Research Article

A Simplified Descriptor System Approach to Delay-Dependent Stability and Robust Performance Analysis for Discrete-Time Systems with Time Delays

Fengying Xu and Daxin Li

Physical Education Department, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Fengying Xu; xufyhanhan@126.com

Received 25 November 2012; Accepted 24 February 2013

Academic Editor: Yang Yi

Copyright © 2013 F. Xu and D. Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A simplified descriptor system approach is proposed for discrete-time systems with delays in terms of linear matrix inequalities. In comparison with the results obtained by combining the descriptor system approach with recently developed bounding technique, our approach can remove the redundant matrix variables while not reducing the conservatism. It is shown that the bounding technique is unnecessary in the derivation of our results. Via the proposed method, delay-dependent results on quadratic cost and H_∞ performance analysis are also presented.

1. Introduction

In the past decades, considerable attention has been paid to the problems of stability analysis and control synthesis of time-delay systems. Many methodologies have been proposed, and a large number of results have been established (see, e.g., [1, 2] and the references therein). All these results can be generally divided into two categories: delay-independent stability conditions [3, 4] and delay-dependent stability conditions [5–11]. The delay-independent stability condition does not take the delay size into consideration and thus is often conservative especially for systems with small delays, while the delay-dependent stability condition makes full use of the delay information and thus is less conservative than the delay-independent one. Very recently, in order to provide less conservative delay-dependent stability criteria, a descriptor system approach was proposed in [12, 13], while a new bounding technique has been presented in [14] (also called Moon's inequality). By combining the descriptor system approach with the bounding technique, novel delay-dependent sufficient conditions for the existence of a memoryless feedback guaranteed cost controller are derived for a class of discrete-time systems with delays in [6, 7].

Although the descriptor system approach proposed in [12, 13] is powerful to deal with the stability analysis of time-delay

systems, there are too many matrix variables introduced. In [15], a simplified but equivalent descriptor system approach to delay-dependent stability analysis was established for the continuous-time systems with delays. It is shown in [15] that the bounding technique in [14] is not necessary when deriving the delay-dependent stability results. It should be pointed out that the result in [15] is only applicable to continuous-time systems with delays. In this paper, we focus our attention upon deriving a simplified descriptor system approach to delay-dependent stability analysis in the context of discrete-time systems with delays. It is shown that the results derived by our approach are also equivalent to those obtained in [6, 7] but with fewer variables to be determined. It is also proved that, for discrete-time systems, the bounding technique in [14] will introduce some redundant variables and thus is unnecessary. Via the proposed method, delay-dependent results on quadratic cost and H_∞ performance analysis are also presented. It is worth mentioning that through the approach proposed in this paper, the delay-dependent guaranteed cost control conditions in [6, 7] obtained by the descriptor system approach and the bounding technique can also be simplified.

Notations. Throughout this paper, for real symmetric matrices X and Y , the notation $X \geq Y$ (resp., $X > Y$) means

that the matrix $X - Y$ is positive semidefinite (resp., positive definite). The superscript “ T ” represents the transpose. I is an identity matrix with appropriate dimension. $\text{diag}(\cdot)$ denotes a diagonal matrix. $l_2[0, \infty)$ refers to the space of square summable infinite vector sequences. In symmetric block matrices, we use an asterisk “ $*$ ” to represent a term that is induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Main Results

In order to introduce the simplified descriptor system approach, we consider the following discrete time-delay system

$$(\Sigma_a) : x(k+1) = \sum_{i=0}^2 A_i x(k-d_i(k)), \quad (1)$$

$$x(k) = \phi(k), \quad \forall k \in [-h, 0],$$

where $x(k) \in \mathbb{R}^n$ is the state, $d_0(k) = 0$, $\phi(k)$ is the initial condition, the scalar $h > 0$ is an upper bound on the time delays $d_i(k)$, $i = 1, 2$, and A_i , $i = 0, 1, 2$, are known real constant matrices.

Throughout this paper, we make the following assumption.

Assumption 1. $d_i(k)$ are unknown but satisfy for all $k \in \mathbb{Z}^+$

$$0 < d_i(k) \leq d_i, \quad i = 1, 2. \quad (2)$$

Now, we are in a position to present the main result of this paper.

