Research Article

A Nonoverlapping Domain Decomposition Method for an Exterior Anisotropic Quasilinear Elliptic Problem in Elongated Domains

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Based on the Kirchhoff transformation, a nonoverlapping domain decomposition method is discussed for solving exterior anisotropic quasilinear problems with elliptic artificial boundary. By the principle of the natural boundary reduction, we obtain the natural integral equation for the anisotropic quasilinear problem on elliptic artificial boundaries and construct the algorithm and analyze its convergence. Moreover, we give the existence and uniqueness result for the original problem. Finally, some numerical examples are presented to illustrate the feasibility of the method.

1. Introduction

When solving a problem modelled by a linear or nonlinear partial differential equation in the bounded or unbounded domain, domain decomposition methods are one of the most efficient techniques. One can refer to [1–4] and references therein for more details. Based on natural boundary reduction [5, 6], the overlapping and nonoverlapping domain decomposition methods can be viewed as effective ways to solve the problems in the unbounded domains. These techniques have been used to solve many linear problems [6–8], and they have also been generalized to linear or nonlinear wave problems [5, 9, 10]. In this paper, we consider a nonoverlapping domain decomposition method for an exterior anisotropic quasilinear elliptic problem with elliptic artificial boundary. By the Kirchhoff transformation, we will discuss some exterior anisotropic quasilinear elliptic problems [11–15] by virtue of the nonoverlapping domain decomposition method.

Let Ω be an elongated, bounded, and simple connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega = \Gamma_0$,

$\Omega^c = \mathbb{R}^2/\overline{\Omega}$. We consider the numerical solution to the exterior anisotropic quasilinear problem

$$-\left(\frac{\partial}{\partial x}\left(\alpha a(\mathbf{x}, u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\beta a(\mathbf{x}, u)\frac{\partial u}{\partial y}\right)\right) = f(\mathbf{x}),$$

$$\text{in } \Omega^c,$$

$$u = 0, \quad \text{on } \Gamma_0,$$
(1)

$$u(\mathbf{x}) = \mathcal{O}(1), \quad \text{as } |\mathbf{x}| \rightarrow \infty,$$
(2)

with $\beta > \alpha > 0$ or $\alpha = \beta = 1$, $\mathbf{x} = (x, y)$, $a(\cdot, \cdot)$, and $f(\mathbf{x})$ are the given functions which will be ranked as below.

Problem (1) has many physical applications in, for example, the field of heat transfer, where a is the thermal conductivity of the medium and u is the temperature field; the field of compressible flow, where a is the density and u is the velocity potential. Problem (1) can also describe a temperature distribution in large transformers whose magnetic cores (consisting of iron tins) are nonlinear anisotropic media

FIGURE 1: The illustration of domains Ω_1 and Ω_2 .

where a is the heat conductivity. One can also refer to [11–14, 16] for more details.

Following [11, 13], suppose that the given function $a(\cdot, \cdot)$ satisfies

$$\begin{aligned} 0 < C_0 \leq a(\mathbf{x}, u) \leq C_1, \\ \forall u \in \mathbb{R}, \text{ and for almost all } \mathbf{x} \in \Omega^c, \end{aligned} \quad (3)$$

Where the two constants $C_0, C_1 \in \mathbb{R}$. In the following, we suppose that the function $f(\mathbf{x}) \in L^2(\Omega^c)$ has compact support, that is, there exists a constant $\gamma_0 > 0$, such that

$$\text{supp } f \subset \Omega_{\gamma_0} = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leq \gamma_0\}. \quad (4)$$

We also assume that

$$a(\mathbf{x}, u) \doteq a_0(u), \quad \text{when } |\mathbf{x}| \geq \gamma_0. \quad (5)$$

Now, we introduce an elliptical artificial boundary Γ_1 enclosing Γ_0 such that

$$\begin{aligned} \Gamma_1 &= \{(\mu, \varphi) \mid \mu = \mu_1 > \gamma_0, \ 0 \leq \varphi \leq 2\pi\}, \\ \text{and } \text{dist}(\Gamma_1, \Gamma_0) &= \delta_0 > 0. \end{aligned} \quad (6)$$

Then, Ω^c is divided into two nonoverlapping subdomains Ω_1 and Ω_2 (see Figure 1), where Ω_1 denotes the bounded domain between Γ_0 and Γ_1 , and Ω_2 refers to the unbounded domain outside Γ_1 . The original problem (1) is decomposed into two subproblems in domains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$. We have the nonoverlapping domain decomposition algorithm.

