Research Article

Analytical and Semianalytical Treatment of the Collinear Points in the Photogravitational Relativistic RTBP

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The problem of restricted three bodies is considered. The treatment is carried out within the framework of the post-Newtonian approximation. The primaries also are assumed to be radiant sources. The locations of the collinear points are computed. Series forms of these locations are obtained as new results. A Mathematica program is constructed so as to draw the locations of collinear points versus the whole range of the mass ratio μ taking into account the photo-gravitational effects and/or the relativistic corrections. Analyses of these figures are addressed.

1. Introduction

The restricted three-body problem (in brief RTBP) is the most celebrated problem in celestial mechanics, wherein one is interested in the motion of a test particle, a body of negligible mass called infinitesimal mass in the presence of two massive bodies called primaries revolving around their common centre of mass. Attempts at its solution laid the foundation for dynamical systems theory and alerted Poincaré in the 1890s to the existence of deterministic chaos within Newtonian mechanics. There is no general analytical solution in terms of simple algebraic functions for the problem of three gravitating bodies of comparable masses. Except in a few very specific cases the problem has to be solved numerically. Are there any points where an infinitesimal mass could orbit around the centre of mass with the same period as the primaries? Euler found three such points, the collinear points, on the line joining the primaries. Lagrange found two additional points, the fourth and fifth triangular points. They are not collinear with m_1 and m_2 but are such that the three masses are at the corners of an equilateral triangle. The five points are called the equilibrium points.

These locations of the equilibrium points can be found by setting the first derivatives of the potential to zero. Note that, in the line joining the two masses, the equivalent potential at the collinear points is a maximum, and therefore these points are unstable; these points are actually saddle points. In several spacecrafts which are in orbit or are planned to be in orbit around the collinear points (e.g., SOHO at the interior collinear point and MAP at L_2), this instability has already used in consuming small expenditure of fuel which is presumably needed to keep them there.

The classical restricted three-body problem concerned with the determination of the locations of the libration points and investigating their linear/nonlinear stability. The RTBP becomes photo-gravitational when one or both of the masses of the primaries is an intense emitter of radiation, Kumar and Ishwar [1]. The restricted three body problem describes the dynamical system more accurately on account of realistic assumptions of the potential field and of the motion of the primaries, Narayan and Kumar [2].

The location and stability of the libration points in the problem of three bodies are subject to modifications when including one or more of additional forces in the potential

of the classical problem as the radiation pressure component of the radiation drag which is the next most powerful component after the gravitational forces, Schuerman [3]; the oblateness of the primaries, Oberti and Vienne [4]; the primaries orbits being elliptic, Kumar and Ishwar [5]; the relativistic corrections to RTBP Abd El-Salam and Abd El-Bar [6], Douskos and Perdios [7], and Ragos et al. [8]; the aerodynamic drag, Markakis et al. [9]; Douskos [10] and the eternal radiation pressure as the effect of the direct solar radiation pressure on the equilibrium points in the Earth-Moon system, Poynting-Robertson (P-R) drag, and so forth.

The literature is wealthy of works on the subject of the restricted three body problem in classical form since its eminent pioneer Lagrange to the very recent date when considering different sources of perturbations and it will be worth noting to sketch some of these most important as well as recent works: Sharma [11], Hallan and Rana [12], Sándor and Érdi [13], Floría [14], Markeev [15], Palacián et al. [16, 17], Bruno and Varin [18], Ammar [19], Narayan et al. [20–22], Kushvah and [23–25], Douskos [26], Shankaran et al. [27].

In this work, we consider the collinear equilibrium points of the relativistic RTBP amended with the photo-gravitational effects of the two primaries. We determine new formulas for approximate positions of the collinear points as series expansions. Terms are ordered in μ and $1/c^2$ under the considered perturbations.

The rest of the paper is organized as follows. In Section 2 we formulated the equations of motion of the infinitesimal mass in the relativistic RTBP in a synodic frame of reference taking into account the radiation of the primaries. In Section 3 we computed the perturbed locations of the collinear libration points. In Section 4 we designed a Mathematica program to draw the locations of collinear points L_i , $i = 1, 2, 3$, versus the whole range of the mass ratio μ taking into account the photogravitational effects and/or the relativistic corrections. In Section 5 we concluded our work and highlighted the differences between our work and previous works. Also we mentioned our forthcoming works for the sake of completeness.

2. Equations of Motion

The equations of motion of the infinitesimal mass in the relativistic RTBP in a synodic frame of reference (ξ, η) , in which the primary coordinates on the ξ -axis $(-\mu, 0)$, $(1-\mu, 0)$ are kept fixed and the origin at the center of mass, are given by Brumberg [28] and Bhatnagar and Hallan [29]:

$$\begin{aligned}\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial U}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{\xi}} \right), \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial U}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{\eta}} \right)\end{aligned}\quad (1)$$

with

$$\begin{aligned}n &= 1 + \frac{1}{2c^2} (\mu(1-\mu) - 3), \\ r &= \sqrt{(\xi^2 + \eta^2)},\end{aligned}\quad (2)$$

where U is the potential-like function of the relativistic RTBP, which can be written as composed of two components U_{c+ph} and U_{r+ph} , namely, on respective the potential of the classical RTBP gravitational potential and the relativistic terms corrected for the radiation of the primaries which lead to mass reduction:

$$U = U_{c+ph} + U_{r+ph}, \quad (3)$$

where U_{c+ph} and U_{r+ph} are given by

$$\begin{aligned}U_{c+ph} &= \frac{r^2}{2} + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}, \\ U_{r+ph} &= \frac{r^2}{2c^2} (\mu(1-\mu) - 3) + \frac{1}{8c^2} \\ &\times \left((\xi + \dot{\eta})^2 + (\eta - \dot{\xi})^2 \right)^2 \\ &+ \frac{3}{2c^2} \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \\ &\times \left((\xi + \dot{\eta})^2 + (\eta - \dot{\xi})^2 \right) \\ &- \frac{1}{2c^2} \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right)^2 - \frac{q_1q_2\mu(1-\mu)}{2c^2} \\ &\times \left[\frac{1}{r_1} + \left(\frac{1}{r_1} - \frac{1}{r_2} \right) (1 - 3\mu - 7\xi - 8\dot{\eta}) \right. \\ &\quad \left. + \eta^2 \left(\frac{q_2\mu}{r_1^3} + \frac{q_1(1-\mu)}{r_2^3} \right) \right], \\ r_1 &= \sqrt{(\xi + \mu)^2 + \eta^2}, \\ r_2 &= \sqrt{(\xi + \mu - 1)^2 + \eta^2},\end{aligned}\quad (4)$$

where $q_i = 1 - (F_p/F_g)_i$, $i = 1, 2$, is the mass reduction factor constant for a given particle. F_p is the force due to radiation pressure, and F_g is the force due to gravitational field, Sharma et al. [30], Simmons et al. [31], and Chernikov [32] with a modification to account for the radiation of the primaries q_i .