Theorem 2. *Under Assumption 1, the time-delay system (Σ_a) is asymptotically stable for all $d_i(k)$, $i = 1, 2$, satisfying (2) if there exist matrices $P > 0$, P_i , R_i , S_i , and L_i , $i = 1, 2$, such that the following LMI holds:*

$$\begin{bmatrix} \Omega & G^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - L_1^T & G^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - L_2^T & -d_1 L_1^T & -d_2 L_2^T \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -d_1 S_1 & 0 \\ * & * & * & * & -d_2 S_2 \end{bmatrix} < 0, \quad (3)$$

where

$$G = \begin{bmatrix} P & 0 \\ P_1 & P_2 \end{bmatrix},$$

$$\Omega = G^T \begin{bmatrix} 0 & I \\ A_0 - I & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_0 - I & -I \end{bmatrix}^T G + \sum_{i=1}^2 \begin{bmatrix} L_i \\ 0 \end{bmatrix}$$

$$+ \sum_{i=1}^2 \begin{bmatrix} L_i \\ 0 \end{bmatrix}^T + \begin{bmatrix} \sum_{i=1}^2 R_i & 0 \\ 0 & P + \sum_{i=1}^2 d_i S_i \end{bmatrix}. \quad (4)$$

Proof. For all $d_i(k)$, $i = 1, 2$, satisfying (2), it can be verified that (3) implies that

$$\Theta(k) := \begin{bmatrix} \Omega & G^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - L_1^T & G^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - L_2^T & -d_1(k) L_1^T & -d_2(k) L_2^T \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -d_1(k) S_1 & 0 \\ * & * & * & * & -d_2(k) S_2 \end{bmatrix} < 0. \quad (5)$$

Let

$$y(k) = x(k+1) - x(k). \quad (6)$$

It is easy to see that

$$x(k-d_i(k)) = x(k) - \sum_{l=k-d_i(k)}^{k-1} y(l). \quad (7)$$

Then, the system (Σ_a) can be transformed into an equivalent descriptor form

$$x(k+1) = x(k) + y(k),$$

$$0 = -y(k) + \left(\sum_{i=0}^2 A_i - I \right) x(k) - \sum_{i=1}^2 A_i \left(\sum_{l=k-d_i(k)}^{k-1} y(l) \right). \quad (8)$$

Now, choose a Lyapunov functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k), \quad (9)$$

where

$$V_1(k) = x^T(k) P x(k),$$

$$V_2(k) = \sum_{i=1}^2 \left\{ \sum_{l=k-d_i(k)}^{k-1} x^T(l) R_i x(l) \right\}, \quad (10)$$

$$V_3(k) = \sum_{i=1}^2 \left\{ \sum_{\theta=-d_i(k)+1}^0 \sum_{l=k-1+\theta}^{k-1} y^T(l) S_i y(l) \right\}.$$

Then,

$$\begin{aligned}
\Delta V_1(k) &= V_1(k+1) - V_1(k) \\
&= 2x^T(k)Py(k) + y^T(k)Py(k) \\
&= 2\bar{x}^T(k)G^T \begin{bmatrix} y(k) \\ 0 \end{bmatrix} + y^T(k)Py(k) \\
&= 2\bar{x}^T(k)G^T \\
&\quad \times \left[-y(k) + \left(\sum_{i=0}^2 A_i - I \right) x(k) - \sum_{i=1}^2 A_i \left(\sum_{l=k-d_i(k)}^{k-1} y(l) \right) \right] \\
&\quad + y^T(k)Py(k) \\
&= 2\bar{x}^T(k)G^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^2 A_i - I & -I \end{bmatrix} \bar{x}(k) \\
&\quad + \bar{x}^T(k) \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \bar{x}(k) \\
&\quad - 2\bar{x}^T(k) \sum_{i=1}^2 G^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} \sum_{l=k-d_i(k)}^{k-1} y(l) \\
&= 2\bar{x}^T(k)G^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^2 A_i - I & -I \end{bmatrix} \bar{x}(k) \\
&\quad + \bar{x}^T(k) \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \bar{x}(k) \\
&\quad + 2\bar{x}^T(k) \sum_{i=1}^2 \left(L_i^T - G^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} \right) \sum_{l=k-d_i(k)}^{k-1} y(l) \\
&\quad - 2\bar{x}^T(k) \sum_{i=1}^2 L_i^T \sum_{l=k-d_i(k)}^{k-1} y(l) \\
&= 2\bar{x}^T(k)G^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^2 A_i - I & -I \end{bmatrix} \bar{x}(k) + \bar{x}^T(k) \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \bar{x}(k) \\
&\quad + 2\bar{x}^T(k) \sum_{i=1}^2 \left(L_i^T - G^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} \right) [x(k) - x(k-d_i(k))] \\
&\quad - 2\bar{x}^T(k) \sum_{i=1}^2 L_i^T \sum_{l=k-d_i(k)}^{k-1} y(l) \\
&= 2\bar{x}^T(k) \left(G^T \begin{bmatrix} 0 & I \\ A_0 - I & -I \end{bmatrix} + \sum_{i=1}^2 \begin{bmatrix} L_i \\ 0 \end{bmatrix}^T \right) \bar{x}(k) \\
&\quad + \bar{x}^T(k) \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \bar{x}(k)
\end{aligned}$$