Step 1. Choose an initial value $\lambda^0 \in H^{1/2}(\Gamma_1)$, and put $k = 0$.

Step 2. Solve a Dirichlet boundary value problem in the exterior domain Ω_2 :

$$\begin{aligned} -\left(\frac{\partial}{\partial x} \left(\alpha a(\mathbf{x}, u_2^k) \frac{\partial u_2^k}{\partial x} \right) + \frac{\partial}{\partial y} \left(\beta a(\mathbf{x}, u_2^k) \frac{\partial u_2^k}{\partial y} \right) \right) \\ = 0, \quad \text{in } \Omega_2, \\ u_2^k = \lambda^k, \quad \text{on } \Gamma_1, \\ u_2^k(\mathbf{x}) = \mathcal{O}(1), \quad \text{as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \quad (7)$$

Step 3. Solve a mixed boundary value problem in the interior domain Ω_1 :

$$\begin{aligned} -\left(\frac{\partial}{\partial x} \left(\alpha a(\mathbf{x}, u_1^k) \frac{\partial u_1^k}{\partial x} \right) + \frac{\partial}{\partial y} \left(\beta a(\mathbf{x}, u_1^k) \frac{\partial u_1^k}{\partial y} \right) \right) &= f(\mathbf{x}), \\ &\text{in } \Omega_1, \\ \frac{\partial u_1^k}{\partial n_1} &= -\frac{\partial u_2^k}{\partial n_2}, \quad \text{on } \Gamma_1, \\ u_1^k &= 0, \quad \text{on } \Gamma_0. \end{aligned} \quad (8)$$

Step 4. Update the boundary value $0 < \theta_k < 1$,

$$\lambda^{k+1} = \theta_k u_1^k + (1 - \theta_k) \lambda^k, \quad \text{on } \Gamma_1. \quad (9)$$

Step 5. Put $k = k + 1$, turn to Step 2.

The relaxation factor θ_k is a suitably chosen real number. Notice that, in Step 3, we solve problem (8) by the standard finite element method and only need the normal derivative of the solution to problem (7) in Step 2. So, we need not to solve (7) directly. Based on the Kirchhoff transformation, the natural integral equation for the quasilinear problem can be obtained by the natural boundary reduction [6, 17, 18]. Particularly, when $a(\mathbf{x}, u) = c$ which is independent of $\mathbf{x}$ and u , [6, 8, 19] have discussed the corresponding problems by this technique. Now, we introduce the so-called Kirchhoff transformation [20]

$$w(\mathbf{x}) = \int_0^{u(\mathbf{x})} a(\xi) d\xi, \quad \mathbf{x} \in \Omega^c. \quad (10)$$

Then, we have

$$\nabla w = a(u) \nabla u, \quad (11)$$

$$\left(\alpha \frac{\partial w}{\partial x}, \beta \frac{\partial w}{\partial y} \right) = \left(\alpha a(u) \frac{\partial u}{\partial x}, \beta a(u) \frac{\partial u}{\partial y} \right). \quad (12)$$

From (7), we have that w satisfies the following problem

$$\begin{aligned} -\left(\alpha \frac{\partial^2 w^k}{\partial x^2} + \beta \frac{\partial^2 w^k}{\partial y^2} \right) &= 0, \quad \text{in } \Omega_2, \\ w^k &= \int_0^{\lambda^k} a_0(\xi) d\xi, \quad \text{on } \Gamma_1. \end{aligned} \quad (13)$$

The rest of the paper is organized as follows. In Section 2, we obtain the natural integral equation for the quasilinear problem in the elliptical unbounded domain. In Section 3, we state the nonoverlapping domain decomposition method and discuss the convergence of the algorithm. We also show the existence and uniqueness of the original problem. At last, in Section 4, we present some numerical examples to illustrate the efficiency and feasibility of our method.

