Research Article

On the Complexity of a Class of Pyramid Graphs and Chebyshev Polynomials

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In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we define a class of pyramid graphs and derive simple formulas of the complexity, number of spanning trees, of these graphs, using linear algebra, Chebyshev polynomials, and matrix analysis techniques.

1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G , a spanning tree in G is a tree which has the same vertex set as G . The number of spanning trees of G , also known as the complexity of the graph, is denoted by $\tau(G)$; this quantity is a well-studied quantity for long time. A classical result of Kirchhoff [1] can be used to determine the number of spanning trees for a graph G . If $V = \{v_1, v_2, \dots, v_n\}$, then the Kirchhoff matrix H is defined as $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G , $H = [a_{ij}]$ defined as follows:

- (i) $a_{ij} = -1$, when v_i and v_j are adjacent and $i \neq j$;
- (ii) a_{ij} equals the degree of vertex v_i if $i = j$;
- (iii) $a_{ij} = 0$ otherwise. All of cofactors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ denote the eigenvalues of H matrix of a p point graph. It is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [2]

have shown that, $\tau(G) = (1/p) \prod_{k=1}^{p-1} \mu_k$. The formula for the number of spanning trees in a d -regular graph G can be expressed as $\tau(G) = (1/p) \prod_{k=1}^{p-1} (d - \lambda_k)$, where $\lambda_0 = d, \lambda_1, \lambda_2, \dots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on n vertices, K_n , has n^{n-2} spanning trees; that is, he showed that $\tau(K_n) = n^{n-2}$, $n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1} q^{p-1}$, $p, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in, for example, [4, 5]. Another result is due to Sedláček [6] who derived a formula for the wheel on $n + 1$ vertices, W_{n+1} ; he showed that $\tau(W_{n+1}) = ((3 + \sqrt{5})/2)^n + ((3 - \sqrt{5})/2)^n - 2$, for $n \geq 3$. Sedláček [7] also later derived a formula for the number of

spanning trees in a Mobius ladder, M_n , $\tau(M_n) = (n/2)[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism graph. See Boesch, et al. [8, 9].

Now, we introduce the following lemma.

Lemma 1 (see [10]). Consider $\tau(G) = (1/n^2) \det(nI - \bar{D} + \bar{A})$, where $\bar{A}$ and $\bar{D}$ are the adjacency and degree matrices of $\bar{G}$, the complement of G , respectively, and I is the $n \times n$ unit matrix.

The advantage of this formula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use in our computations.

We begin with their definitions; see Zhang et al. [11].

Let $A_n(x)$ be $n \times n$ matrix such that

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix}. \quad (1)$$

Further, we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x). \quad (2)$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n} \frac{d}{dx} T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}. \quad (3)$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \quad (4)$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), \quad n \geq 1. \quad (5)$$

Furthermore by using standard methods for solving the recursion (4), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}} \left[\left(x + \sqrt{x^2-1} \right)^{n+1} - \left(x - \sqrt{x^2-1} \right)^{n+1} \right], \quad n \geq 1, \quad (6)$$

where the identity is true for all complex x (except at $x = \pm 1$, where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1} \prod_{j=1}^{n-1} \left(x - \cos \frac{j\pi}{n} \right). \quad (7)$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1} U_{n-1}(x). \quad (8)$$

These two results yield another formula for $U_n(x)$:

$$U_{n-1}^2(x) = 4^{n-1} \prod_{j=1}^{n-1} \left(x^2 - \cos^2 \frac{j\pi}{n} \right). \quad (9)$$

Finally, a simple manipulation of the above formula yields the following formula (10), which is extremely useful to us latter:

$$U_{n-1}^2 \left(\sqrt{\frac{x+2}{4}} \right) = \prod_{j=1}^{n-1} \left(x - 2 \cos \frac{2j\pi}{n} \right). \quad (10)$$

Furthermore, one can show that

$$\begin{aligned} U_{n-1}^2(x) &= \frac{1}{2(1-x^2)} [1 - T_{2n}] \\ &= \frac{1}{2(1-x^2)} [1 - T_n(2x^2 - 1)]. \end{aligned} \quad (11)$$

And

$$T_n(x) = \frac{1}{2} \left[\left(x + \sqrt{x^2-1} \right)^n + \left(x - \sqrt{x^2-1} \right)^n \right]. \quad (12)$$

Now we introduce the following important two lemmas.