3. Location of the Libration Points

From the equations of motion (1), it is apparent that an equilibrium solution exists relative to the rotating frame when the partial derivatives of the pseudopotential function (U_ξ, U_η, U_ζ) are all zero; that is, $U = \text{const}$. These points correspond to the positions in the rotating frame at which the gravitational forces and the centrifugal force associated with the rotation of the synodic frame all cancel, with the result that a particle located at one of these points appears stationary in the synodic frame. There are five equilibrium points in the circular RTBP, also known as Lagrange points or libration points. Three of them (the collinear points) lie along the ξ -axis: one interior point L_1 between the two primaries, and one point on the far side of each primary

FIGURE 1: The five Lagrangian points with Sun-Earth system as an example.

with respect to the barycenter L_2, L_3 . The other two libration points (the triangular points) are each located at the apex of an equilateral triangle formed with the primaries; see Figure 1. The triangular points are designated as L_4 and L_5 , with L_4 moving ahead of the ξ -axis and L_5 trailing ξ -axis, along the orbit of the small primary as the synodic frame rotates uniformly relative to the inertial frame (as shown in Figure 1).

Remark 1. All the five libration points lie in the ξ - η plane, that is, in the plane of motion.

The libration points are obtained from equations of motion after setting $\ddot{\xi} = \ddot{\eta} = \dot{\xi} = \dot{\eta} = 0$. These points represent particular solutions of equations of motion

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= \frac{\partial U_{c+ph}}{\partial \xi} + \frac{\partial U_{r+ph}}{\partial \xi} = 0, \\ \frac{\partial U}{\partial \eta} &= \frac{\partial U_{c+ph}}{\partial \eta} + \frac{\partial U_{r+ph}}{\partial \eta} = 0. \end{aligned} \quad (5)$$

The explicit required partial derivatives are (remembering that $U = U_{c+ph} + U_{r+ph}$)

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= \xi - \frac{q_1(1-\mu)(\xi+\mu)}{r_1^3} - \frac{q_2\mu(\xi+\mu-1)}{r_2^3} + \frac{1}{c^2} \\ &\times \left\{ (\mu - \mu^2 - 3)\xi + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \right. \\ &\times \left(\frac{q_1(1-\mu)(\xi+\mu)}{r_1^3} + \frac{q_2\mu(\xi+\mu-1)}{r_2^3} \right) \\ &+ \frac{1}{2}(\eta^2 + \xi^2)\xi - \frac{3}{2} \\ &\times \left(\frac{q_1(1-\mu)(\xi+\mu)}{r_1^3} + \frac{q_2\mu(\xi+\mu-1)}{r_2^3} \right) \\ &\times (\eta^2 + \xi^2) + 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \xi \end{aligned}$$

$$\begin{aligned} &- \frac{1}{2}q_1q_2\mu(1-\mu) \\ &\times \left[-\frac{1}{r_1^3}(\xi+\mu) \right. \\ &+ \left(\frac{(\xi+\mu)}{r_1^3} - \frac{(\xi+\mu-1)}{r_2^3} \right) \\ &\times (-1+3\mu+7\xi) - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) - 3\eta^2 \\ &\times \left. \left(\frac{q_2\mu(\xi+\mu)}{r_1^5} + \frac{q_1(1-\mu)(\xi+\mu-1)}{r_2^5} \right) \right] \Bigg\} \\ &= 0, \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{\partial U}{\partial \eta} &= \eta - \frac{q_1(1-\mu)}{r_1^3}\eta - \frac{q_2\mu}{r_2^3}\eta \\ &+ \frac{1}{c^2} \left\{ (\mu - \mu^2 - 3)\eta + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \right. \\ &\times \left(\frac{q_1(1-\mu)}{r_1^3} + \frac{q_2\mu}{r_2^3} \right) \eta + \frac{1}{2}(\xi^2 + \eta^2)\eta \\ &- \frac{3}{2}(\xi^2 + \eta^2) \left(\frac{q_1(1-\mu)}{r_1^3} + \frac{q_2\mu}{r_2^3} \right) \eta \\ &+ 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \eta + \frac{q_1q_2\mu(1-\mu)}{2}\eta \\ &+ \frac{1}{2}q_1q_2\mu(1-\mu)\eta \\ &\times \left[\left(-\frac{1}{r_1^3} + \frac{1}{r_2^3} \right) (-1+3\mu+7\xi) \right. \\ &+ 3 \left(\frac{q_1(1-\mu)}{r_2^5} + \frac{q_2\mu}{r_1^5} \right) \eta^2 \\ &+ \frac{1}{r_1^3} - 2 \left(\frac{q_2\mu}{r_1^3} + \frac{q_1(1-\mu)}{r_2^3} \right) \Bigg] \Bigg\} = 0. \end{aligned} \quad (7)$$

4. Location of Collinear Libration Points

Any point of the collinear points must, by definition, have $\zeta = \eta = 0$, and the solution of the classical RTBP satisfies

$$\begin{aligned} \sigma_1 r_1 + \sigma_2 r_2 &= 1, \\ r_1 &= \sigma_1(\xi + \mu), \\ r_2 &= -\sigma_2(\mu + \xi - 1), \end{aligned} \quad (8)$$

where (see Figures 2, 3, and 4) we have

$$\begin{aligned} L_1 : \sigma_1 &= 1, \sigma_2 = 1, \\ L_2 : \sigma_1 &= 1, \sigma_2 = -1, \\ L_3 : \sigma_1 &= -1, \sigma_2 = 1. \end{aligned} \quad (9)$$

4.1. *Location of L_1 .* Substituting from (8) with the corresponding values of $\sigma_1 = 1$ and $\sigma_2 = 1$ into (6) and noting that $\partial r_1 / \partial \xi = -\partial r_2 / \partial \xi = 1$, we get

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= \xi - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) + \frac{1}{c^2} \\ &\times \left\{ (\mu - \mu^2 - 3)\xi + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \right. \\ &\times \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &- \frac{3}{2} \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \xi^2 \\ &+ 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \xi + \frac{1}{2} \xi^3 \\ &- \frac{1}{2} q_1 q_2 \mu (1-\mu) \\ &\times \left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) (-1 + 3\mu + 7\xi) \right. \\ &\left. \left. - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0 \end{aligned} \quad (10)$$

which can be re-written as a function of r_1, r_2 as

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu - r_2 - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &+ \frac{1}{c^2} \left\{ \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \right. \\ &+ (\mu - \mu^2 - 3) \times (1 - \mu - r_2) \\ &+ \frac{1}{2} (1 - \mu - r_2)^3 - \frac{3}{2} \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &\times (1 - \mu - r_2)^2 + 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \end{aligned}$$

FIGURE 2: The Location of L_1 .

FIGURE 3: The Location of L_2 .

FIGURE 4: The Location of L_3 .

$$\begin{aligned} &\times (1 - \mu - r_2) - \frac{1}{2} q_1 q_2 \mu (1 - \mu) \\ &\times \left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \right. \\ &\times (-1 + 3\mu + 7(1 - \mu - r_2)) \\ &\left. \left. - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0. \end{aligned} \quad (11)$$

Then it may be reasonable in our case to assume that positions of the equilibrium points L_1 are the same as given by classical RTBP but perturbed due to the inclusion of the relativistic correction by quantities $\varepsilon_1 \equiv \mathcal{O}(1/c^2)$. See Soffel [33] and Douskos and Perdios [7].

