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Research Article

Numerical Solution of Second-Order Fuzzy Differential Equation Using Improved Runge-Kutta Nystrom Method

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We develop the Fuzzy Improved Runge-Kutta Nystrom (FIRKN) method for solving second-order fuzzy differential equations (FDEs) based on the generalized concept of higher-order fuzzy differentiability. The scheme is two-step in nature and requires less number of stages which leads to less number of function evaluations in comparison with the existing Fuzzy Runge-Kutta Nystrom method. Therefore, the new method has a lower computational cost which effects the time consumption. We assume that the fuzzy function and its derivative are Hukuhara differentiable. FIRKN methods of orders three, four, and five are derived with two, three, and four stages, respectively. The numerical examples are given to illustrate the efficiency of the methods.

1. Introduction

Fuzzy differential equations serve as mathematical models for many exciting *real-world* problems, not only in science and technology but also in such diverse fields as population models [1], civil engineering [2], and modelling hydraulic [3].

Initially, the derivative of fuzzy-valued functions was first introduced by Chang and Zadeh [4]. It was followed by Dubois and Prade [5], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [6]; they generalized and extended the concept of Hukuhara differentiability (H-derivative) from set-valued mappings to the class of fuzzy mappings.

Subsequently, using H-derivative, Kaleva [7, 8] and Seikkala [9] developed the theory of fuzzy differential equations.

In the last few years, many researchers have worked on theoretical and numerical solution of FDEs [10–23], specially some authors considered the second-order fuzzy differential equations [24–26].

To our best knowledge up to now, a few investigations have been devoted to the numerical solution of second order fuzzy differential equations. In this paper, a novel Runge-kutta Nystrom method is proposed for solving fuzzy differential equations of second order which is constructed based on the used of the previous values of $y(x)$ acquired in the last stage. This Algorithm initially has been proposed by Rabiei et al. [27] for solving second-order ordinary differential equations which was the extension of the crisp concept of this method in solving first-order ODEs given in [28, 29]. The most important advantage of this method is that it has a lower computational cost in comparison with the previous findings for methods of the same order which improved the efficiency.

The aim of this paper is to introduce the fuzzy extension of Improved Runge-Kutta Nystrom method in solving second-order ODEs given in [27]. The second-order FODEs are assumed under Hukuhara differentiability. Actually, we suppose that fuzzy function and its derivative are H-differentiable. We, therefore, motivated our interest in the examples under this assumption. The accuracy of the proposed algorithms is demonstrated by test problems. Additionally, many formulas corresponding to mentioned references previously are applied for solving other kinds

of FODEs and are traceless in the literature for second-order ODEs in the fuzzy sense which is another motivation for developing Improved Runge-Kutta Nystrom method for solving second order FODEs.

The paper is organized as follows. In Section 2, we give some basic definitions and theorem on FDEs. In Section 3, Fuzzy Improved Runge-Kutta Nystrom method of orders 3, 4, and 5 are proposed. In Section 4, the numerical examples are provided to illustrate the validity and applicability of the new method. Finally, some conclusions are given.

2. Preliminaries

We give some definitions and introduce the necessary notation which will be used throughout the paper; see [30, 31].

We consider $\mathbb{R}$, the set of all real numbers. A fuzzy number is mapping $u : \mathbb{R} \to [0,1]$ with the following properties:

(a) u is upper semicontinuous;

(b) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$;

(c) u is normal, that is, $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$;

(d) supp $u = \{x \in \mathbb{R} \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy numbers on r. The r-level set of a fuzzy number $u \in \mathbb{E}$, $0 \leq r \leq 1$, denoted by $[u]^r$, is defined as

$$[u]_r = \begin{cases} \{x \in \mathbb{R} \mid u(x) \geq r\} & \text{if } 0 < r \leq 1, \\ \operatorname{cl}(\operatorname{supp} u) & \text{if } r = 0. \end{cases} \tag{1}$$

It is clear that r-level set of a fuzzy number is a closed and bounded interval $[u_1(r), u_2(r)]$, where $u_1(r)$ denotes the left-hand endpoint of $[u]^r$ and $u_2(r)$ denotes the right-hand endpoint of $[u]^r$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{y}$ is defined by

$$\tilde{y}(t) = \begin{cases} 1 & \text{if } t = y, \\ 0 & \text{if } t \neq y. \end{cases} \tag{2}$$

For $u, v \in \mathbb{E}$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda \odot u$ are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$

$[\lambda \odot u]^\alpha = \lambda[u]^\alpha$, for all $\alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means that usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The Hausdorff distance fuzzy numbers are given by $D : \mathbb{E} \times \mathbb{E} \to \mathbb{R}_+ \bigcup 0$:

$$D(u,v) = \sup_{r \in [0,1]} \max\left\{\left|u_1(r) - v_1(r)\right|, \left|u_2(r) - v_2(r)\right|\right\}. \tag{3}$$

It is easy to see that D is a metric in $\mathbb{E}$ and has the following properties:

(i) $D(u \oplus w, v \oplus w) = D(u,v)$, for all $u, v, w \in \mathbb{E}$,

(ii) $D(k \odot u, k \odot v) = |k| D(u,v)$, for all $k \in \mathbb{R}, u, v \in \mathbb{E}$,

(iii) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$, for all $u, v, w \in \mathbb{E}$,

(iv) $(D, \mathbb{E})$ is a complete metric space.

Definition 1. Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and $\epsilon > 0$, $\delta > 0$ such that

$$|t - t_0| < \delta \Longrightarrow D(f(t), f(t_0)) < \epsilon, \tag{4}$$

f is said to be continuous.