Lemma 2 (see [10]). Let $B_n(x)$ be $n \times n$ circulant matrix such that

$$B_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & x \end{pmatrix}. \quad (13)$$

Then for $n \geq 3$, $x \geq 4$, one has

$$\det(B_n(x)) = \frac{2(x+n-3)}{x-3} \left[T_n \left(\frac{x-1}{2} \right) - 1 \right]. \quad (14)$$

Lemma 3 (see [12]). If $A \in F^{n \times n}$, $B \in F^{n \times m}$, $C \in F^{m \times n}$, and $D \in F^{m \times m}$, assuming that A and D are nonsingular matrices, then

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(A - BD^{-1}C) \det D \\ &= \det A \det(D - CA^{-1}B). \end{aligned} \quad (15)$$

This lemma gives a sort of symmetry for some matrices which facilitates one's calculations of the complexities of some special graphs.

3. Main Results

Definition 4. The pyramid graph $P_n^{(m)}$ is the graph formed from the wheel graph W_{m+1} with vertices $\{v_0, v_1, v_2, \dots, v_m\}$ and m sets of vertices, say, $\{u_1^1, u_2^1, \dots, u_n^1\}, \{u_1^2, u_2^2, \dots, u_n^2\}, \dots, \{u_1^m, u_2^m, \dots, u_n^m\}$, such that for all $i = 1, 2, \dots, n$ the vertex u_i^j is adjacent to v_j and v_{j+1} , where $j = 1, 2, \dots, m-1$, and u_i^m is adjacent to v_1 and v_m . See Figures 1(a) and 1(b).

Theorem 5. For $n \geq 0$,

$$\tau(P_n^{(3)}) = 2^{3n-2} \times (3n+8)^2. \quad (16)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} \tau(P_n^{(3)}) &= \frac{1}{(3n+4)^2} \det((3n+4)I - \overline{D} + \overline{A}) \\ &= \frac{1}{(3n+4)^2} \det \begin{pmatrix} 4 & 0 & 0 & 0 & 1 & \dots & \dots & \dots & \dots & \dots & 1 \\ 0 & 2(n+2) & 0 & 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 2(n+2) & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & 2(n+2) & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 1 & 3 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 1 & 0 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 3 \end{pmatrix}. \end{aligned} \quad (17)$$

Let $j = (1 \dots 1)$ be the $1 \times n$ matrix with all one and J_n the $n \times n$ matrix with all one. Set $a = 2n+4$ and $b = 3n+4$. Then we have

$$\begin{aligned} \tau(P_n^{(3)}) &= \frac{1}{b^2} \det \begin{pmatrix} 4 & 0 & 0 & 0 & j & j & j \\ 0 & & & & 0 & j & 0 \\ 0 & aI_3 & 0 & 0 & 0 & 0 & j \\ 0 & & j & 0 & 0 & 0 & 0 \\ j^t & 0 & 0 & j^t & & & \\ j^t & j^t & 0 & 0 & 2I_{3n} + J_{3n} & & \\ j^t & 0 & j^t & 0 & & & \end{pmatrix} \\ &= \frac{1}{b^2} \det \begin{pmatrix} b & 0 & 0 & 0 & j & j & j \\ b & & & & 0 & j & 0 \\ b & & aI_3 & 0 & 0 & 0 & j \\ b & & & j & 0 & 0 & 0 \\ bj^t & 0 & 0 & j^t & & & \\ bj^t & j^t & 0 & 0 & 2I_{3n} + J_{3n} & & \\ bj^t & 0 & j^t & 0 & & & \end{pmatrix} \\ &= \frac{1}{b} \det \begin{pmatrix} 1 & 0 & 0 & 0 & j & j & j \\ 1 & & & & 0 & j & 0 \\ 1 & & aI_3 & 0 & 0 & 0 & j \\ 1 & & & j & 0 & 0 & 0 \\ 1j^t & 0 & 0 & j^t & & & \\ 1j^t & j^t & 0 & 0 & 2I_{3n} + J_{3n} & & \\ 1j^t & 0 & j^t & 0 & & & \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b} \det \begin{pmatrix} 1 & 0 & 0 & 0 & j & j & j \\ 0 & & & -j & 0 & -j \\ 0 & & aI_3 & -j & -j & 0 \\ 0 & & 0 & 0 & -j & -j \\ 0 & 0 & 0 & j^t & & \\ 0 & j^t & 0 & 0 & & 2I_{3n} \\ 0 & 0 & j^t & 0 & & \end{pmatrix} \\
&= \frac{1}{b} \det \begin{pmatrix} & & -j & 0 & -j \\ & aI_3 & -j & -j & 0 \\ & 0 & -j & -j \\ 0 & 0 & j^t & & \\ j^t & 0 & 0 & & 2I_{3n} \\ 0 & j^t & 0 & & \end{pmatrix}.
\end{aligned}$$