The solution of the classical part of (10) including the photogravitational correction can be obtained easily using the potential $U_{c+ph} = q_1(1-\mu)(r_1^2+2/r_1)+q_2\mu(r_2^2+2/r_2)$ as follows:

$$\begin{aligned} \frac{\partial U_{c+ph}}{\partial \xi} &= \frac{\partial U_{c+ph}}{\partial r_1} \frac{\partial r}{\partial \xi} + \frac{\partial U_{c+ph}}{\partial r_2} \frac{\partial r_2}{\partial \xi} = 0, \\ \implies \frac{\partial U_{c+ph}}{\partial \xi} &= \frac{\partial U_{c+ph}}{\partial r_1} - \frac{\partial U_{c+ph}}{\partial r_2} = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial U_{c+ph}}{\partial r_1} &= \frac{\partial U_{c+ph}}{\partial r_2} \\ \Rightarrow \frac{q_2 \mu}{q_1 (1 - \mu)} &= \frac{(3r_2 - 3r_2^2 + r_2^3) r_2^2}{(1 - r_2)^2 (1 - r_2^3)}. \end{aligned} \quad (12)$$

If $(q_2 \mu)/(q_1 (1 - \mu))$ is small, this equation has a root in the vicinity of $r_2 = \alpha$, where $\alpha = ((q_2 \mu)/(3q_1 (1 - \mu)))^{1/3}$; this α can be written as

$$\alpha = r_2 \left(1 + \frac{1}{3} r_2 + \frac{1}{3} r_2^2 + \frac{53}{81} r_2^3 + \frac{101}{243} r_2^4 + \mathcal{O}(r_2^5) + \dots \right). \quad (13)$$

Let successive approximations of r_2 be r_{2i} :

$$\begin{aligned} r_{20} &= \alpha, \\ \alpha &= r_{21} \left(1 + \frac{1}{3} \alpha \right), \\ r_{21} &= \alpha \left(1 - \frac{1}{3} \alpha \right), \\ \alpha &= r_{22} \left[1 + \frac{1}{3} \alpha + \frac{2}{9} \alpha^2 + \dots \right], \\ &\vdots \end{aligned} \quad (14)$$

Then successive approximation yields

$$r_{24} = \alpha \left(1 - \frac{\alpha}{3} - \frac{\alpha^2}{9} + \frac{2}{27} \alpha^3 + \frac{2}{81} \alpha^4 + \mathcal{O}(\alpha^5) \right). \quad (15)$$

Let us set the following notation:

$$\begin{aligned} r_1 &= a_1 + \varepsilon_1, \\ r_2 &= b_1 - \varepsilon_1, \\ a_1 &= 1 - b_1, \end{aligned} \quad (16)$$

where a_1 and b_1 are unperturbed relativistically positions of r_1 and r_2 , respectively, and b_1 is given after some successive approximation by the relation

$$\begin{aligned} b_1 &= r_{24} \\ &= \alpha \left(1 - \frac{\alpha}{3} - \frac{\alpha^2}{9} + \frac{2}{27} \alpha^3 + \frac{2}{81} \alpha^4 + \mathcal{O}(\alpha^5) \right). \end{aligned} \quad (17)$$

Substituting from (16) into (11) and retaining the terms up to the first order in the small quantities ε_1 , we can get

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu - b_1 + \varepsilon_1 \\ &\quad - \frac{q_1 (1 - \mu)}{(1 - b_1)^2} \left(1 - \frac{2\varepsilon_1}{1 - b_1} \right) \\ &\quad + \frac{q_2 \mu}{b_1^2} \left(1 + \frac{2\varepsilon_1}{b_1} \right) + \frac{1}{c^2} \\ &\quad \times \left\{ (\mu - \mu^2 - 3)(1 - \mu - b_1) \right. \\ &\quad + \left(\frac{q_1 (1 - \mu)}{(1 - b_1)} + \frac{q_2 \mu}{b_1} \right) \left(\frac{q_1 (1 - \mu)}{(1 - b_1)^2} - \frac{q_2 \mu}{b_1^2} \right) \\ &\quad + \frac{1}{2} (1 - \mu - b_1)^3 - \frac{3}{2} \\ &\quad \times \left(\frac{q_1 (1 - \mu)}{(1 - b_1)^2} - \frac{q_2 \mu}{b_1^2} \right) \times (1 - \mu - b_1)^2 \\ &\quad + 3 \left(\frac{q_1 (1 - \mu)}{(1 - b_1)} + \frac{q_2 \mu}{b_1} \right) \\ &\quad \times (1 - \mu - b_1) - \frac{1}{2} q_1 q_2 \mu (1 - \mu) \\ &\quad \times \left[-\frac{1}{(1 - b_1)^2} - 7 \left(\frac{1}{(1 - b_1)} - \frac{1}{b_1} \right) \right. \\ &\quad + \left. \left(\frac{1}{(1 - b_1)^2} + \frac{1}{b_1^2} \right) \right. \\ &\quad \left. \left. \times (6 - 7b_1 - 4\mu) \right] \right\} = 0. \end{aligned} \quad (18)$$

Equation (18) can be solved for ε_1 ; then substituting back into (16) and (8) yields, after some lengthy algebraic manipulations, the location of L_1 as

$$\begin{aligned} \xi_{0,L_1} &= 1 - \mu \\ &\quad - \sum_{k=0}^5 \left[\mathcal{M}_k^{L_1} + \left(\frac{1}{c^2} \right) \mathcal{N}_k^{L_1} \right] \left(\frac{\mu}{3} \right)^{k/3} + \mathcal{O} \left(\frac{\mu}{3} \right)^2, \end{aligned} \quad (19)$$

where the nonvanishing coefficients $\mathcal{M}_k^{L_1}$ and $\mathcal{N}_k^{L_1}$, are collected in Appendix A.

4.2. Location of L_2 . The collinear point L_2 is located outside the small massive primary of mass μ . We now drive an approximate location for this point under the considered perturbations. Substituting from (8) with the corresponding

values of $\sigma_1 = 1$ and $\sigma_2 = -1$ into (6) and noting that $\partial r_1 / \partial \xi = \partial r_2 / \partial \xi = 1$, we get

$$\begin{aligned} r_1 - r_2 &= 1, \\ r_1 &= \xi + \mu, \\ r_2 &= \xi + \mu - 1, \\ \frac{\partial r_1}{\partial \xi} &= \frac{\partial r_2}{\partial \xi} = 1. \end{aligned} \quad (20)$$

Substituting from (20) into (6), we get

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= \xi - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) + \frac{1}{c^2} \\ &\times \left\{ (\mu - \mu^2 - 3)\xi + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \right. \\ &\times \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &- \frac{3}{2} \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \xi^2 \\ &+ 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \xi + \frac{1}{2} \xi^3 \\ &- \frac{1}{2} q_1 q_2 \mu (1-\mu) \\ &\times \left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \right. \\ &\times \left. \left. (-1 + 3\mu + 7\xi) - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0 \end{aligned} \quad (21)$$

which can be rewritten as a function of r_1 and r_2 as

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu + r_2 - \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &+ \frac{1}{c^2} \left\{ (\mu - \mu^2 - 3)\xi + \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \right. \\ &\times \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \\ &- \frac{3}{2} \left(\frac{q_1(1-\mu)}{r_1^2} - \frac{q_2\mu}{r_2^2} \right) \xi^2 \\ &+ 3 \left(\frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} \right) \xi + \frac{1}{2} \xi^3 \\ &- \frac{1}{2} q_1 q_2 \mu (1-\mu) \\ &\times \left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \right. \\ &\times \left. \left. (-1 + 3\mu + 7\xi) - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0 \end{aligned}$$

$$\begin{aligned} &- \frac{1}{2} q_1 q_2 \mu (1-\mu) \\ &\times \left[-\frac{1}{r_1^2} + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \right. \\ &\times \left. \left. (-1 + 3\mu + 7\xi) - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0. \end{aligned} \quad (22)$$