Initially the H-derivative (Hukuhara differentiability) for fuzzy mappings was introduced by Puri and Ralescu [6] which is based on the H-difference sets, as follows.

Definition 2. Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" stands for H-difference, and also note that $x \ominus y \neq x + (-1)y$.

Definition 3. Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy function. We say f is differentiable at $t_0 \in \mathbb{R}$, if there exists an element $f'(t_0) \in \mathbb{E}$ such that limits

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h}, \qquad \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} \tag{5}$$

exist and are equal to $f'(t_0)$. Here the limits are taken in the metric space $(\mathbb{E}, D)$, since we have defined $h^{-1} \odot (f(t_0) \ominus f(t_0 - h))$ and $h^{-1} \odot (f(t_0 + h) \ominus f(t_0))$.

Definition 4 (see [32]). Let $f : (a,b) \to \mathbb{E}$ and $x_0 \in (a,b)$. We say that f is strongly generalized differential at x_0. If there exists an element $f'(x) \in \mathbb{E}$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d)

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} &= \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} \\ &= f'(x_0), \end{aligned} \tag{6}$$

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d)

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{-h} &= \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{-h} \\ &= f'(x_0), \end{aligned} \tag{7}$$

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0), \tag{8}$$

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0). \tag{9}$$

Remark 5. In [32], the authors consider four cases for derivatives. Here we only consider the two first cases of Definition 4. In the other cases, the derivative is trivial because it is reduced to a crisp element. We say f is (1)-differentiable on (a, b) if f is differentiable with the meaning (i) of Definition 4 and also (2)-differentiable that f satisfies in the Definition 7 case (ii).

Theorem 6 (see [33]). *Let $f : (a, b) \to \mathbb{E}$ be a function and denote $[F(t)]^r = [f_r(t), g_r(t)]$, for each $r \in [0, 1]$. Then*

(1) *if f is (1)-differentiable, then $f_r(t)$ and $g_r(t)$ are differentiable functions and*

$$\left[F'(t)\right]^r = \left[f_r'(t), g_r'(t)\right], \tag{10}$$

(2) *if f is (2)-differentiable, then $f_r(t)$ and $g_r(t)$ are differentiable functions and*

$$\left[F'(t)\right]^r = \left[g_r'(t), f_r'(t)\right]. \tag{11}$$

2.1. Fuzzy Initial Value Problem. Consider the second-order fuzzy initial value problem:

$$y''(x) = f(t, y(t)), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \quad t \in [t_0\ T], \tag{12}$$

where f is a fuzzy function with r-level sets of initial values

$$[y_0]_r = [y_1(0; r), y_2(0; r)], \quad \left[y_0'\right]_r = \left[y_1'(0; r), y_2'(0; r)\right], \quad r \in [0, 1]. \tag{13}$$

We write $y(t, y) = [y_1(t; r), y_2(t; r)]$, $y'(t, y) = [y_1'(t; r), y_2'(t; r)]$ and $f(t, y) = [f_1(t, y), f_2(t, y)]$ where

$$f_1(t, y) = F[t, y_1(t; r), y_2(t; r)], \quad f_2(t, y) = G[t, y_1(t; r), y_2(t; r)]. \tag{14}$$

By using the extension principle, when $y(t)$ is a fuzzy number we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \mid s = f(t, \tau)\}, \quad s \in \mathbb{R}. \tag{15}$$

It follows that

$$[f(t, y)]^r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in [0\ 1], \tag{16}$$

where

$$f_1(t, y; r) = \min\{f(t, u) \mid u \in [y_1(r), y_2(r)]\}, \quad f_2(t, y; r) = \max\{f(t, u) \mid u \in [y_1(r), y_2(r)]\}. \tag{17}$$

Definition 7 (see [26]). Let $f : (t_0, T) \times \mathbb{E} \to \mathbb{E}$ and $x_0 \in (t_0, T)$. We say that f is strongly generalized differentiable of the second order at x_0, if there exists an element $f''(x_0) \in \mathbb{E}$, such that

(i) for all $h > 0$ sufficiently small, $\exists f'(x_0 + h) \ominus f'(x_0)$, $\exists f'(x_0) \ominus f'(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{f'(x_0 + h) \ominus f'(x_0)}{h} = \lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 - h)}{h} = f''(x_0), \tag{18}$$

or (ii) for all $h > 0$ sufficiently small, $\exists f'(x_0) \ominus f'(x_0 + h)$, $\exists f'(x_0 - h) \ominus f'(x_0)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f'(x_0 - h) \ominus f'(x_0)}{-h} = f''(x_0), \tag{19}$$

or (iii) for all $h > 0$ sufficiently small, $\exists f'(x_0 + h) \ominus f'(x_0)$, $\exists f'(x_0 - h) \ominus f'(x_0)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{f'(x_0 + h) \ominus f'(x_0)}{h} = \lim_{h \to 0^+} \frac{f'(x_0 - h) \ominus f'(x_0)}{-h} = f''(x_0), \tag{20}$$

or (iv) for all $h > 0$ sufficiently small, $\exists f'(x_0) \ominus f'(x_0 + h)$, $\exists f'(x_0) \ominus f'(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 - h)}{h} = f''(x_0). \tag{21}$$

Proposition 8. *For a supposed fuzzy function f, one has two possibilities, according to Definitions 4 and 7, to obtain the derivative of f over t: $D_1^{(1)} f(t)$ and $D_2^{(1)} f(t)$. Then for each of these two derivatives, one has again two possibilities: $D_1^{(1)}(D_1^{(1)} f(t))$, $D_2^{(1)}(D_1^{(1)} f(t))$ and $D_1^{(1)}(D_2^{(1)} f(t))$, $D_2^{(1)}(D_2^{(1)} f(t))$, respectively.*

Remark 9. In the rest of paper, we assume that f and f' are (1)-differentiable.