2. Natural Boundary Reduction

In this section, by virtue of the Poisson integral formula and natural integral equation for the linear problem, we will obtain the corresponding results for the quasilinear problem in Ω_2 . For this purpose, we need to discuss some properties between elliptic coordinates (μ, φ) and Cartesian coordinates (x, y) first. The relationship between the two coordinates can be expressed as below

$$\begin{aligned} x &= f_0 \cosh \mu \cos \varphi, \\ y &= f_0 \sinh \mu \sin \varphi, \end{aligned} \quad (14)$$

with $f_0 = \sqrt{a^2 - b^2}$, $a = f_0 \cosh \mu_1$, and $b = f_0 \sinh \mu_1$. Following from [18], we have the following.

Theorem 1. *The transformation between elliptic coordinates and Cartesian coordinates in (14) possesses the following property.*

(1) *The Jacobian determinant of (14) is*

$$\begin{aligned} J &= f_0^2 \cosh^2 \mu \sin^2 \varphi + f_0^2 \sinh^2 \mu \cos^2 \varphi \\ &= f_0^2 (\cosh^2 \mu - \cos^2 \varphi), \end{aligned} \quad (15)$$

$J = 0$ if and only if $(x, y) = (\pm f_0, 0)$.

(2) *For $u \in C^2(\mathbb{R}^2)$, the following holds*

$$\frac{\partial^2 u}{\partial \mu^2} + \frac{\partial^2 u}{\partial \varphi^2} = J \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (16)$$

(3) *For the exterior domain Ω_2 , we have*

$$\frac{\partial u}{\partial \nu} = -\frac{1}{\sqrt{J}} \frac{\partial u}{\partial \mu}, \quad (17)$$

where ν refers to the unit exterior normal vector on Γ_{μ_1} (regarded as the inner boundary of Ω_2).

Proof. The conclusions 1 and 2 can be obtained by direct computation. And 3 follows from the property

$$\nu = -\frac{1}{\sqrt{J}} (f_0 \sinh \mu \cos \varphi, f_0 \cosh \mu \sin \varphi). \quad (18)$$

□

2.1. Natural Integral Equation for $\alpha = \beta = 1$. Assume that $w(\mathbf{x})$ is the solution to problem (13) and the value $w|_{|\mu|=\mu_1}$ is given, that is,

$$w|_{|\mu|=\mu_1} = w_0(\varphi). \quad (19)$$

Then, based on the natural boundary reduction, there are the Poisson integral formulas

$$\begin{aligned} w(\mu, \varphi) &= \frac{e^{2\mu} - e^{2\mu_1}}{2\pi} \\ &\times \int_0^{2\pi} \frac{w_0(\varphi')}{e^{2\mu} + e^{2\mu_1} - 2e^{\mu+\mu_1} \cos(\varphi - \varphi')} d\varphi', \\ &\mu > \mu_1, \end{aligned} \quad (20)$$

or

$$\begin{aligned} w(\mu, \varphi) &= \frac{1}{\pi} \sum_{j=1}^{\infty} e^{j(\mu_1 - \mu)} \\ &\times \int_0^{2\pi} \cos j(\varphi - \varphi') w_0(\varphi') \\ &+ \frac{1}{2\pi} \int_0^{2\pi} w_0(\varphi') d\varphi', \quad \mu > \mu_1, \end{aligned} \quad (21)$$

and the natural integral equation

$$\frac{\partial w}{\partial n} = \frac{1}{\sqrt{J_0}} \left[-\frac{1}{4\pi \sin^2(\varphi/2)} * w_0(\varphi) \right], \quad \mu = \mu_1, \quad (22)$$

or

$$\begin{aligned} \frac{\partial w}{\partial n} &= \frac{1}{\pi \sqrt{J_0}} \sum_{j=1}^{\infty} j \\ &\times \int_0^{2\pi} \cos j(\varphi - \varphi') w_0(\mu_1, \varphi') d\varphi, \\ &\mu = \mu_1, \end{aligned} \quad (23)$$

the definition of J_0 can be found in the following. The Poisson integral formulas (20)-(21) and the natural integral equation (22)-(23) can also be expressed in the Fourier series forms

$$w(\mu, \varphi) = \sum_{j=-\infty}^{+\infty} a_j e^{j(\mu_1 - \mu) + ij\varphi}, \quad \mu > \mu_1, \quad (24)$$