(18)

Theorem 6. For $n \geq 0$,

Using Lemma 3 yields

$$\begin{aligned}
\tau(P_n^{(3)}) &= \frac{1}{b} \det \begin{pmatrix} aI_3 & B \\ C & 2I_{3n} \end{pmatrix} \\
&= \frac{1}{b} \det \left(aI_3 - B \cdot \frac{1}{2I_{3n}} C \right) \cdot 2^{3n}
\end{aligned}$$

Proof. Applying Lemma 1, we have

$$\tau(P_n^{(4)}) = 2^{4n} \times (n+3)^2 \times (2n+5). \quad (20)$$

$$\tau(P_n^{(4)})$$

$$= \frac{1}{(4n+5)^2} \det((4n+5)I - \overline{D} + \overline{A})$$

$$= \frac{1}{(4n+5)^2}$$

$$\times \det \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 1 \\ 0 & 2(n+2) & 0 & 1 & 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 2(n+2) & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & 2(n+2) & 0 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & 0 & 2(n+2) & 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 1 & 0 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 1 & 0 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & 1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 1 & 1 & 0 & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 1 & 3 \end{pmatrix} \quad (21)$$

□

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one and J_n the $n \times n$ matrix with all one. Set $a = 2n + 4$ and $b = 4n + 5$. Then we have

$$\begin{aligned}
 & \tau(P_n^{(3)}) \\
 &= \frac{1}{b^2} \\
 & \times \det \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & j & j & j & j \\ 0 & a & 0 & 1 & 0 & 0 & j & j & 0 \\ 0 & 0 & a & 0 & 1 & 0 & 0 & j & j \\ 0 & 1 & 0 & a & 0 & j & 0 & 0 & j \\ 0 & 0 & 1 & 0 & a & j & j & 0 & 0 \\ j^t & 0 & 0 & j^t & j^t & & & & \\ j^t & j^t & 0 & 0 & j^t & & & & \\ j^t & j^t & j^t & 0 & 0 & & & & \\ j^t & 0 & j^t & j^t & 0 & & & & \end{pmatrix} \\
 &= \frac{1}{b^2} \\
 & \times \det \begin{pmatrix} b & 0 & 0 & 0 & 0 & j & j & j & j \\ b & a & 0 & 1 & 0 & 0 & j & j & 0 \\ b & 0 & a & 0 & 1 & 0 & 0 & j & j \\ b & 1 & 0 & a & 0 & j & 0 & 0 & j \\ b & 0 & 1 & 0 & a & j & j & 0 & 0 \\ bj^t & 0 & 0 & j^t & j^t & & & & \\ bj^t & j^t & 0 & 0 & j^t & & & & \\ bj^t & j^t & j^t & 0 & 0 & & & & \\ bj^t & 0 & j^t & j^t & 0 & & & & \end{pmatrix} \\
 &= \frac{1}{b^2} \\
 & \times \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & j & j & j & j \\ 1 & a & 0 & 1 & 0 & 0 & j & j & 0 \\ 1 & 0 & a & 0 & 1 & 0 & 0 & j & j \\ 1 & 1 & 0 & a & 0 & j & 0 & 0 & j \\ 1 & 0 & 1 & 0 & a & j & j & 0 & 0 \\ 1j^t & 0 & 0 & j^t & j^t & & & & \\ 1j^t & j^t & 0 & 0 & j^t & & & & \\ 1j^t & j^t & j^t & 0 & 0 & & & & \\ 1j^t & 0 & j^t & j^t & 0 & & & & \end{pmatrix} \\
 &= \frac{1}{b} \\
 & \times \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & j & j & j & j \\ 0 & a & 0 & 1 & 0 & -j & 0 & 0 & -j \\ 0 & 0 & a & 0 & 1 & -j & -j & 0 & 0 \\ 0 & 1 & 0 & a & 0 & 0 & -j & -j & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & -j & -j \\ 0 & 0 & 0 & j^t & j^t & & & & \\ 0 & j^t & 0 & 0 & j^t & & & & \\ 0 & j^t & j^t & 0 & 0 & & & & \\ 0 & 0 & j^t & j^t & 0 & & & & \end{pmatrix} \\
 &= \frac{1}{b}
 \end{aligned}$$

$$\times \det \begin{pmatrix} a & 0 & 1 & 0 & -j & 0 & 0 & -j \\ 0 & a & 0 & 1 & -j & -j & 0 & 0 \\ 1 & 0 & a & 0 & 0 & -j & -j & 0 \\ 0 & 1 & 1 & a & 0 & 0 & -j & -j \\ 0 & 0 & j^t & j^t & & & & \\ j^t & 0 & 0 & j^t & & & & \\ j^t & j^t & 0 & 0 & & & & \\ 0 & j^t & j^t & 0 & & & & \end{pmatrix} \cdot 2I_{4n} \quad (22)$$