3. Fuzzy Improved Runge-Kutta Nystrom Method (FIRKN)

Improved Runge-Kutta (IRK) method for solving first-order ordinary differential equations was constructed by Rabiei and Ismail [28, 29]. Later, they developed the Improved Runge-Kutta Nystrom (IRKN) method for solving special second order ODEs. This scheme is two-step in nature and requires less number of stages which leads to less number of function evaluations compared with existing Runge-Kutta Nystrom methods. The general form of IRKN method with s-stages is given by (see [27])

$$\begin{aligned}
y_{n+1} &= y_n + \frac{3h}{2} y'_n - \frac{h}{2} y'_{n-1} + h^2 \sum_{i=2}^{s} \overline{b}_i (k_i - k_{-i}), \\
y'_{n+1} &= y'_n + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i (k_i - k_{-i}) \right), \\
k_1 &= f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}), \\
k_i &= f\left(x_n + c_i h, y_n + h c_i y'_n + h^2 \sum_{j=1}^{i-1} a_{ij} k_j \right), \\
&\qquad i = 2, \ldots, s, \\
k_{-i} &= f\left(x_{n-1} + c_i h, y_{n-1} + h c_i y'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} k_{-j} \right), \\
&\qquad i = 2, \ldots, s,
\end{aligned} \tag{22}$$

where $c_2, \ldots, c_s \in [0 \;\; 1]$ and f depends on both t and y while k_i and k_{-i} depend on the values of k_j and k_{-j} for $j = 1, \ldots, i-1$. Here s is the number of function evaluations performed at each step and increases with the order of local accuracy of the IRK method. In each step we only need to evaluate the values of $k_1, k_2, \ldots$, while $k_{-1}, k_{-2}, \ldots$ are calculated from the previous step.

Based on IRKN method in (22), we proposed the fuzzy version of IRKN method which is denoted by FIRKN. Here, FIRKN methods of orders three, four, and five with two, three, and four stages, respectively, are derived. The coefficients of FIRKN methods are the same as the coefficients of the IRKN methods.

Let the exact solution $[Y(t)]^r = [Y_1(t;r), Y_2(t;r)]$ be approximated by $[y(t)]^r = [y_1(t;r), y_2(t;r)]$. We define

$$[k_i(t, y(t;r))]^r = [k_{i1}(t, y(t;r)),\ k_{i2}(t, y(t;r))], \qquad i = 1, \ldots, s. \tag{23}$$

Note that the values of $k_{-i1}((t_{n-1}, y(t_{n-1};r))$ and $k_{-i2}(t_{n-1}, y(t_{n-1};r))$, $i = 1, \ldots, s$ in each step are replaced by $k_{i1}(t_n, y(t_n;r))$ and $k_{i2}(t_n, y(t_n;r))$, $i = 1, \ldots, s$ from the previous step; therefore, there is no need to evaluate them again.

3.1. Fuzzy Improved Runge-Kutta Nystrom Method of Order Three. Based on formulas (22), we define the FIRKN3 with two stages ($s = 2$) as follows:

$$\begin{aligned}
y_1(t_{n+1};r) &= y_1(t_n;r) + \frac{3}{2} h y'_1(t_n;r) \\
&\quad - \frac{h}{2} y'_1(t_{n-1};r) \\
&\quad + h^2 \big(\overline{b}_2 \{ k_{21}(t_n, y(t_n;r)) \\
&\qquad - k_{-21}(t_{n-1}, y(t_{n-1};r)) \} \big), \\
y_2(t_{n+1};r) &= y_2(t_n;r) + \frac{3}{2} h y'_2(t_n;r) \\
&\quad - \frac{h}{2} y'_2(t_{n-1};r) \\
&\quad + h^2 \big(\overline{b}_2 \{ k_{22}(t_n, y(t_n;r)) \\
&\qquad - k_{-22}(t_{n-1}, y(t_{n-1};r)) \} \big), \\
y'_1(t_{n+1};r) &= y'_1(t_n;r) \\
&\quad + h \big(b_1 k_{11}(t_n, y(t_n;r)) \\
&\qquad - b_{-1} k_{-11}(t_{n-1}, y(t_{n-1};r)) \\
&\qquad + b_2 \{ k_{21}(t_n, y(t_n;r)) \\
&\qquad - k_{-21}(t_{n-1}, y(t_{n-1};r)) \} \big), \\
y'_2(t_{n+1};r) &= y'_2(t_n;r) \\
&\quad + h \big(b_1 k_{12}(t_n, y(t_n;r)) \\
&\qquad - b_{-1} k_{-12}(t_{n-1}, y(t_{n-1};r)) \\
&\qquad + b_2 \{ k_{22}(t_n, y(t_n;r)) \\
&\qquad - k_{-22}(t_{n-1}, y(t_{n-1};r)) \} \big),
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
&k_{11}(t_n, y(t_n;r)) \\
&\quad = \min \{ f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)] \}, \\
&k_{12}(t_n, y(t_n;r)) \\
&\quad = \max \{ f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)] \}, \\
&k_{21}(t_n, y(t_n;r)) \\
&\quad = \min \{ f(t_n + c_2 h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))] \},
\end{aligned}$$

$$
\begin{aligned}
&k_{22}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_2h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))]\}, \\
&z_{11}(t_n, y(t_n;r)) = y_1(t_n;r) + hc_2y_1'(t_n;r) \\
&\qquad + h^2a_{21}k_{11}(t_n, y(t_n;r)), \\
&z_{12}(t_n, y(t_n;r)) = y_2(t_n;r) + hc_2y_2'(t_n;r) \\
&\qquad + h^2a_{21}k_{12}(t_n, y(t_n;r)).
\end{aligned} \tag{25}
$$