$$\frac{\partial w}{\partial n} = \frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j e^{ij\varphi}, \quad \mu = \mu_1, \quad (25)$$

with $a_j = (1/2\pi) \int_0^{2\pi} w_0(\varphi') e^{-ij\varphi'} d\varphi'$, $J_0 = f_0^2 (\cosh^2 \mu_1 \sin^2 \varphi + \sinh^2 \mu_1 \cos^2 \varphi) = f_0^2 (\cosh^2 \mu_1 - \cos^2 \varphi)$, $i = \sqrt{-1}$.

From (12), we obtain

$$\frac{\partial w}{\partial n} = a_0(u) \frac{\partial u}{\partial n}. \quad (26)$$

Combining (11), (25), and (26), we get the exact artificial boundary condition of u on Γ_{μ_1}

$$\begin{aligned} a_0(u) \frac{\partial u(\mu, \varphi)}{\partial n} \Big|_{\mu=\mu_1} &= \frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j e^{ij\varphi} \\ &= \frac{1}{\pi \sqrt{J_0}} \sum_{j=1}^{+\infty} j \int_0^{2\pi} \cos j(\varphi - \varphi') \left(\int_0^{u(\mu_1, \varphi)} a_0(y) dy \right) d\varphi \\ &\doteq \mathcal{K}_1(u(\mu_1, \varphi)), \end{aligned} \quad (27)$$

with $a_j = (1/2\pi) \int_0^{2\pi} \left(\int_0^{u(\mu_1, \varphi')} a_0(y) dy \right) e^{-ij\varphi'} d\varphi'$, $J_0 = f_0^2 (\cosh^2 \mu_1 - \cos^2 \varphi)$, $i = \sqrt{-1}$.

2.2. *Natural Integral Equation for $\beta > \alpha > 0$.* Now, we assume that Γ_{μ_1} can be expressed in the form: $\Gamma_{\mu_1} = \{(x, y) | px^2 + qy^2 = R^2\}$, with $\beta q > \alpha p > 0$. We also assume that $w(x)$ is the solution to problem (13) and the value $w|_{|\mu|=\mu_1}$ is given, namely,

$$w|_{|\mu|=\mu_1} = w_0(\varphi). \quad (28)$$

Let $x = \sqrt{\alpha}\xi$ and $y = \sqrt{\beta}\eta$. Then, the boundary Γ_{μ_1} is changed by the elliptic boundary $\tilde{\Gamma}_1 = \{(\xi, \eta) | \alpha p \xi^2 + \beta q \eta^2 = R^2\}$; the unit exterior normal vector on $\tilde{\Gamma}_1$ is

$$\nu = -\frac{(\sqrt{\alpha p} \cos \theta, \sqrt{\beta q} \sin \theta)}{\sqrt{\alpha p \cos^2 \theta + \beta q \sin^2 \theta}}. \quad (29)$$

By the above transformation, problem (13) changes into

$$-\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) = 0, \quad \text{in } \tilde{\Omega}_2, \quad (30)$$

$$w = w_0, \quad \text{on } \tilde{\Gamma}_1. \quad (31)$$

This is the right problem we talked in Section 2.1. Similar with (14), we let

$$\xi = f_0 \cosh \mu \cos \varphi, \quad \eta = f_0 \sinh \mu \sin \varphi, \quad (32)$$

with

$$f_0 = \sqrt{\frac{\beta q - \alpha p}{\alpha p \beta q}} R, \quad \mu_0 = \ln \left(\frac{\sqrt{\beta q} + \sqrt{\alpha p}}{\sqrt{\beta q} - \sqrt{\alpha p}} \right), \quad (33)$$

$$\tilde{\Gamma}_1 = \{(\mu, \varphi) | \mu = \mu_0, \varphi \in [0, 2\pi]\},$$

$$\tilde{\Omega}_2 = \{(\mu, \varphi) | \mu > \mu_0, \varphi \in [0, 2\pi]\}.$$

Then, just the same as the problem discussed in Section 2.1, we have the natural integral equation on Γ_{μ_1}