Then, it may be reasonable in our case to assume that the position of the equilibrium point L_2 is the same as given by classical RTBP but perturbed due to the inclusion of the relativistic correction by quantities $\varepsilon_2 \equiv \mathcal{O}(1/c^2)$:

$$\begin{aligned} r_1 &= a_2 + \varepsilon_2, \\ r_2 &= b_2 + \varepsilon_2, \\ a_2 &= 1 + b_2 \end{aligned} \quad (23)$$

with a_2 and b_2 being the unperturbed positions of r_1 and r_2 , respectively, where b_2 is given after some successive approximation as

$$\begin{aligned} b_2 &= \alpha \left(1 + \frac{\alpha}{3} - \frac{\alpha^2}{9} - \frac{2}{27} \alpha^3 + \frac{2}{81} \alpha^4 + \mathcal{O}(\alpha^5) \right), \\ \alpha &= \left(\frac{q_2 \mu}{3 q_1 (1-\mu)} \right)^{1/3}. \end{aligned} \quad (24)$$

Substituting from (23) into (22) and retaining the terms up to the first order in the small quantities ε_2 , we can get

$$\begin{aligned} \frac{\partial U}{\partial \xi} &= 1 - \mu + b_2 + \varepsilon_2 - \frac{q_1(1-\mu)}{(1+b_2)^2} \\ &\times \left(1 - \frac{2\varepsilon_2}{1+b_2} \right) - \frac{q_2\mu}{b_2^2} \left(1 - \frac{2\varepsilon_2}{b_2} \right) \\ &+ \frac{1}{c^2} \left\{ (\mu - \mu^2 - 3)(1 - \mu + b_2) \right. \\ &+ \left(\frac{q_1(1-\mu)}{(1+b_2)} + \frac{q_2\mu}{b_2} \right) \\ &\times \left(\frac{q_1(1-\mu)}{(1+b_2)^2} + \frac{q_2\mu}{b_2^2} \right) \\ &+ \frac{1}{2} (1 - \mu + b_2)^3 \\ &- \frac{3}{2} \left(\frac{q_1(1-\mu)}{(1+b_2)^2} + \frac{q_2\mu}{b_2^2} \right) \\ &\times (1 - \mu + b_2)^2 + 3 \\ &\times \left(\frac{q_1(1-\mu)}{(1+b_2)} + \frac{q_2\mu}{b_2} \right) \\ &\times \left(\frac{q_1(1-\mu)}{(1+b_2)^2} + \frac{q_2\mu}{b_2^2} \right) \\ &- \frac{3}{2} \left(\frac{q_1(1-\mu)}{(1+b_2)^2} + \frac{q_2\mu}{b_2^2} \right) \xi^2 \\ &+ 3 \left(\frac{q_1(1-\mu)}{(1+b_2)} + \frac{q_2\mu}{b_2} \right) \xi + \frac{1}{2} \xi^3 \\ &- \frac{1}{2} q_1 q_2 \mu (1-\mu) \\ &\times \left[-\frac{1}{(1+b_2)^2} + \left(\frac{1}{(1+b_2)^2} - \frac{1}{b_2^2} \right) \right. \\ &\times \left. \left. (-1 + 3\mu + 7\xi) - 7 \left(\frac{1}{1+b_2} - \frac{1}{b_2} \right) \right] \right\} = 0 \end{aligned}$$

$$\begin{aligned}
& \times (1 - \mu + b_2) - \frac{1}{2} q_1 q_2 \mu (1 - \mu) \\
& \times \left[-\frac{1}{(1 + b_2)^2} - 7 \left(\frac{1}{(1 + b_2)} - \frac{1}{b_2} \right) \right. \\
& \quad \left. + \left(\frac{1}{(1 + b_2)^2} - \frac{1}{b_2^2} \right) (6 + 7b_2 - 4\mu) \right] \Bigg\} \\
& = 0.
\end{aligned} \tag{25}$$

Equation (25) can be solved for ε_2 ; then substituting back into (23) and (20) yields, after some lengthy algebraic manipulations, the location of L_2 as

$$\begin{aligned}
\xi_{0,L_2} &= 1 - \mu \\
&+ \sum_{k=0}^5 \left[\mathcal{M}_k^{L_2} + \left(\frac{1}{c^2} \right) \mathcal{N}_k^{L_2} \right] \left(\frac{\mu}{3} \right)^{k/3} \\
&+ \mathcal{O} \left(\frac{\mu}{3} \right)^2,
\end{aligned} \tag{26}$$

where the non-vanishing coefficients $\mathcal{M}_k^{L_2}$ and $\mathcal{N}_k^{L_2}$, are collected in Appendix B.

4.3. Location of L_3 . The L_3 point lies on the negative ξ -axis. We now derive an approximate location for this point. Substituting from (8) with the corresponding values of $\sigma_1 = -1$ and $\sigma_2 = 1$ into (6) and noting that $\partial r_1 / \partial \xi = \partial r_2 / \partial \xi = -1$, we get

$$\begin{aligned}
r_2 - r_1 &= 1, \\
r_1 &= -(\xi + \mu), \\
r_2 &= 1 - \mu - \xi, \\
\frac{\partial r_1}{\partial \xi} &= \frac{\partial r_2}{\partial \xi} = -1.
\end{aligned} \tag{27}$$

Hence substituting from (27) into (6), we get

$$\begin{aligned}
\frac{\partial U}{\partial \xi} &= \xi + \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \\
&+ \frac{1}{c^2} \left\{ (\mu - \mu^2 - 3) \xi - \left(\frac{q_1 (1 - \mu)}{r_1} + \frac{q_2 \mu}{r_2} \right) \right. \\
&\quad \times \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \\
&\quad + \frac{1}{2} \xi^3 + \frac{3}{2} \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \xi^2 \\
&\quad \left. + 3 \left(\frac{q_1 (1 - \mu)}{r_1} + \frac{q_2 \mu}{r_2} \right) \xi \right\}
\end{aligned}$$

which can be rewritten as a function of r_1 and r_2 as

$$\begin{aligned}
\frac{\partial U}{\partial \xi} &= 1 - \mu - r_2 + \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \\
&+ \frac{1}{c^2} \left\{ (\mu - \mu^2 - 3) (1 - \mu - r_2) \right. \\
&\quad - \left(\frac{q_1 (1 - \mu)}{r_1} + \frac{q_2 \mu}{r_2} \right) \\
&\quad \times \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \\
&\quad + \frac{1}{2} (1 - \mu - r_2)^3 \\
&\quad + \frac{3}{2} \left(\frac{q_1 (1 - \mu)}{r_1^2} + \frac{q_2 \mu}{r_2^2} \right) \\
&\quad \times (1 - \mu - r_2)^2 \\
&\quad + 3 \left(\frac{q_1 (1 - \mu)}{r_1} + \frac{q_2 \mu}{r_2} \right) (1 - \mu - r_2) \\
&\quad - \frac{1}{2} q_1 q_2 \mu (1 - \mu) \\
&\quad \times \left[\frac{1}{r_1^2} - \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) (-1 + 3\mu + 7\xi) \right. \\
&\quad \left. \left. - 7 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \right\} = 0.
\end{aligned} \tag{29}$$