Here, the coefficients of FIRKN3 are the same as the coefficients of IRKN3 which are given as follows:

$$
\begin{gathered}
c_2 = \frac{1}{2}, \qquad a_{21} = \frac{1}{8}, \qquad b_{-1} = \frac{-1}{2}, \\
b_1 = \frac{2}{3}, \qquad b_2 = \frac{5}{6}, \qquad \bar{b}_2 = \frac{5}{12}.
\end{gathered} \tag{26}
$$

3.2. Fuzzy Improved Runge-Kutta Nystrom Method of Order Four. From formulas (22), consider the FIRKN4 with three stages ($s = 3$) as follows:

$$
\begin{aligned}
y_1(t_{n+1};r) &= y_1(t_n;r) + \frac{3}{2}hy_1'(t_n;r) \\
&\quad - \frac{h}{2}y_1'(t_{n-1};r) \\
&\quad + h^2\left(\sum_{i=2}^{3} b_i\{k_{i1}(t_n, y(t_n;r)) \right. \\
&\qquad \left. - k_{-i1}(t_{n-1}, y(t_{n-1};r))\}\right), \\
y_2(t_{n+1};r) &= y_2(t_n;r) + \frac{3}{2}hy_2'(t_n;r) \\
&\quad - \frac{h}{2}y_2'(t_{n-1};r) \\
&\quad + h^2\left(\sum_{i=2}^{3} b_i\{k_{i2}(t_n, y(t_n;r)) \right. \\
&\qquad \left. - k_{-i2}(t_{n-1}, y(t_{n-1};r))\}\right), \\
y_1'(t_{n+1};r) &= y_1'(t_n;r) \\
&\quad + h(b_1k_{11}(t_n, y(t_n;r)) \\
&\quad - b_{-1}k_{-11}(t_{n-1}, y(t_{n-1};r)) \\
&\quad + \left(\sum_{i=2}^{3} b_i\{k_{i1}(t_n, y(t_n;r)) \right. \\
&\qquad \left. - k_{-i1}(t_{n-1}, y(t_{n-1};r))\}\right), \\
y_2'(t_{n+1};r) &= y_2'(t_n;r) \\
&\quad + h(b_1k_{12}(t_n, y(t_n;r)) \\
&\quad - b_{-1}k_{-12}(t_{n-1}, y(t_{n-1};r)) \\
&\quad + \sum_{i=2}^{3} b_i\{k_{i2}(t_n, y(t_n;r)) \\
&\qquad - k_{-i2}(t_{n-1}, y(t_{n-1};r))\}),
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
&k_{11}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)]\}, \\
&k_{12}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)]\}, \\
&k_{21}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n + c_2h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))]\}, \\
&k_{22}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_2h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))]\}, \\
&k_{31}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n + c_3h, u) \mid u \in [z_{21}(t_n, y(t_n;r)), z_{22}(t_n, y(t_n;r))]\}, \\
&k_{32}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_3h, u) \mid u \in [z_{21}(t_n, y(t_n;r)), z_{22}(t_n, y(t_n;r))]\}, \\
&z_{11}(t_n, y(t_n;r)) \\
&\qquad = y_1(t_n;r) + hc_2y_1'(t_n;r) + h^2a_{21}k_{11}(t_n, y(t_n;r)), \\
&z_{12}(t_n, y(t_n;r)) \\
&\qquad = y_2(t_n;r) + hc_2y_2'(t_n;r) + h^2a_{21}k_{12}(t_n, y(t_n;r)), \\
&z_{21}(t_n, y(t_n;r)) \\
&\qquad = y_1(t_n;r) + hc_3y_1'(t_n;r) \\
&\qquad\quad + h^2\{a_{31}k_{11}(t_n, y(t_n;r)) + a_{32}k_{21}(t_n, y(t_n;r))\}, \\
&z_{22}(t_n, y(t_n;r)) \\
&\qquad = y_2(t_n;r) + hc_3y_2'(t_n;r) \\
&\qquad\quad + h^2\{a_{31}k_{12}(t_n, y(t_n;r)) + a_{32}k_{22}(t_n, y(t_n;r))\}.
\end{aligned} \tag{28}
$$

The coefficients of FIRKN4 are the same as the coefficients of IRKN4 which are given as follows:

$$
\begin{gathered}
c_2 = \frac{1}{4}, \qquad c_3 = \frac{3}{4}, \qquad a_{21} = \frac{1}{32}, \\
a_{31} = 0, \qquad a_{32} = \frac{9}{32},
\end{gathered}
$$

$$b_{-1} = \frac{-1}{18}, \qquad b_1 = \frac{19}{18}, \qquad b_2 = \frac{-1}{6},$$
$$b_3 = \frac{11}{18}, \qquad \bar{b}_2 = \frac{7}{24}, \qquad \bar{b}_3 = \frac{1}{8}. \tag{29}$$