$$\begin{aligned} & \alpha n_x \frac{\partial w}{\partial x} + \beta n_y \frac{\partial w}{\partial y} \\ &= -\sqrt{\frac{\alpha p \beta q}{p \cos^2 \varphi + q \sin^2 \varphi}} \left[\frac{1}{4\pi R \sin^2(\varphi/2)} * w_0(\varphi) \right], \end{aligned} \quad (34)$$

where $(n_x, n_y) = (x/R, y/R)$ is the unit exterior normal vector on Γ_{μ_1} . From (13), we obtain

$$\alpha n_x \frac{\partial w}{\partial x} + \beta n_y \frac{\partial w}{\partial y} = \alpha n_x a_0(u) \frac{\partial u}{\partial x} + \beta n_y a_0(u) \frac{\partial u}{\partial y}. \quad (35)$$

Combining (11), (34), and (35), we obtain the exact artificial boundary condition of u on Γ_{μ_1}

$$\begin{aligned} & \left(\alpha n_x a_0(u) \frac{\partial u}{\partial x} + \beta n_y a_0(u) \frac{\partial u}{\partial y} \right) \Big|_{\mu=\mu_1} \\ &= -\sqrt{\frac{\alpha p \beta q}{p \cos^2 \varphi + q \sin^2 \varphi}} \\ & \times \left[\frac{1}{4\pi R \sin^2(\varphi/2)} * \left(\int_0^{u(\mu_1, \varphi)} a_0(y) dy \right) \right] \\ &= \sqrt{\frac{\alpha p \beta q}{p \cos^2 \varphi + q \sin^2 \varphi}} \sum_{j=-\infty}^{+\infty} \frac{j}{2\pi R} \\ & \times \int_0^{2\pi} \cos j(\varphi - \varphi') \left(\int_0^{u(\mu_1, \varphi)} a_0(y) dy \right) d\varphi \\ & \doteq \mathcal{K}_1(u(\mu_1, \varphi)). \end{aligned} \quad (36)$$

3. Variational Problem and Convergence Analysis of the Algorithm

3.1. *The Equivalent Variational Problem.* Now, we consider problem (8). We will use $W^{m,p}$ denoting the standard Sobolev spaces. $\|\cdot\|$ and $|\cdot|$ refer to the corresponding norms and seminorms. Especially, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$, and $|\cdot|_{m,\Omega} = |\cdot|_{m,2,\Omega}$. Let us introduce the space

$$V = \{v \in H^1(\Omega_1) | v|_{\Gamma_0} = 0\}, \quad (37)$$

and the corresponding norms

$$\|v\|_{0,\Omega_1} = \sqrt{\int_{\Omega_1} |v|^2 d\mathbf{x}}, \quad (38)$$

$$\|v\|_{1,\Omega_1} = \sqrt{\int_{\Omega_1} (|v|^2 + |\nabla v|^2) d\mathbf{x}}.$$

The boundary value problem (8) is equivalent to the following variational problem.

$$\text{Find } u \in V, \text{ such that} \quad (39)$$

$$D(u; u, v) + \widehat{D}(u; u, v) = F(v), \quad \forall v \in V,$$

with

$$D(t; u, v) = \int_{\Omega_1} a(\mathbf{x}, t) \left(\alpha \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \beta \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) d\mathbf{x}, \quad (40)$$

$$\begin{aligned} \widehat{D}(t; u, v) &= \sum_{j=1}^{+\infty} \frac{\sqrt{\alpha\beta}}{j\pi} \times \int_0^{2\pi} \int_0^{2\pi} a_0(t(\mu_1, \varphi')) \\ & \times \frac{\partial u(\mu_1, \varphi')}{\partial \varphi'} \frac{\partial v(\mu_1, \varphi)}{\partial \varphi} \\ & \times \cos j(\varphi' - \varphi) d\varphi' d\varphi, \end{aligned} \quad (41)$$

where $\widehat{D}(t; u, v)$ follows from Green's formula, (27) with $ds = \sqrt{J_0} d\varphi$, and (36) with $ds = (R/\sqrt{pq})\sqrt{p\cos^2\varphi + q\sin^2\varphi} d\varphi$.
And,

$$F(v) = \int_{\Omega_1} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}. \quad (42)$$