Then it may be reasonable in our case to assume that the position of the equilibrium point L_3 is the same as given by classical restricted three-body problem but perturbed due to the inclusion of the relativistic correction by quantities $\varepsilon_3 \equiv \mathcal{O}(1/c^2)$:

$$\begin{aligned}
r_1 &= a_3 + \varepsilon_3, \\
r_2 &= b_3 + \varepsilon_3, \\
a_3 &= b_3 - 1,
\end{aligned} \tag{30}$$

where a_3 and b_3 are the unperturbed values of r_1 and r_2 , respectively, and b_3 is given after some successive approximation by the relation

$$\begin{aligned} b_3 = & 2 - \frac{7}{4} \left(\frac{q_2}{q_1} \right) \left(\frac{\mu}{3} \right) + \frac{21(-q_1 + q_2)\mu^2}{4q_2} \left(\frac{q_2}{q_1} \right)^2 \left(\frac{\mu}{3} \right)^2 \\ & + \frac{147(-q_1 + q_2)}{16q_2} \left(\frac{q_2}{q_1} \right)^3 \left(\frac{\mu}{3} \right)^3 \\ & + \frac{1029(-q_1 + q_2)}{64q_2} \left(\frac{q_2}{q_1} \right)^4 \left(\frac{\mu}{3} \right)^4 \\ & - \frac{7203(q_1 - q_2)}{256q_2} \left(\frac{q_2}{q_1} \right)^5 \left(\frac{\mu}{3} \right)^5 + \mathcal{O}\left(\frac{\mu}{3}\right)^6. \end{aligned} \quad (31)$$

Substituting from (30) into (29) and retaining the terms up to the first order in the small quantities ε_3 , we can get

$$\begin{aligned} \frac{\partial U}{\partial \xi} = & 1 - \mu - b_3 - \varepsilon_3 + \frac{q_1(1 - \mu)}{(1 - b_3)^2} \\ & \times \left(1 + \frac{2\varepsilon_3}{1 - b_3} \right) + \frac{q_2\mu}{b_3^2} \left(1 - \frac{2\varepsilon_3}{b_3} \right) + \frac{1}{c^2} \\ & \times \left\{ (\mu - \mu^2 - 3)(1 - \mu - b_3) \right. \\ & - \left(\frac{q_1(1 - \mu)}{(1 - b_3)} - \frac{q_2\mu}{b_3} \right) \left(\frac{q_1(1 - \mu)}{(1 - b_3)^2} + \frac{q_2\mu}{b_3^2} \right) \\ & + \frac{1}{2}(1 - \mu - b_3)^3 + \frac{3}{2} \left(\frac{q_1(1 - \mu)}{(1 - b_3)^2} + \frac{q_2\mu}{b_3^2} \right) \\ & \times (1 - \mu - b_3)^2 - 3 \left(\frac{q_1(1 - \mu)}{(1 - b_3)} - \frac{q_2\mu}{b_3} \right) \\ & \times (1 - \mu - b_3) - \frac{1}{2}q_1q_2\mu(1 - \mu) \\ & \times \left[\frac{1}{(1 - b_3)^2} - \left(\frac{1}{(1 - b_3)^2} - \frac{1}{b_3^2} \right) \right. \\ & \left. \left. \times (6 - 7b_3 - 4\mu) - 7 \left(\frac{1}{(1 - b_3)} - \frac{1}{b_3} \right) \right] \right\} = 0. \end{aligned} \quad (32)$$

Equation (32) can be solved for ε_3 ; then substituting back into (30) and (27) yields, after some lengthy algebraic manipulations, the location of L_3 as

$$\xi_{0,L_3} = 1 - \mu - \sum_{k=0}^5 \left[\mathcal{M}_k^{L_3} + \left(\frac{1}{c^2} \right) \mathcal{N}_k^{L_3} \right] \left(\frac{\mu}{3} \right)^k + \mathcal{O}\left(\frac{\mu}{3}\right)^6, \quad (33)$$

where the non-vanishing coefficients $\mathcal{M}_k^{L_3}$ and $\mathcal{N}_k^{L_3}$ are collected in Appendix C.

Remark 2. It is worth noting that there is no contribution in other two coordinates; that is, $\eta_{0,L_i} = \zeta_{0,L_i} = 0$, $i = 1, 2, 3$. This can be interpreted due to the fact that the concerned relativistic field is conservative.

5. Numerical Representations and Analyses

In this section we design a Mathematica program to draw the locations of collinear points L_i , $i = 1, 2, 3$, versus the whole range of the mass ratio μ taking into account the photogravitational effects and/or the relativistic corrections. In the case of one body radiates we considering the following values of the photogravitational parameter: $q_i = 1, 0.8, 0.6, 0.4, 0.2$, $i = 1, 2$, and they are represented on figures as black (no radiation exist), green, blue, orange, and red, respectively. But in the case two primaries radiating we adopted the following values, $q_i = 1, 0.6, 0.2$, $i = 1, 2$, and they are black (no radiation exist), blue, and red, respectively, this is actually to avoid the high density of the curves represented on these figures.

5.1. Analysis of L_1 Location. The black curve represents the case when the radiation pressure is totally neglected. In Figures 5(a) and 5(b), it seems that when $\mu = 0.5$ the location of L_1 lies exactly in the barycenter, and when $\mu \rightarrow 0$ the location of L_1 is shifted towards the center of the less massive primary and then the problem reduced to the classical two-body problem. The same case also exists in Figure 5(c) when there is no radiation from the two primaries, but when one or two primaries radiate, the situation is changed.

In Figure 5(a), we plot the location of L_1 under the effect of changing q_1 (the radiation coefficient of the massive primary $1 - \mu$) with different mass ratios. The green, blue, orange, and red curves represent the effect of successive increase in radiation of the massive primary, respectively. As is cleared from these curves illustrate, the greater the effect of the radiation (mass loss) the greater the negative displacement of the location of L_1 away from less massive primary, that is, towards the massive primary. This effect is of nonlinear nature. When the mass loss is significantly large especially in the case of binary systems of comparable masses (near $\mu = 0.4$ and greater), the location of L_1 crosses the barycenter towards the massive primary, that is, along the negative ξ -axis.

In Figure 5(b) we plot the location of L_1 under the effect of changing q_2 (the radiation coefficient of the less massive primary μ) with different mass ratios; the greater the effect of the radiation (mass loss) the greater the positive displacement of the location of L_1 towards less massive primary. When the mass loss is significantly large especially in the case of binary systems of comparable masses (near $\mu = 0.4$ and greater), the location of L_1 never crosses the barycenter towards the less massive primary. This may be interpreted as follows: as the emitter in this case is the less massive body, within the domain of classical stability $\mu \in (0, 0.038)$ the location of L_1 is not affected so much in contrast to the greater mass ratios.