3.3. Fuzzy Improved Runge-Kutta Nystrom Method of Order Five. From formulas (22), consider Fuzzy Improved Runge-Kutta Nystrom method of order five (FIRKN5) with four stages ($s = 4$) as follows:

$$\begin{aligned}
y_1(t_{n+1};r) &= y_1(t_n;r) + \frac{3}{2}hy_1'(t_n;r) \\
&\quad - \frac{h}{2}y_1'(t_{n-1};r) \\
&\quad + h^2\left(\sum_{i=2}^{4} b_i \{k_{i1}(t_n, y(t_n;r)) \right. \\
&\qquad \left. - k_{-i1}(t_{n-1}, y(t_{n-1};r))\}\right), \\
y_2(t_{n+1};r) &= y_2(t_n;r) + \frac{3}{2}hy_2'(t_n;r) \\
&\quad - \frac{h}{2}y_2'(t_{n-1};r) \\
&\quad + h^2\left(\sum_{i=2}^{4} b_i \{k_{i2}(t_n, y(t_n;r)) \right. \\
&\qquad \left. - k_{-i2}(t_{n-1}, y(t_{n-1};r))\}\right), \\
y_1'(t_{n+1};r) &= y_1'(t_n;r) \\
&\quad + h(b_1 k_{11}(t_n, y(t_n;r)) \\
&\qquad - b_{-1}k_{-11}(t_{n-1}, y(t_{n-1};r)) \\
&\qquad + \sum_{i=2}^{4} b_i \{k_{i1}(t_n, y(t_n;r)) \\
&\qquad\qquad - k_{-i1}(t_{n-1}, y(t_{n-1};r))\}), \\
y_2'(t_{n+1};r) &= y_2'(t_n;r) \\
&\quad + h(b_1 k_{12}(t_n, y(t_n;r)) \\
&\quad - b_{-1}k_{-12}(t_{n-1}, y(t_{n-1};r)) \\
&\quad + \sum_{i=2}^{4} b_i \{k_{i2}(t_n, y(t_n;r)) \\
&\qquad - k_{-i2}(t_{n-1}, y(t_{n-1};r))\}),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&k_{11}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)]\}, \\
&k_{12}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n, u) \mid u \in [y_1(t_n;r), y_2(t_n;r)]\}, \\
&k_{21}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n + c_2h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))]\}, \\
&k_{22}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_2h, u) \mid u \in [z_{11}(t_n, y(t_n;r)), z_{12}(t_n, y(t_n;r))]\}, \\
&k_{31}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n + c_3h, u) \mid u \in [z_{21}(t_n, y(t_n;r)), z_{22}(t_n, y(t_n;r))]\}, \\
&k_{32}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_3h, u) \mid u \in [z_{21}(t_n, y(t_n;r)), z_{22}(t_n, y(t_n;r))]\}, \\
&k_{41}(t_n, y(t_n;r)) \\
&\quad = \min\{f(t_n + c_4h, u) \mid u \in [z_{31}(t_n, y(t_n;r)), z_{32}(t_n, y(t_n;r))]\}, \\
&k_{42}(t_n, y(t_n;r)) \\
&\quad = \max\{f(t_n + c_4h, u) \mid u \in [z_{31}(t_n, y(t_n;r)), z_{32}(t_n, y(t_n;r))]\}, \\
&z_{11}(t_n, y(t_n;r)) \\
&\quad = y_1(t_n;r) + hc_2y_1'(t_n;r) + h^2a_{21}k_{11}(t_n, y(t_n;r)), \\
&z_{12}(t_n, y(t_n;r)) \\
&\quad = y_2(t_n;r) + hc_2y_2'(t_n;r) + h^2a_{21}k_{12}(t_n, y(t_n;r)), \\
&z_{21}(t_n, y(t_n;r)) \\
&\quad = y_1(t_n;r) + hc_3y_1'(t_n;r) \\
&\qquad + h^2\{a_{31}k_{11}(t_n, y(t_n;r)) + a_{32}k_{21}(t_n, y(t_n;r))\}, \\
&z_{22}(t_n, y(t_n;r)) \\
&\quad = y_2(t_n;r) + hc_3y_2'(t_n;r) \\
&\qquad + h^2\{a_{31}k_{12}(t_n, y(t_n;r)) + a_{32}k_{22}(t_n, y(t_n;r))\}, \\
&z_{31}(t_n, y(t_n;r)) \\
&\quad = y_1(t_n;r) + hc_4y_1'(t_n;r) \\
&\qquad + h^2\{a_{41}k_{11}(t_n, y(t_n;r)) + a_{42}k_{21}(t_n, y(t_n;r)) \\
&\qquad\qquad + a_{43}k_{31}(t_n, y(t_n;r))\}, \\
&z_{32}(t_n, y(t_n;r)) \\
&\quad = y_2(t_n;r) + hc_4y_2'(t_n;r) \\
&\qquad + h^2\{a_{41}k_{12}(t_n, y(t_n;r)) + a_{42}k_{22}(t_n, y(t_n;r)) \\
&\qquad\qquad + a_{43}k_{32}(t_n, y(t_n;r))\}.
\end{aligned} \tag{31}$$

Table 1: The absolute error for y_1 at $t_N = 1$, $N = 10$ in solving Problem 1.