3.2. Convergence Analysis of the Method in Continuous Case.
From (10)-(11), the original problem (1) can be changed to

$$\begin{aligned} -\left(\alpha \frac{\partial^2 w}{\partial x^2} + \beta \frac{\partial^2 w}{\partial y^2}\right) &= f(\mathbf{x}), \quad \text{in } \Omega^c, \\ w &= 0, \quad \text{on } \Gamma_0, \\ w(\mathbf{x}) &= \mathcal{O}(1), \quad \text{as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \quad (43)$$

Then, we let $x = \sqrt{\alpha}\xi$ and $y = \sqrt{\beta}\eta$. Problem (43) becomes

$$\begin{aligned} -\Delta w &= -\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) = f(\mathbf{x}), \quad \text{in } \widetilde{\Omega}^c, \\ w &= 0, \quad \text{on } \widetilde{\Gamma}_0, \\ w(\mathbf{x}) &= \mathcal{O}(1), \quad \text{as } |\mathbf{x}| \rightarrow \infty, \end{aligned} \quad (44)$$

where $\widetilde{\Omega}^c$ and $\widetilde{\Gamma}_0$ are the corresponding changes from Ω^c and Γ_0 , respectively. Let g be extended to $\widetilde{\Omega}^c$, $w = u - g$, $f = \Delta g$. Then, problem (44) is equivalent to

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \widetilde{\Omega}^c, \\ u &= g, \quad \text{on } \widetilde{\Gamma}_0, \\ u(\mathbf{x}) &= \mathcal{O}(1), \quad \text{as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \quad (45)$$

Since it is difficult to estimate the convergence rate for a general unbounded domain Ω^c , we here let Ω^c be an exterior domain of an ellipse Γ_0 , with $\mu = \mu_0$, and Γ_1 is taken as stated in Section 1. We introduce the following conclusion first.

Lemma 2. *If u is the solution of*

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \Omega_1, \\ u &= u_0, \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} &= u_n, \quad \text{on } \Gamma_1, \end{aligned} \quad (46)$$

where Ω_1 is the elliptical ring domain between Γ_0 and Γ_1 ,

$$\begin{aligned} u_0 &= \sum_{j=-\infty}^{+\infty} c_j e^{ij\phi} \in H^{1/2}(\Gamma_0), \\ u_n &= \frac{1}{\sqrt{J_0}} \left(\sum_{-\infty, j \neq 0}^{+\infty} d_j |j| e^{ij\phi} + d_0 \right), \\ &\in H^{-1/2}(\Gamma_1), \end{aligned} \quad (47)$$

then there exist a unique $u \in H^1(\Omega_1)$ and

$$\begin{aligned} u(\mu, \phi) &= \sum_{-\infty, j \neq 0}^{+\infty} \left(c_j \left(e^{lj(\mu-\mu_1)} + e^{lj(\mu_1-\mu)} \right) + d_j \right. \\ &\quad \times \left(e^{lj(\mu-\mu_0)} - e^{lj(\mu_0-\mu)} \right) \\ &\quad \times \left(e^{lj(\mu_1-\mu_0)} + e^{lj(\mu_0-\mu_1)} \right)^{-1} \Big) \\ &\quad \times e^{ij\phi} + c_0 + d_0 (\mu - \mu_0). \end{aligned} \quad (48)$$

Proof. The result can be obtained directly from (46) by the separation of variables. $\square$

Theorem 3. *If $0 < \theta_k < 1$, then the nonoverlapping domain decomposition method (7)-(9) is convergent.*

Proof. We assume that the exact solution to problem (1) is u , and we let $\lambda = u|_{\Gamma_1}$, $u_k = u|_{\Omega_k}$, and $k = 1, 2$. Then, following (7)-(8), we let $e_1^k = \lambda - u_1^k$ and $e_1^k|_{\Gamma_1} = \lambda - \lambda^k \doteq e_2^k$. From (24), we assume that $e_2^k = \sum_{j=-\infty}^{+\infty} a_j e^{ij\phi} \in H^{1/2}(\Gamma_1)$. By the natural integral equation (27), we have

$$\frac{\partial e_1^k}{\partial n} = -\mathcal{K}_1(e_2^k) = -\frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j e^{ij\phi}. \quad (49)$$