FIGURE 5: (a) The location of L_1 against mass ratio μ under the effect of changing q_1 and constant q_2 without relativistic corrections. (b) The location of L_1 against mass ratio μ under the effect of changing q_2 and constant q_1 without relativistic corrections. (c) The location of L_1 against mass ratio μ under the effect of changing q_1 and q_2 without relativistic corrections. (d) The location of L_1 against mass ratio μ under the effect of changing q_1 and constant q_2 with relativistic corrections. (e) The location of L_1 against mass ratio μ under the effect of changing q_2 and constant q_1 with relativistic corrections. (f) The location of point L_1 against mass ratio μ under the effect of changing q_1 and q_2 with relativistic corrections.

In Figure 5(c), we plot the location of L_1 under the effect of changing q_1 and q_2 (the radiation coefficient of the two primaries). The situation is a combination of the two effects depicted in Figures 5(a) and 5(b). Since the point L_1 is closer to less massive primary, the effect of the radiation of it is dominant. With the observation that when the mass loss is significantly large especially in the case of binary systems of

comparable masses (near $\mu = 0.4$ and greater), the location of L_1 crosses the barycenter towards the massive primary.

Figures 5(d)–5(f) represent the location of Lagrangian point L_1 under the effect of changing q_1 and/or q_2 with relativistic corrections. In Figure 5(d) we plot the location of L_1 under the effect of changing q_1 (the radiation coefficient of the less massive primary μ) with relativistic corrections

FIGURE 6: (a) The location of L_2 against mass ratio μ under the effect of changing q_1 and constant q_2 without relativistic corrections. (b) The location of L_2 against mass ratio μ under the effect of changing q_2 and constant q_1 without relativistic corrections. (c) The location of L_2 against mass ratio μ under the effect of changing q_1 and q_2 without relativistic corrections. (d) The location of L_2 against mass ratio μ under the effect of changing q_1 and constant q_2 with relativistic corrections. (e) The location of L_2 against mass ratio μ under the effect of changing q_2 and constant q_1 with relativistic corrections. (f) The location of L_2 against mass ratio μ under the effect of changing q_1 and q_2 with relativistic corrections.

versus different mass ratios. The greater the effect of the radiation (mass loss), the greater the positive displacement of the location of L_1 towards less massive primary in the domain $\mu \in (0, 0.038)$. The most observable thing is that L_1 crosses to the right of the less massive primary especially in the very small mass ratios. Beyond this domain, that is, when $\mu \in (0.038, 0.5)$, the situation converses, but the nonlinearity becomes more visible. In Figure 5(e), the dynamics is similar

to that in Figure 5(b) but with clear different sizes of perturbations. In Figure 5(f), we have combined effects of those depicted in Figures 5(d) and 5(e).

5.2. Analysis of L_2 Location. The black curve represents the case when the radiation pressure is totally neglected. In Figures 6(a) and 6(b), it seems that when $\mu = 0.5$ the location of L_2 lies exactly at 1.2, and when $\mu \rightarrow 0$ the location of

FIGURE 7: (a) The location of L_3 against mass ratio μ under the effect of changing q_1 and constant q_2 without relativistic corrections. (b) The location of L_3 against mass ratio μ under the effect of changing q_2 and constant q_1 without relativistic corrections. (c) The location of L_3 against mass ratio μ under the effect of changing q_1 and q_2 without relativistic corrections. (d) The location of L_1 against mass ratio μ under the effect of changing q_1 and constant q_2 with relativistic corrections. (e) The location of L_3 against mass ratio μ under the effect of changing q_2 and constant q_1 with relativistic corrections. (f) The location of L_3 against mass ratio μ under the effect of changing q_1 and q_2 with relativistic corrections.

L_2 is shifted to the left, that is, towards the center of the less massive primary and lies exactly at a normalized distance equal to 1 on the right of the barycenter. Then the problem reduced to the classical two body problem. The same case also exists in Figure 6(c) when there is no radiation from the two primaries, but when one or two primaries radiate, the situation is changed.

In Figure 6(a), we plot the location of L_2 under the effect of changing q_1 (the radiation coefficient of the massive primary $1 - \mu$) versus different mass ratios. The green, blue, orange, and red curves represent the effect of successive increase in radiation of the massive primary, respectively. As these curves illustrate, the greater the effect of the radiation (mass loss), the greater the negative nonlinear displacement

of the location of L_2 . As is observed at high radiation and greater mass ratios specifically at $\mu > 0.038L_2$ crosses to the left of less massive primary. When $\mu \in (0, 0.2)$, the locations of L_2 have positive displacements, while this effect is reversed when $\mu \in (0.2, 0.5)$. In Figure 6(b) the situation is similar to that in Figure 6(b) but with different size of perturbations.

In Figure 6(c), we plot the location of L_2 under the effect of changing q_1 and q_2 (the radiation coefficient of the two primaries). The situation is a combination of the two effects depicted in Figures 6(a) and 6(b). Since the point L_2 is closer to less massive primary, the effect of the radiation of it is dominant.

Figures 6(d)–6(f) represent the location of Lagrangian point L_2 under the effect of changing q_1 and/or q_2 with relativistic corrections. In Figure 6(d) we plot the location of L_2 under the effect of changing q_1 (the radiation coefficient of the less massive primary μ) with relativistic corrections versus different mass ratios. The greater the effect of the radiation (mass loss), the greater the positive displacement of the location of L_2 in the domain $\mu \in (0, 0.038)$. Beyond this domain, that is, when $\mu \in (0.038, 0.5)$, the situation converses, but the nonlinearity becomes more visible. In Figure 6(e), the dynamics is similar to that in Figure 6(b) but it is clear with different size of perturbations. In Figure 6(f), we have combined effects of those depicted in Figures 6(d), and 6(e).

5.3. Analysis of L_3 Location. The black curve represents the case when the radiation pressure is totally neglected. In Figure 7(a) and in Figure 7(b), it seems that when $\mu = 0.5$ the location of L_3 lies exactly at a normalized distance equal to 1.2 to the left of the barycenter, and when $\mu \rightarrow 0$ the location of L_3 lies exactly at a normalized distance equal to 1 to the left of the barycenter, and then the problem reduced to the classical two body problem. The same case also exists in Figure 7(c) when there is no radiation from the two primaries, but when one or two primaries radiate, the situation is changed.

In Figure 7(a), we plot the location of L_3 under the effect of changing q_1 (the radiation coefficient of the massive primary $1 - \mu$) against different mass ratios. The green, blue, orange, and red curves represent the effect of successive increase in radiation of the massive primary, respectively. As these curves illustrate, the greater the effect of the radiation (mass loss) the greater the positive displacement of the location of L_3 , that is, towards the massive primary.

In the case of sever radiation (the red curve) and mass ratios $\mu > 0.15$ the order of the primaries RTBP may be reversed; for example, when $q_1 > 0.6$ the location of L_3 is weakly nonlinearly decreased. When $q_1 \geq 0.8$ and $0.15 < \mu < 0.35$ the location of L_3 is exponentially decreased while when $0.35 < \mu < 0.5$ the location of L_3 is exponentially increased.

In Figure 7(b) we plot the location of L_3 under the effect of changing q_2 (the radiation coefficient of the less massive primary μ) against different mass ratios. The greater the effect of the radiation (mass loss), the greater the positive displacement of the location of L_3 , linearly during $q < 0.4$ and nonlinearly during $q > 0.4$. This effect becomes very clear for significantly large mass ratios. When comparing

the two most left and the two most right parts on Figure 7(a) and Figure 7(b), they seemed in contrast to each other when $q < 0.4$ due to the fact that the location of L_3 is closer to the massive primary than the less massive primary.