$$\begin{aligned}
&+ 2\bar{x}^T(k) \sum_{i=1}^2 \left(G^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} - L_i^T \right) x(k-d_i(k)) \\
&- 2\bar{x}^T(k) \sum_{i=1}^2 L_i^T \sum_{l=k-d_i(k)}^{k-1} y(l),
\end{aligned}$$

(11)

where $\bar{x}(k) = [x^T(k) \ y^T(k)]^T$.

Furthermore, from (11), we obtain

$$\begin{aligned}
\Delta V_1(k) &= \frac{1}{d_1(k)d_2(k)} \\
&\quad \times \sum_{\alpha_2=k-d_2(k)}^{k-1} \sum_{\alpha_1=k-d_1(k)}^{k-1} \left[2\bar{x}^T(k) \right. \\
&\quad \times \left(G^T \begin{bmatrix} 0 & I \\ A_0 - I & -I \end{bmatrix} \right. \\
&\quad \left. \left. + \sum_{i=1}^2 \begin{bmatrix} L_i \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}P \end{bmatrix} \right) \bar{x}(k) \right. \\
&\quad \left. + 2\bar{x}^T(k) \sum_{i=1}^2 \left(G^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} - L_i^T \right) \right. \\
&\quad \times x(k-d_i(k)) \\
&\quad \left. - 2d_1(k)\bar{x}^T(k)L_1^T y(\alpha_1) \right. \\
&\quad \left. - 2d_2(k)\bar{x}^T(k)L_2^T y(\alpha_2) \right].
\end{aligned}$$

(12)

After some manipulations, we get

$$\begin{aligned}
\Delta V_2(k) + \Delta V_3(k) &\leq \sum_{i=1}^2 \left[x^T(k)R_i x(k) + d_i y^T(k)S_i y(k) \right. \\
&\quad \left. - x^T(k-d_i(k))R_i x(k-d_i(k)) \right. \\
&\quad \left. - \sum_{l=k-d_i(k)}^{k-1} y^T(l)S_i y(l) \right] \\
&= \frac{1}{d_1(k)d_2(k)}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\alpha_2=k-d_2(k)}^{k-1} \sum_{\alpha_1=k-d_1(k)}^{k-1} \left\{ \bar{x}^T(k) \right. \\
& \quad \times \begin{bmatrix} \sum_{i=1}^2 R_i & 0 \\ 0 & \sum_{i=1}^2 d_i S_i \end{bmatrix} \bar{x}(k) \\
& \quad - d_2(k) y^T(\alpha_2) S_2 y(\alpha_2) \\
& \quad - d_1(k) y^T(\alpha_1) S_1 y(\alpha_1) \\
& \quad - \sum_{i=1}^2 x^T(k-d_i(k)) \\
& \quad \left. \times R_i x(k-d_i(k)) \right\}. \tag{13}
\end{aligned}$$

Combining (12) with (13) yields

$$\begin{aligned}
& \Delta V(k) \\
& \leq \frac{1}{d_1(k) d_2(k)} \\
& \quad \times \sum_{\alpha_2=k-d_2(k)}^{k-1} \sum_{\alpha_1=k-d_1(k)}^{k-1} \eta^T(k, \alpha_1, \alpha_2) \Theta(k) \eta(k, \alpha_1, \alpha_2), \tag{14}
\end{aligned}$$

where $\Theta(k)$ is given in (5) and

$$\begin{aligned}
& \eta(k, \alpha_1, \alpha_2) \\
& = \begin{bmatrix} \bar{x}^T(k) & x^T(k-d_1(k)) & x^T(k-d_2(k)) & y^T(\alpha_1) & y^T(\alpha_2) \end{bmatrix}^T. \tag{15}
\end{aligned}$$

Therefore, the time-delay system (Σ_a) is asymptotically stable for all $d_i(k)$, $i = 1, 2$, satisfying (2) by the Lyapunov stability theory. This completes the proof. $\square$

Remark 3. It is noted that only two time delays are considered for the sake of simplicity. However, the results in Theorem 2 can be extended to the case of multiple delays. The simplified approach in Theorem 2 can also be used to tackle with the discrete time-delay systems with uncertainties, such as norm-bounded parameter uncertainties and linear fractional uncertainties.