Using Lemma 3 yields

$$\begin{aligned}
 & \tau(P_n^{(4)}) \\
 &= \frac{1}{b} \det \begin{pmatrix} A & B \\ B & 2I_{4n} \end{pmatrix} \\
 &= \frac{1}{b} \det \left(A - B \cdot \frac{1}{2I_{4n}} C \right) \cdot 2^{4n} \\
 &= \frac{1}{b} 2^{4n} \cdot 2^{-4} \\
 & \times \det \begin{pmatrix} 2a & n & 2(n+1) & n \\ n & 2a & n & 2(n+1) \\ 2(n+1) & n & 2a & n \\ n & 2(n+1) & n & 2a \end{pmatrix} \\
 &= \frac{1}{b} 2^{4n-4} \\
 & \times \det \begin{pmatrix} 2a+4n+2 & n & 2(n+1) & n \\ 2a+4n+2 & 2a & n & 2(n+1) \\ 2a+4n+2 & n & 2a & n \\ 2a+4n+2 & 2(n+1) & n & 2a \end{pmatrix} \\
 &= \frac{1}{b} 2^{4n} \cdot 2^{-4} \cdot 2b \\
 & \times \det \begin{pmatrix} 1 & n & 2(n+1) & n \\ 1 & 2a & n & 2(n+1) \\ 1 & n & 2a & n \\ 1 & 2(n+1) & n & 2a \end{pmatrix} \\
 &= 2^{4n-3} \\
 & \times \det \begin{pmatrix} 1 & n & 2n & n \\ 0 & 2a-n & -n-2 & n+2 \\ 0 & 0 & 2a-2n-2 & 0 \\ 0 & n+2 & -n-2 & 2a-n \end{pmatrix} \\
 &= 2^{4n-3} \cdot 8(a+1)(a-n-1)^2 \\
 &= 2^{4n} (2n+5)(n+3)^2. \quad (23)
 \end{aligned}$$

□

Theorem 7. For $n \geq 0, m \geq 3$,

$$\begin{aligned}
 \tau(P_n^{(m)}) &= 2^{m-m} \left[(n+3+\sqrt{2n+5})^m \right. \\
 & \quad \left. + (n+3-\sqrt{2n+5})^m - 2(n+2)^m \right]. \quad (24)
 \end{aligned}$$

$$\begin{aligned} & \tau(P_n^{(m)}) \\ &= \frac{1}{(mn+m+1)^2} \det((mn+m+1)I - \overline{D} + \overline{A}) \\ &= \frac{1}{(mn+m+1)^2} \end{aligned}$$

$\times \det$

{

$(m+1)$	0	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	$2(n+2)$	0	1	$\cdots$	$\cdots$	$\cdots$	1	0	0	$\cdots$	0	1	$\cdots$	1	$\cdots$
$\vdots$	0	$2(n+2)$	0	1	$\ddots$	$\ddots$	$\ddots$	1	0	$\cdots$	0	0	$\cdots$	0	$\cdots$
$\vdots$	1	0	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$	1	$\cdots$	1	0	$\cdots$	0	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$	1	$\cdots$	1	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	1	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	0	1	$\cdots$	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	0	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	1	0	$2(n+2)$	1	$\cdots$	1	$\cdots$	$\cdots$	$\cdots$
1	0	0	1	1	$\vdots$	$\vdots$	1	1	3	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
$\vdots$	0	0	1	1	$\vdots$	$\vdots$	1	1	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
$\vdots$	1	0	0	1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
$\vdots$	1	0	0	1	$\vd$										

(25)

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one and J_n the $n \times n$ matrix with all one. Set $a = 2n + 4$ and $b = mn + m + 1$. Then we have

FIGURE 1: (a) The triangular pyramid graph $P_3^{(3)}$. (b) The square pyramid graph $P_4^{(4)}$.