The analysis of Figure 7(d) is that the greater the effect of the radiation (mass loss), the greater the negative displacement of the location of L_3 ; that is to say, for $q < 0.4$ the analysis of Figure 7(d) is in weak contrast to the analysis of Figure 7(a), while for $q > 0.4$ and significantly large mass ratios we have again positive displacements.

The analyses of Figure 7(e) are in strong contrast to that of Figure 7(b) but with different sizes of perturbations due to the inclusion of the relativistic effects in addition to the photogravitational effects.

Figures 7(c)–7(f) represent the location of Lagrangian point L_3 under the effect of changing q_1 and q_2 without/with relativistic corrections, respectively. They represent the combined effects of those depicted in Figures 7(a) and 7(b) and in Figures 7(d) and 7(e), respectively.

6. Conclusion

In a very important two works, Douskos and Perdios [7] and Ragos et al. [8] determined the stability of the equilibrium points and approximated the positions of the collinear points by series by expansions and determined the stability of the triangular as well as collinear points without taking into account the photo-gravitational effects of the two primaries. In our here presented work, we assumed the primaries to be radiant sources. The corrected locations of the collinear points are computed. Series forms of these location are obtained as new analytical results. In order to introduce a semianalytical view, A Mathematica program is constructed so as to draw the locations of collinear points versus the whole range of the mass ratio μ taking into account the photo-gravitational effects and/or the relativistic corrections. We analyzed all the obtained figures.

In a forthcoming works we aim to complete this work by performing the same computation to the triangular equilibrium points. Also we planned to study the stability of all equilibrium points; this would actually enhance the scientific significance of the current study.

Appendices

A.

The non-vanishing coefficients $\mathcal{M}_k^{L_1}$ and $\mathcal{N}_k^{L_1}$, of (19) are as follows:

$$\begin{aligned}\mathcal{M}_0^{L_1} &= \frac{1 - q_1}{1 + 8q_1}, \\ \mathcal{M}_1^{L_1} &= \frac{3q_1}{(1 + 8q_1)^2} (-1 + 28q_1) \left(\frac{q_2}{q_1}\right)^{1/3}, \\ \mathcal{M}_2^{L_1} &= \frac{q_1}{(1 + 8q_1)^3} [-7 + 4(17 - 76q_1)q_1] \left(\frac{q_2}{q_1}\right)^{2/3},\end{aligned}$$

$$\mathcal{M}_3^{L_1} = \frac{1}{9(1+8q_1)^4} \times [27(-1+q_1)(1+8q_1)^2 - (35+2q_1(-777+8q_1(213+292q_1)))q_2],$$

$$\mathcal{M}_4^{L_1} = \frac{2}{9(1+8q_1)^5} \times [27q_1(1+8q_1)^2(7+2q_1(-19+56q_1)) + (-11+q_1(1013+8q_1(-753+4q_1(-317+884q_1)))) \times q_2] \left(\frac{q_2}{q_1}\right)^{1/3},$$

$$\mathcal{N}_0^{L_1} = \frac{1}{2(1+8q_1)}(-1+q_1)(5+2q_1),$$

$$\mathcal{N}_1^{L_1} = \frac{3}{2(1+8q_1)^2} \times \left[(1+q_1(31+6q_1-56q_1^2)) \right] \left(\frac{q_2}{q_1}\right)^{1/3},$$

$$\mathcal{N}_2^{L_1} = \frac{1}{(1+8q_1)^3} \times [1+q_1(41+q_1(-153+172q_1(-1+4q_1)))] \times \left(\frac{q_2}{q_1}\right)^{2/3},$$

$$\mathcal{N}_3^{L_1} = -\frac{1}{18q_1(1+8q_1)^4} \times [27q_1(1+8q_1)^2(-5+q_1(3+4q_1(1+4q_1))) + 2(15+q_1(367+q_1(11041+2q_1(16125+8q_1(677+284q_1))))q_2],$$

$$\mathcal{N}_4^{L_1} = \frac{1}{18q_1(1+8q_1)^5} \left(\frac{q_2}{q_1}\right)^{1/3} \times [27q_1(1+8q_1)^2 \times (5+2q_1(-10+q_1(231+8q_1(-29+104q_1)))) + 2(4+q_1(222+q_1(-4381+2q_1(25441+8q_1(20607+4q_1)))) \times (5671+1740q_1))q_2] \times \left(\frac{q_2}{q_1}\right)^{1/3},$$

$$\mathcal{N}_5^{L_1} = -\frac{1}{54q_1(1+8q_1)^6} \times [27q_1(1+8q_1)^2(9+2q_1 \times (319+4q_1(25+q_1 \times (2523+8q_1(-23+412q_1)))) + 2(-9+2q_1(-228+q_1 \times (7285+q_1(-184181+16q_1 \times (-62741+q_1(-34747+8q_1 \times (10231+8644q_1))))))q_2] \left(\frac{q_2}{q_1}\right)^{2/3}. \quad (\text{A.1})$$

B.

The non-vanishing coefficients $\mathcal{M}_k^{L_2}$ and $\mathcal{N}_k^{L_2}$, of (26) are as follows:

$$\mathcal{M}_0^{L_2} = \frac{-1+q_1}{1+8q_1}$$

$$\mathcal{M}_1^{L_2} = \frac{3q_1}{(1+8q_1)^2}(-1+28q_1) \left(\frac{q_2}{q_1}\right)^{1/3}$$

$$\mathcal{M}_2^{L_2} = \frac{q_1}{(1+8q_1)^3} \times (7+4q_1(-17+76q_1)) \left(\frac{q_2}{q_1}\right)^{2/3},$$

$$\mathcal{M}_3^{L_2} = -\frac{1}{9(1+8q_1)^4} \times [27(-1+q_1)(1+8q_1)^2 + (35+2q_1(-777+8q_1(213+292q_1)))q_2],$$

$$\mathcal{M}_4^{L_2} = \frac{2}{9(1+8q_1)^5} \times [27q_1(1+8q_1)^2(7+2q_1(-19+56q_1)) + (11+q_1(-1013+8q_1(753+4(317-884q_1) \times q_1)))q_2] \left(\frac{q_2}{q_1}\right)^{1/3},$$

$$\begin{aligned}\mathcal{M}_5^{L_2} &= \frac{1}{9(1+8q_1)^6} \\ &\times [9q_1(1+8q_1)^2 \\ &\times (-55+16q_1(24+q_1 \\ &\times (-279+304q_1))) + 2(1+q_1 \\ &\times (1287+16q_1(-778+q_1 \\ &\times (-397+8q_1(1131+236q_1)))))) \\ &\times q_2] \left(\frac{q_2}{q_1}\right)^{2/3},\end{aligned}$$