Remark 4. Note that the delays considered here satisfy (2). From the proof of Theorem 2, the delay-dependent results in this paper can be extended to the case of interval delays (see

TABLE 1: the maximum delay bound of d_1 .

References	[18]	[17]	[6]	Theorem 2
d_1	—	12	16	18

[16] for more details), where the delays vary between a lower bound (may be not zero) and an upper bound.

By the method proposed in Theorem 2, the quadratic cost analysis result derived by using the descriptor system approach, together with the inequality in [14] as shown in [6, 7], can also be simplified. To make it clear, introduce the following quadratic cost function

$$J = \sum_{k=0}^{\infty} x^T(k) Q x(k). \tag{16}$$

Then, by Theorem 2, we have the following result.

Theorem 5. *There exist matrices $P > 0$, P_i , R_i , S_i , and L_i , $i = 1, 2$, such that the following LMI holds:*

$$\begin{bmatrix} \Omega_1 & G^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - L_1^T & G^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - L_2^T & -d_1 L_1^T & -d_2 L_2^T \\ * & -R_1 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 \\ * & * & * & -d_1 S_1 & 0 \\ * & * & * & * & -d_2 S_2 \end{bmatrix} < 0, \tag{17}$$

where $\Omega_1 = \Omega + \text{diag}(Q, 0)$, with G and Ω being defined in (4), then the system (Σ_a) is asymptotically stable, and the cost function in (16) satisfies

$$\begin{aligned}
& J \leq J_0 \\
& = x^T(0) P x(0) \\
& \quad + \sum_{i=1}^2 \left\{ \sum_{l=-d_i}^{-1} x^T(l) R_i x(l) + \sum_{\theta=-d_i+1}^0 \sum_{l=-1+\theta}^{-1} y^T(l) S_i y(l) \right\}, \tag{18}
\end{aligned}$$

where $y(l) = x(l+1) - x(l)$.

In the next, via the method proposed in Theorem 2, we will present the H_∞ performance analysis result.

Consider the following time-delay system:

$$\begin{aligned}
& (\Sigma_b) : x(k+1) = \sum_{i=0}^2 A_i x(k-d_i(k)) + B \omega(k), \\
& z(k) = C x(k) + D \omega(k), \tag{19}
\end{aligned}$$

where $z(k) \in \mathbb{R}^p$ is the output and $\omega(k) \in \mathbb{R}^q$ is the disturbance signal which is assumed to be in $l_2[0, \infty)$.

Then, the following delay-dependent result on H_∞ performance analysis can be obtained by Theorem 2.

Theorem 6. Given a scalar $\gamma > 0$. Then, under Assumption 1, the time-delay system (Σ_b) :

- (i) is asymptotically stable with $\omega(k) = 0$,
- (ii) satisfies

$$\|z\|_2 < \gamma \|\omega\|_2, \quad (20)$$

under zero-initial condition for all nonzero $\omega \in l_2[0, \infty)$ if there exist matrices $P > 0$, P_i , $R_i > 0$, $S_i > 0$, L_i , $i = 1, 2$, such that the following LMI holds:

$$\begin{bmatrix} \Omega_2 & G^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - L_1^T & G^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - L_2^T & -d_1 L_1^T & -d_2 L_2^T & G^T \begin{bmatrix} 0 \\ B \end{bmatrix} & \begin{bmatrix} C^T \\ 0 \end{bmatrix} \\ * & -R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 & 0 & 0 \\ * & * & * & -d_1 S_1 & 0 & 0 & 0 \\ * & * & * & * & -d_2 S_2 & 0 & 0 \\ * & * & * & * & * & -\gamma I & D^T \\ * & * & * & * & * & * & -\gamma I \end{bmatrix} < 0, \quad (21)$$

where $\Omega_2 = \Omega + \text{diag}(C^T C, 0)$, with G and Ω being defined in (4).