r	FIRKN3 ($s = 2$)	FIRKN4 ($s = 3$)	FIRKN5 ($s = 4$)	FRKN4 ($s = 4$)
0	5.58×10^{-6}	9.31×10^{-8}	2.18×10^{-10}	1.02×10^{-6}
0.1	5.64×10^{-6}	9.41×10^{-8}	2.20×10^{-10}	1.03×10^{-6}
0.2	5.57×10^{-6}	9.52×10^{-8}	2.23×10^{-10}	1.04×10^{-6}
0.3	5.76×10^{-6}	9.62×10^{-8}	2.25×10^{-10}	1.06×10^{-6}
0.4	5.83×10^{-6}	9.72×10^{-8}	2.27×10^{-10}	1.07×10^{-6}
0.5	5.89×10^{-6}	9.83×10^{-8}	2.30×10^{-10}	1.08×10^{-6}
0.6	5.95×10^{-6}	9.93×10^{-8}	2.32×10^{-10}	1.09×10^{-6}
0.7	6.01×10^{-6}	1.00×10^{-7}	2.35×10^{-10}	1.10×10^{-6}
0.8	6.07×10^{-6}	1.01×10^{-7}	2.37×10^{-10}	1.11×10^{-6}
0.9	6.14×10^{-6}	1.02×10^{-7}	2.40×10^{-10}	1.12×10^{-6}
1.0	6.20×10^{-6}	1.03×10^{-7}	2.42×10^{-10}	1.13×10^{-6}

Table 2: The absolute error for y_2 at $t_N = 1$, $N = 10$ in solving Problem 1.

r	FIRKN3 ($s = 2$)	FIRKN4 ($s = 3$)	FIRKN5 ($s = 4$)	FRKN4 ($s = 4$)
0	6.82×10^{-6}	1.13×10^{-7}	2.66×10^{-10}	1.25×10^{-6}
0.1	6.76×10^{-6}	1.12×10^{-7}	2.64×10^{-10}	1.24×10^{-6}
0.2	6.69×10^{-6}	1.11×10^{-7}	2.61×10^{-10}	1.23×10^{-6}
0.3	6.63×10^{-6}	1.10×10^{-7}	2.59×10^{-10}	1.21×10^{-6}
0.4	6.57×10^{-6}	1.09×10^{-7}	2.57×10^{-10}	1.20×10^{-6}
0.5	6.51×10^{-6}	1.08×10^{-7}	2.54×10^{-10}	1.19×10^{-6}
0.6	6.45×10^{-6}	1.07×10^{-7}	2.52×10^{-10}	1.18×10^{-6}
0.7	6.38×10^{-6}	1.06×10^{-7}	2.49×10^{-10}	1.17×10^{-6}
0.8	6.32×10^{-6}	1.05×10^{-7}	2.47×10^{-10}	1.16×10^{-6}
0.9	6.26×10^{-5}	1.04×10^{-7}	2.44×10^{-10}	1.15×10^{-6}
1.0	6.20×10^{-6}	1.03×10^{-7}	2.42×10^{-10}	1.13×10^{-6}

Again, the coefficients of FIRKN5 are the same as the coefficients of IRKN5 which are given as the following:

$$\begin{gathered} c_2 = \frac{1}{4}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{3}{4}, \\ a_{21} = \frac{1}{32}, \qquad a_{31} = \frac{-1}{6}, \qquad a_{32} = \frac{7}{24}, \\ a_{41} = \frac{4}{32}, \qquad a_{42} = 0, \qquad a_{43} = \frac{5}{32}, \\ b_{-1} = \frac{1}{45}, \qquad b_1 = \frac{46}{45}, \qquad b_2 = \frac{-1}{15}, \\ b_3 = \frac{-1}{10}, \qquad b_4 = \frac{29}{45}, \qquad \bar{b}_2 = \frac{49}{180}, \\ \bar{b}_3 = \frac{7}{180}, \qquad b_4 = \frac{19}{180}. \end{gathered} \tag{32}$$

4. Numerical Examples

In this section, we solved the fuzzy initial value problems to show the efficiency and accuracy of the proposed methods.

Table 3: The absolute error for y_1 at $t_N = 1$, $N = 10$ in solving Problem 2.

r	FIRKN3 ($s = 2$)	FIRKN4 ($s = 3$)	FIRKN5 ($s = 4$)	FRKN4 ($s = 4$)
0	3.08×10^{-5}	3.81×10^{-8}	3.13×10^{-8}	1.73×10^{-6}
0.1	3.11×10^{-5}	3.96×10^{-8}	3.19×10^{-8}	1.76×10^{-6}
0.2	3.13×10^{-5}	4.12×10^{-8}	3.25×10^{-8}	1.79×10^{-6}
0.3	3.16×10^{-5}	4.28×10^{-8}	3.31×10^{-8}	1.82×10^{-6}
0.4	3.19×10^{-5}	4.44×10^{-8}	3.37×10^{-8}	1.85×10^{-6}
0.5	3.22×10^{-5}	4.59×10^{-8}	3.43×10^{-8}	1.89×10^{-6}
0.6	3.24×10^{-5}	4.75×10^{-8}	3.49×10^{-8}	1.92×10^{-6}
0.7	3.27×10^{-5}	4.91×10^{-8}	3.55×10^{-8}	1.95×10^{-6}
0.8	3.30×10^{-5}	5.06×10^{-8}	3.61×10^{-8}	1.98×10^{-6}
0.9	3.32×10^{-5}	5.22×10^{-8}	3.67×10^{-8}	2.01×10^{-6}
1.0	3.35×10^{-5}	5.38×10^{-8}	3.73×10^{-8}	2.05×10^{-6}

Table 4: The absolute error for y_2 at $t_N = 1$, $N = 10$ in solving Problem 2.