$$\mathcal{N}_0^{L_2} = \frac{1}{2+16q_1} [7+11q_1(1+2q_1)],$$

$$\begin{aligned}\mathcal{N}_1^{L_2} &= \frac{3}{2(1+8q_1)^2} \\ &\times [1+q_1(25+2q_1 \\ &\times (7+12q_1)(-3+16q_1))] \\ &\times \left(\frac{q_2}{q_1}\right)^{1/3},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_2^{L_2} &= \frac{1}{(1+8q_1)^3} \\ &\times [2+q_1(39+q_1 \\ &\times (847-48q_1(-51+q_1 \\ &\times (-89+64q_1))))] \left(\frac{q_2}{q_1}\right)^{2/3}, \\ \mathcal{N}_3^{L_2} &= \frac{1}{18q_1(1+8q_1)^4} \\ &\times [-27q_1(1+8q_1)^2(11+q_1 \\ &\times (67+4q_1(59+44q_1))) \\ &+ 2(3+2q_1(123+q_1 \\ &\times (-957+q_1(15203+16q_1 \\ &\times (4475+9926q_1+9216q_1^2)))))) q_2],\end{aligned}$$

$$\begin{aligned}\mathcal{N}_4^{L_2} &= -\frac{1}{18q_1(1+8q_1)^5} \left(\frac{q_2}{q_1}\right)^{1/3} \\ &\times [27q_1(1+8q_1)^2(-5+2q_1 \\ &\times (-14+q_1(-699+8q_1 \\ &\times (-149+16q_1(23+84q_1)))))) \\ &+ 2(7+q_1(343+4q_1(-382+q_1 \\ &\times (17073+8q_1(8977+8q_1 \\ &\times (2332+q_1(6521+9216q_1)))))) q_2],\end{aligned}$$

$$\begin{aligned}\mathcal{N}_5^{L_2} &= \frac{1}{54q_1(1+8q_1)^6} \\ &\times [27q_1(1+8q_1)^2 \\ &\times (3+2q_1(179+8q_1 \\ &\times (-278+q_1(-803+12q_1 \\ &\times (-949+12q_1(-123+128q_1)))))) \\ &+ 2(-9+2q_1(-219+q_1 \\ &\times (-15493+q_1(65081+16q_1 \\ &\times (2888+q_1(-39569+16q_1 \\ &\times (17815+2q_1 \\ &\times (38069+54528q_1)))))) q_2] \left(\frac{q_2}{q_1}\right)^{2/3}.\end{aligned}\tag{B.1}$$

C.

The non-vanishing coefficients $\mathcal{M}_k^{L_3}$ and $\mathcal{N}_k^{L_3}$ of (33) are as follows:

$$\begin{aligned}\mathcal{M}_0^{L_3} &= \frac{1}{3}(5+q_1), \\ \mathcal{M}_1^{L_3} &= \frac{1}{18}(-19-5q_1+3q_2), \\ \mathcal{M}_2^{L_3} &= \frac{7}{96q_2^2}(-2q_1^2(19+5q_1) \\ &\quad + q_1q_2(73+23q_1) \\ &\quad - 4(7+5q_1)q_2^2) \left(\frac{q_2}{q_1}\right)^2, \\ \mathcal{M}_3^{L_3} &= \frac{7}{2304q_2^3} \\ &\times [-84q_1^3(19+5q_1) \\ &\quad + 336q_1^2(19+5q_1)q_2 \\ &\quad - 768q_1(11+4q_1)q_2^2 \\ &\quad + (3248+1741q_1)q_2^3] \\ &\times \left(\frac{q_2}{q_1}\right)^3,\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_4^{L_3} &= \frac{7}{9216q_2^4} \\
&\times \left[-588q_1^4 (19 + 5q_1) \right. \\
&\quad + 414q_1^3 (19 + 5q_1) q_2 \\
&\quad + 9q_1^2 (5419 + 2173q_1) q_2^2 \\
&\quad - q_1 (84371 + 46834q_1) q_2^3 \\
&\quad \left. + 56 (625 + 512q_1) q_2^4 \right] \\
&\times \left(\frac{q_2}{q_1} \right)^4, \\
\mathcal{M}_5^{L_3} &= \frac{49}{147465q_2^5} \\
&\times \left[-2352q_1^5 (19 + 5q_1) \right. \\
&\quad - 6096q_1^4 (19 + 5q_1) q_2 \\
&\quad + 12q_1^3 (59527 + 22817q_1) q_2^2 \\
&\quad - 12q_1^2 (96175 + 46304q_1) q_2^3 \\
&\quad + q_1 (795713 + 420985q_1) q_2^4 \\
&\quad \left. - (214592 + 93715q_1) q_2^5 \right] \\
&\times \left(\frac{q_2}{q_1} \right)^4, \\
\mathcal{N}_0^{L_3} &= \frac{1}{6} (5 + q_1 (-3 + 2q_1)), \\
\mathcal{N}_1^{L_3} &= \frac{1}{72} \left[47 - 34q_1^2 + 15q_2 \right. \\
&\quad \left. - \frac{91q_2}{q_1} + q_1 (15 + 8q_2) \right], \\
\mathcal{N}_2^{L_3} &= -\frac{1}{1728q_1^2} \\
&\times \left[-2q_1^2 (-103 + q_1) \right. \\
&\quad \times (537 + 50q_1)) + q_1 \\
&\quad \times (1855 + q_1 (579 + 4882q_1)) q_2 \\
&\quad \left. + 6 (-413 + q_1 (21 + 82q_1)) q_2^2 \right], \\
\mathcal{N}_3^{L_3} &= \frac{1}{124416q_1^3} \\
&\times \left[12q_1^3 (1007 + 25q_1 (-57 + 14q_1)) \right. \\
&\quad + 72q_1^2 (2163 + q_1 \\
&\quad \times (-1338 + 893q_1)) q_2 + 84q_1 \\
&\quad \times (-3040 + q_1 \times (2679 + 2615q_1)) q_2^2 \\
&\quad \left. - 7 (-15092 + q_1 \right. \\
&\quad \times (18417 + 1594q_1)) q_2^3 \left. \right], \\
\mathcal{N}_4^{L_3} &= \frac{1}{1492992q_1^4} \\
&\times \left[84q_1^4 (1007 + 25q_1 \right. \\
&\quad \times (-57 + 14q_1)) \\
&\quad + 6q_1^3 (357511 + q_1 \\
&\quad \times (-27069 + 205210q_1)) q_2 \\
&\quad - 63q_1^2 (108347 - 7701q_1 \\
&\quad + 6582q_1^2) q_2^2 - 7q_1 \\
&\quad \times (-865669 + 2q_1 \\
&\quad \times (8751 + 69208q_1)) q_2^3 \\
&\quad + 28 (-53648 + q_1 \\
&\quad \times (-8295 + 5113q_1)) q_2^4 \left. \right], \\
\mathcal{N}_5^{L_3} &= \frac{7}{71663616q_1^5} \\
&\times \left[336q_1^5 (1007 + 25q_1 \right. \\
&\quad \times (-57 + 14q_1)) + 48q_1^4 \\
&\quad \times (135337 + q_1 \\
&\quad \times (29127 + 110680q_1)) q_2 \\
&\quad - 108q_1^3 (285685 + 3q_1 \\
&\quad \times (21979 + 84386q_1)) q_2^2 \\
&\quad + 12q_1^2 (4088959 + 2q_1 \\
&\quad \times (505899 + 2124101q_1)) \\
&\quad \times q_2^3 - 3q_1 (11569257 + q_1 \\
&\quad \times (2427243 + 9605126q_1)) q_2^4 \\
&\quad + 7 (1461824 \\
&\quad \left. + (122055 - 169714q_1) q_1) q_2^5 \right].
\end{aligned} \tag{C.1}$$

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