3. A Numerical Example

In this section, we present a numerical example to the effectiveness of the proposed algorithm. In order to show the comparison, we choose $A_2 = 0$ and $d_2 = 0$.

Example 7. Consider the system (Σ_a) with

$$A_0 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}. \quad (22)$$

Based on Theorem 2, we seek the maximum value of d_1 . Compared with three methods, which are in [6, 17, 18], respectively; we can illustrate the advantage of the proposed algorithm in this paper. Table 1 presents the result of comparison.

4. Conclusions

In this paper, we have proposed a simplified delay-dependent stability condition for discrete-time systems with delays. The given condition has fewer variables compared with those established using the descriptor system approach with Moon's bounding technique. It has been shown that Moon's bounding technique is unnecessary when deriving the delay-dependent stability conditions. By the proposed method in this paper, the delay-dependent results on quadratic cost and H_∞ performance analysis have also been provided.

Acknowledgment

This work was supported by the Specialized Research Fund for the Doctoral Program of Higher Education under Grant 20113705120003.

References

- [1] J. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 2nd edition, 1977.
- [2] Q.-L. Han, "Robust stability of uncertain delay-differential systems of neutral type," *Automatica*, vol. 38, no. 4, pp. 719–723, 2002.
- [3] M. S. Mahmoud and N. F. Al-Muthairi, "Quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties," *IEEE Transactions on Automatic Control*, vol. 39, no. 10, pp. 2135–2139, 1994.
- [4] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H_∞ controllers for state delayed systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 159–162, 1994.
- [5] S. Xu, J. Lam, and C. Yang, " H_∞ and positive-real control for linear neutral delay systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1321–1326, 2001.
- [6] W. Chen, Z. Guan, and X. Lu, "Delay-dependent guaranteed cost control for uncertain discretetime systems with delay," *IEEE Proceedings Control Theory & Applications*, vol. 150, no. 4, pp. 412–416, 2003.
- [7] W.-H. Chen, Z.-H. Guan, and X. Lu, "Delay-dependent guaranteed cost control for uncertain discrete-time systems with both state and input delays," *Journal of the Franklin Institute*, vol. 341, no. 5, pp. 419–430, 2004.
- [8] G. Zong, L. Hou, and J. Li, "A descriptor system approach to l_2-l_∞ filtering for uncertain discrete-time switched system with mode-dependent time-varying delays," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 5, pp. 2213–2224, 2011.
- [9] G. Zong, L. Hou, and Y. Wu, "Exponential l_2-l_∞ filtering for discrete-time switched systems under a new framework," *International Journal of Adaptive Control and Signal Processing*, vol. 26, no. 2, pp. 124–137, 2012.
- [10] T. Li, L. Guo, and X. Xin, "Improved delay-dependent bounded real lemma for uncertain time-delay systems," *Information Sciences*, vol. 179, no. 20, pp. 3711–3719, 2009.
- [11] T. Li, L. Guo, and L. Wu, "Simplified approach to the asymptotical stability of linear systems with interval time-varying delay," *IET Control Theory & Applications*, vol. 3, no. 2, pp. 252–260, 2009.

- [12] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Systems & Control Letters*, vol. 43, no. 4, pp. 309–319, 2001.
- [13] E. Fridman and U. Shaked, "A descriptor system approach to H_∞ control of linear time-delay systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 253–270, 2002.
- [14] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *International Journal of Control*, vol. 74, no. 14, pp. 1447–1455, 2001.
- [15] S. Xu, J. Lam, and Y. Zou, "Simplified descriptor system approach to delay-dependent stability and performance analysis for time-delay systems," *IEE Proceedings Control Theory & Applications*, vol. 152, no. 2, pp. 17–151, 2005.
- [16] S. Xu and T. Chen, "Robust H_∞ control for uncertain discrete-time systems with time-varying delays via exponential output feedback controllers," *Systems & Control Letters*, vol. 51, no. 3–4, pp. 171–183, 2004.
- [17] Y. S. Lee and W. H. Kwon, "Delay-Dependent robust stabilization of uncertain discrete-time state-delayed systems," in *Proceedings of the 15th IFAC Congress on Automation and Control*, Barcelona, Spain, 2002.
- [18] S.-H. Song, J.-K. Kim, C.-H. Yim, and H.-C. Kim, " H_∞ control of discrete-time linear systems with time-varying delays in state," *Automatica*, vol. 35, no. 9, pp. 1587–1591, 1999.