$$\begin{aligned}
 \tau \left(P_n^{(m)} \right) &= \frac{1}{b^2} \det \begin{pmatrix} m+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\ 0 & a & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\ \vdots & 0 & a & 0 & 1 & \cdots & 1 & 0 & 0 & j & \ddots & \ddots & j \\ \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & 0 & \ddots & \ddots & \ddots & j \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & j \\ 0 & 0 & 1 & \cdots & 1 & 0 & a & j & \cdots & \cdots & j & 0 & 0 \\ j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\ j^t & j^t & 0 & 0 & \ddots & \ddots & j^t & & & & & & \\ \vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & j^t & & & & & & \\ \vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\ j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & & \end{pmatrix} \\
 &= \frac{1}{b^2} \det \begin{pmatrix} b & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\ b & a & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\ \vdots & 0 & a & 0 & 1 & \cdots & 1 & 0 & 0 & j & \ddots & \ddots & j \\ \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & 0 & \ddots & \ddots & \ddots & j \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\ b & 0 & 1 & \cdots & 1 & 0 & a & j & \cdots & \cdots & j & 0 & 0 \\ bj^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\ bj^t & j^t & 0 & 0 & \ddots & \ddots & j^t & & & & & & \\ \vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & j^t & & & & & & \\ \vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\ bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & & \end{pmatrix}
 \end{aligned}$$

[illegible]

Using Lemma 3 yields

$$\begin{aligned}
 \tau(P_n^{(m)}) &= \frac{1}{b} \det \begin{pmatrix} A & B \\ B & 2I_{mn} \end{pmatrix} \\
 &= \frac{1}{b} \det \left(A - B \cdot \frac{1}{2I_{mn}} B \right) \cdot 2^{mn} \\
 &= \frac{1}{b} 2^{mn} \\
 &\quad \cdot 2^{-m} \det \begin{pmatrix} 2a & n & 2(n+1) & \cdots & 2(n+1) & n \\ n & 2a & n & 2(n+1) & \cdots & 2(n+1) \\ 2(n+1) & n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2(n+1) \\ 2(n+1) & \ddots & \ddots & \ddots & \ddots & n \\ n & 2(n+1) & \cdots & 2(n+1) & n & 2a \end{pmatrix}.
 \end{aligned} \tag{27}$$

By straightforward induction using properties of determinants, we have

$$\begin{aligned}
 \tau(P_n^{(m)}) &= \frac{1}{b} 2^{mn-m} \cdot \frac{2a + n(m-2) + 2(m-3)}{mn + 2m + 2} \\
 &\quad \times \det \begin{pmatrix} 2a-n & 0 & (n+2) & \cdots & (n+2) & 0 \\ 0 & 2a-n & 0 & (n+2) & \cdots & (n+2) \\ (n+2) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+2) \\ (n+2) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+2) & \cdots & (n+2) & 0 & 2a-n \end{pmatrix} \\
 &= \frac{1}{b} 2^{mn-m} \cdot \frac{2b}{mn + 2m + 2} \\
 &\quad \times \det \begin{pmatrix} 2a-n & 0 & (n+2) & \cdots & (n+2) & 0 \\ 0 & 2a-n & 0 & (n+2) & \cdots & (n+2) \\ (n+2) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+2) \\ (n+2) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+2) & \cdots & (n+2) & 0 & 2a-n \end{pmatrix} \\
 &= 2^{mn-m+1} \cdot \frac{(n+2)^m}{mn + 2m + 2} \\
 &\quad \times \det \begin{pmatrix} \frac{2a-n}{n+2} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{2a-n}{n+2} & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{2a-n}{n+2} \end{pmatrix}.
 \end{aligned} \tag{28}$$

Using Lemma 2 yields

$$\begin{aligned}
 \tau \left(P_n^{(m)} \right) &= 2^{mn-m+1} \cdot \frac{(n+2)^m}{mn+2m+2} \cdot \frac{2((2a-n)/(n+2)+m-3)}{(2a-n)/(n+2)-3} \\
 &\quad \times \left[T_m \left(\frac{(2a-n)/(n+2)-1}{2} \right) - 1 \right] \\
 &= 2^{mn-m+1} \cdot \frac{(n+2)^m}{mn+2m+2} \cdot (mn+2m+2) \\
 &\quad \times \left[T_m \left(\frac{2a-2n-2}{2(n+2)} \right) - 1 \right] \\
 &= 2^{mn-m+1} \cdot (n+2)^m \left[T_m \left(\frac{n+3}{n+2} \right) - 1 \right].
 \end{aligned} \tag{29}$$

Using (12) yields

$$\begin{aligned}
 \tau \left(P_n^{(m)} \right) &= 2^{mn-m} \left[(n+3+\sqrt{2n+5})^m \right. \\
 &\quad \left. + (n+3-\sqrt{2n+5})^m - 2(n+2)^m \right].
 \end{aligned} \tag{30}$$

□

4. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and lemmas and their proofs.

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