alpha(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - \beta(\gamma - \alpha)] [(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - (\gamma - \alpha)]},$$

$$\alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \quad m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}},$$

u_{33}

$$= \pm \sqrt{-\frac{3\beta k^2}{2\alpha}} \\ \times \frac{2\alpha_1 (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) \operatorname{sn}(\alpha_1 \xi) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{[\alpha(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - \beta(\gamma - \alpha)] [(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - (\gamma - \alpha)]} \\ \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha} \frac{\alpha(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - \beta(\gamma - \alpha)}{(\gamma - \beta) \operatorname{sn}^2(\alpha_1 \xi) - (\gamma - \alpha)}}, \\ \frac{c}{\beta k^2} = -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma), \quad \alpha_1 = \frac{1}{2} \sqrt{A\beta(\alpha - \gamma)}, \\ m = \sqrt{\frac{\alpha(\gamma - \beta)}{\beta(\gamma - \alpha)}}, \quad (23)$$

where $\xi = k(x - ct)$, $A < 0$ and $0 < \alpha < \beta < \gamma$.

Family 4. We obtained the following exact solutions:

$$u_{41} = \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma) \\ \pm \sqrt{-\frac{6\beta k^2 A}{\alpha} \frac{\beta\gamma}{\gamma + (\beta - \gamma) \operatorname{sn}^2(\alpha_1 \xi)}},$$

$$m = \sqrt{\frac{\alpha(\beta - \gamma)}{\gamma(\beta - \alpha)}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{-A\gamma(\beta - \alpha)},$$

$$\frac{c}{\beta k^2} = -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2),$$

$$u_{42} = \pm \sqrt{-\frac{6\beta k^2}{\alpha} \frac{2\alpha_1 (\gamma - \beta) \operatorname{sn}(\alpha_1 \xi) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\gamma + (\beta - \gamma) \operatorname{sn}^2(\alpha_1 \xi)}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{-A\gamma(\beta - \alpha)}, \quad m = \sqrt{\frac{\alpha(\beta - \gamma)}{\gamma(\beta - \alpha)}},$$

$$u_{43} = \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha} \frac{\beta\gamma}{\gamma + (\beta - \gamma) \operatorname{sn}^2(\alpha_1 \xi, m)}} \\ \pm \sqrt{-\frac{3\beta k^2}{2\alpha} \frac{2\alpha_1 (\gamma - \beta) \operatorname{sn}(\alpha_1 \xi) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\gamma + (\beta - \gamma) \operatorname{sn}^2(\alpha_1 \xi)}},$$

$$\alpha_1 = \frac{1}{2} \sqrt{-A\gamma(\beta - \alpha)}, \quad m = \sqrt{\frac{\alpha(\beta - \gamma)}{\gamma(\beta - \alpha)}},$$

$$\frac{c}{\beta k^2} = -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma),$$

(24)

where $\xi = k(x - ct)$, $A < 0$, and $\alpha < 0 < \beta < \gamma$.

Family 5. We obtained the following exact solutions:

$$\begin{aligned}
 u_{51} &= \mp \frac{1}{4} \sqrt{-\frac{6\beta k^2 A}{\alpha}} (\alpha + \beta + \gamma) \\
 &\quad \pm \sqrt{-\frac{6\beta k^2 A}{\alpha}} \frac{\beta \gamma \operatorname{sn}^2(\alpha_1 \xi)}{\beta - \gamma \operatorname{cn}^2(\alpha_1 \xi)}, \\
 m &= \sqrt{\frac{\gamma(\alpha - \beta)}{\alpha(\gamma - \beta)}}, \\
 \alpha_1 &= \frac{1}{2} \sqrt{A\alpha(\gamma - \beta)}, \\
 \frac{c}{\beta k^2} &= -\frac{A}{8} (3\beta^2 - 2\alpha\beta - 2\beta\gamma + 3\alpha^2 - 2\alpha\gamma + 3\gamma^2), \\
 u_{52} &= \pm \sqrt{-\frac{6\beta k^2 A}{\alpha}} \frac{2\alpha_1(\beta - \gamma) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) [\beta - \gamma \operatorname{cn}^2(\alpha_1 \xi)]}, \quad (25) \\
 \alpha_1 &= \frac{1}{2} \sqrt{A\alpha(\gamma - \beta)}, \quad m = \sqrt{\frac{\gamma(\alpha - \beta)}{\alpha(\gamma - \beta)}}, \\
 u_{53} &= \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha}} \frac{\beta \gamma \operatorname{sn}^2(\alpha_1 \xi)}{\beta - \gamma \operatorname{cn}^2(\alpha_1 \xi)} \\
 &\quad \pm \sqrt{-\frac{3\beta k^2 A}{2\alpha}} \frac{2\alpha_1(\beta - \gamma) \operatorname{cn}(\alpha_1 \xi) \operatorname{dn}(\alpha_1 \xi)}{\operatorname{sn}(\alpha_1 \xi) [\beta - \gamma \operatorname{cn}^2(\alpha_1 \xi)]}, \\
 \alpha_1 &= \frac{1}{2} \sqrt{A\alpha(\gamma - \beta)}, \quad m = \sqrt{\frac{\gamma(\alpha - \beta)}{\alpha(\gamma - \beta)}}, \\
 \frac{c}{\beta k^2} &= -\frac{1}{2} A (\alpha\beta + \beta\gamma + \alpha\gamma),
 \end{aligned}$$

where $\xi = k(x - ct)$, $A < 0$, and $\alpha < \beta < 0 < \gamma$.

Remark 2. The previous solutions obtained here, to our knowledge, are all new families of rational formal solutions to mKdV equation.

4. Conclusion and Discussions

In summary, we have presented the extended elliptic equation method and its algorithm. As a result, some new exact traveling wave solutions for the mKdV equation are obtained which may be useful for describing certain nonlinear physical phenomena. The method which we have proposed in this paper is a standard, direct, and computerized method, which allow us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDE in mathematical physics. In addition, a natural problem is that whether the extended elliptic equation method can be further extended to generate more types of solutions of NLPDE in nonlinear science. Hence, further study of the extended elliptic equation method is needed.

Conflict of Interests

The authors declare that they have no conflict of interests.

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