r	FIRKN3 ($s = 2$)	FIRKN4 ($s = 3$)	FIRKN5 ($s = 4$)	FRKN4 ($s = 4$)
0	3.63×10^{-5}	6.95×10^{-8}	4.32×10^{-8}	2.37×10^{-6}
0.1	3.60×10^{-5}	6.80×10^{-8}	4.26×10^{-8}	2.33×10^{-6}
0.2	3.57×10^{-5}	6.64×10^{-8}	4.20×10^{-8}	2.30×10^{-6}
0.3	3.54×10^{-5}	6.48×10^{-8}	4.15×10^{-8}	2.27×10^{-6}
0.4	3.52×10^{-5}	6.32×10^{-8}	4.09×10^{-8}	2.24×10^{-6}
0.5	3.49×10^{-5}	6.17×10^{-8}	4.03×10^{-8}	2.21×10^{-6}
0.6	3.46×10^{-5}	6.01×10^{-8}	3.97×10^{-8}	2.17×10^{-6}
0.7	3.43×10^{-5}	5.85×10^{-8}	3.91×10^{-8}	2.14×10^{-6}
0.8	3.41×10^{-5}	5.69×10^{-8}	3.85×10^{-8}	2.11×10^{-6}
0.9	3.38×10^{-5}	5.54×10^{-8}	3.79×10^{-8}	2.08×10^{-6}
1.0	3.35×10^{-5}	5.38×10^{-8}	3.73×10^{-8}	2.05×10^{-6}

Let the exact solution be $[Y(t)]^r = [Y_1(t;r), Y_2(t;r)]$. The absolute error formula, considered in Tables 1–4, is as follows:

$$\begin{aligned} ABS_1 &= \left|y_1(1;r) - Y_1(1;r)\right|, \\ ABS_2 &= \left|y_2(1;r) - Y_2(1;r)\right|. \end{aligned} \tag{33}$$

Problem 1 (see [13]). Consider

$$\begin{gathered} y''(t) = -y(t), \quad t \geq 0, \\ y(0) = 0, \qquad y'(0) = [0.9 + 0.1r, 1.1 - 0.1r]. \end{gathered} \tag{34}$$

The exact solution using (1)-differentiability is:

$$Y(t;r) = [(0.9 + 0.1r)\sin(t), (1.1 - 0.1r)\ \sin(t)]. \tag{35}$$

Problem 2 (see [34]). Consider

$$\begin{aligned} y''(t) &= -y(t) + t, \quad t \geq 0, \\ y(0) &= [0.9 + 0.1r, 1.1 - 0.1r], \\ y'(0) &= [1.8 + 0.2r, 2.2 - 0.2r]. \end{aligned} \tag{36}$$

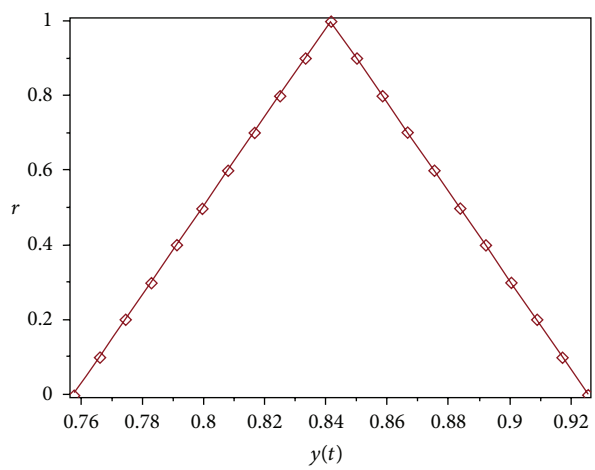

Figure 1: The approximated solution of $y_1(t)$ and $y_2(t)$ (solid line) and exact solution (points) with $h = 0.1$, $t \in [0 \quad 1]$ for Problem 1.

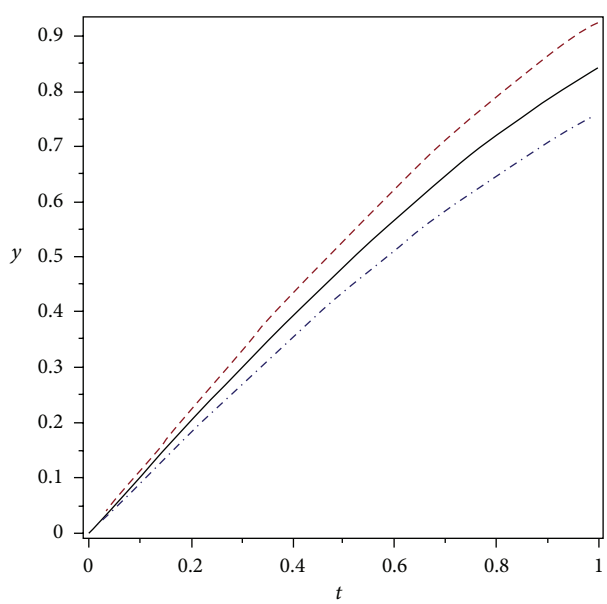

Figure 2: Approximated solutions of $y_1(t;0)$ (dash and dot line), $y_2(t;0)$ (dash line), and approximated solution $y(t;1)$ (solid line) in solving Problem 1.

The exact solution under (1)-differentiability: $Y(t) = [Y_1(t;r), Y_2(t;r)]$ where

$$Y_1(t;r) = \left(\frac{4}{5} + \frac{1}{5}r\right)\sin(t) + \left(\frac{9}{10} + \frac{1}{10}r\right)\cos(t) + t,$$
$$Y_2(t;r) = \left(\frac{6}{5} - \frac{1}{5}r\right)\sin(t) + \left(\frac{11}{10} - \frac{1}{10}r\right)\cos(t) + t. \tag{37}$$

To illustrate the efficiency of FIRKN methods, we compared the numerical results of the new methods with the existing method and numerical results are given in Tables 1–4 and Figures 1–4. In addition, the following abbreviations are used in Tables 1–4.

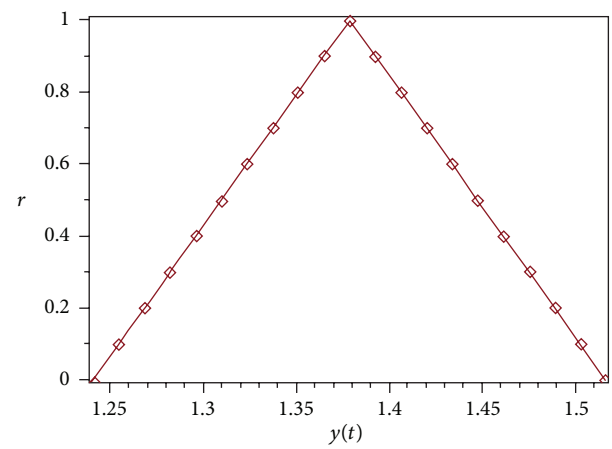

Figure 3: The approximated solution of $y_1(t)$ and $y_2(t)$ (solid line) and exact solution (points) with $h = 0.1$, $t \in [0 \quad 1]$ for Problem 2.

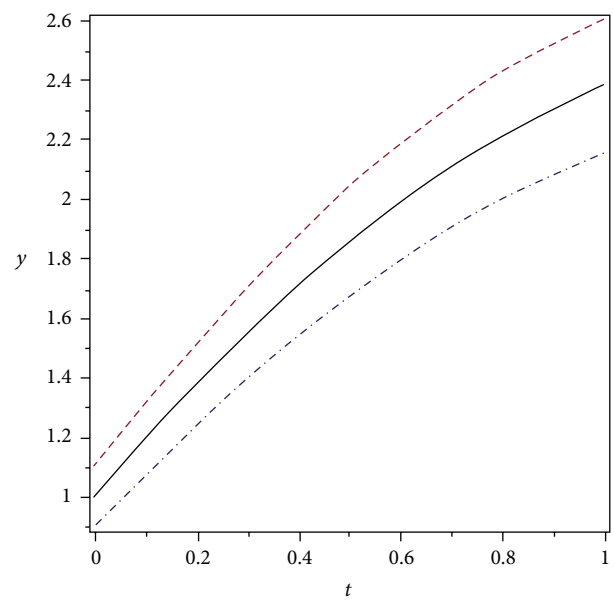

Figure 4: Approximated solutions of $y_1(t;0)$ (dash and dot line), $y_2(t;0)$ (dash line), and approximated solution $y(t;1)$ (solid line) in solving Problem 2.

(i) FIRKN3: third-order Fuzzy Improved Runge-Kutta Nystrom method with two stages derived in this paper.

(ii) FIRKN4: fourth-order Fuzzy Improved Runge-Kutta Nystrom method with three stages derived in this paper.

(iii) FIRKN5: fifth-order Fuzzy Improved Runge-Kutta Nystrom method with four stages derived in this paper.

(iv) FRKN4: fourth-order Fuzzy Runge-Kutta Nystrom method with four stages based on the fourth-order Runge-Kutta Nystrom method with order ten dispersion and dissipation order five, derived by Houwen and Sommeijer (see [35]).

(v) s: number of stages.

For Problem 1, the absolute error of FIRKN3, FIRKN4, and FIRKN5 compared with FRKN4 for y_1 and y_2 are given in Tables 1 and 2, respectively. FIRKN3 used only two stages, but the numerical results for both y_1 and y_2 are comparable with the results produced by FRKN4 with 4 stages. We can say that FIRKN3 is as accurate as FRKN4 but with less number of stages; hence, it is computationally more efficient.

FIRKN4 with three stages given accuracy up to 10^{-8} which is two orders higher than FRKN4. Hence, FIRKN4 is more accurate with less number of functions of evaluations compared to FRKN4.

FIRKN5 with four stapes gives accuracy up to 10^{-10} while FRKN4 with the same number of stages is accurate up to 10^{-6}, that indicates the efficiency of FIRKN methods in solving Problem 1.

Figure 1, shows the approximate solutions using FIRKN methods tend the exact solutions. In Figure 2, the graph of fuzzy function $y(t;r)$ is plotted and we can see that the approximated solutions of $y_1(t;1) = y_2(t;1)$ are between the fuzzy solutions $y_1(t;0)$ and $y_2(t;0)$; thus, the approximated solutions are valid for fuzzy level set.

From the numerical results of Problem 2 in Tables 3 and 4, it is observed that FIRKN3 which used less number of stages than FRKN4 gives a suitable accuracy. FIRKN4 and FIRKN5 give produced accuracy up to 10^{-8} while the accuracy of FRKN4 is only up to 10^{-6}. Also, Figure 3, shows that the approximated solutions by FIRKN methods are close to the exact solutions and the validity of the approximated solutions of $y_1(t;1) = y_2(t;1)$ for fuzzy level set in solving Problem 2 is given by Figure 4.

5. Conclusion

In this paper we developed the Fuzzy Improved Runge-Kutta Nystrom methods for solving second-order fuzzy differential equations. The scheme is based on the Improved Runge-Kutta Nystrom method for solving second-order ordinary differential equations. The methods of orders three, four, and five with two, three and four stages, respectively, are extended.

FIRKN methods used less number of stages which leads to the less number of function evaluations have a lower computational cost. Therefore, we can conclude that the Fuzzy Improved Runge-Kutta Nystrom methods with high accuracy and less number of function evaluations compared with the existing fuzzy Runge-Kutta Nystrom methods are more efficient for solving second-order fuzzy differential equations.

The presented method in this research may be useful if the coefficients, initial values are fuzzy and it can be applied by choosing the different types of fuzzy derivatives except (1)-differentiability.

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