So, e_1^k satisfies

$$\begin{aligned} -\Delta e_1^k &= 0, \quad \text{in } \Omega_1, \\ e_1^k &= 0, \quad \text{on } \Gamma_0, \end{aligned} \quad (50)$$

$$\frac{\partial e_1^k}{\partial n} = \frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j e^{ij\phi}, \quad \text{on } \Gamma_1.$$

By Lemma 2, one obtains

$$e_1^k = - \sum_{j=-\infty}^{+\infty} a_j H_j(\mu) e^{ij\phi}, \quad (51)$$

with $H_j(\mu) = (e^{lj(\mu-\mu_0)} - e^{lj(\mu_0-\mu)}) / (e^{lj(\mu_1-\mu_0)} + e^{lj(\mu_0-\mu_1)})$.

From (51), e_1^k confines on Γ_1 can be expressed as

$$e_1^k|_{\Gamma_1} = - \sum_{j=-\infty}^{+\infty} a_j H_j(\mu_1) e^{ij\phi}, \quad (52)$$

$$\mathcal{K}_1(e_1^k) = -\frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j H_j(\mu_1) e^{ij\phi}.$$

Then, we have

$$\begin{aligned} \frac{\partial e_1^{k+1}}{\partial n} &= -\mathcal{K}_1(\lambda - \lambda^{k+1}) \\ &= \mathcal{K}_1(\theta_k u_1^k + (1 - \theta_k) \lambda^k - \lambda) \\ &= -\theta_k \mathcal{K}_1(e_1^k) - (1 - \theta_k) \mathcal{K}_1(e_2^k) \\ &= \frac{1}{\sqrt{J_0}} \sum_{j=-\infty}^{+\infty} |j| a_j (\theta_k H_j(\mu_1) - 1 + \theta_k) e^{ij\phi}. \end{aligned} \quad (53)$$

Let $E^m \doteq \|\partial e_1^k / \partial n\|_{-1/2, \Gamma_1}^2$, then $E^m = 2\pi \sum_{j=-\infty}^{+\infty} (j^2 / \sqrt{1+j^2}) |a_j|^2$ and

$$\begin{aligned} E^{m+1} &= 2\pi \sum_{j=-\infty}^{+\infty} \frac{j^2}{\sqrt{1+j^2}} |a_j|^2 (\theta_k H_j(\mu_1) - 1 + \theta_k)^2 \\ &= (1 - \theta_k)^2 E^m + 2\pi \sum_{j=-\infty}^{+\infty} \frac{j^2}{\sqrt{1+j^2}} |a_j|^2 \\ &\quad \times \theta_k H_j(\mu_1) [\theta_k (H_j(\mu_1) + 2) - 2]. \end{aligned} \quad (54)$$

Assuming that $\delta_1 = \inf_{j \in \mathbb{Z} \setminus \{0\}} 2/(2 + H_j(\mu_1))$, then $1 > \delta_1 \geq 2/3$.

If $0 < \theta_k \leq \delta_1$, $k = 0, 1, 2, \dots$, then

$$E^{m+1} < (1 - \theta_k)^2 E^m, \quad (55)$$

or equally

$$E^{m+1} < \prod_{j=1}^m (1 - \theta_j)^2 E^1 \leq r^m E^1, \quad \frac{1}{9} \leq r < 1. \quad (56)$$

By the trace theorem, we have

$$\|e_1^k\|_{1, \Omega_1}^2 \leq C E^m \longrightarrow 0, \quad m \longrightarrow \infty. \quad (57)$$

From (54), one also has

$$\begin{aligned} E^{m+1} &= 2\pi \sum_{j=-\infty}^{+\infty} \frac{j^2}{\sqrt{1+j^2}} |a_j|^2 (\theta_k H_j(\mu_1) - 1 + \theta_k)^2 \\ &= (1 - 2\theta_k)^2 E^m + 2\pi \sum_{j=-\infty}^{+\infty} \frac{j^2}{\sqrt{1+j^2}} |a_j|^2 \\ &\quad \times \theta_k I_j(\mu_1) [\theta_k (I_j(\mu_1) - 2) + 1], \end{aligned} \quad (58)$$

with $I_j(\mu_1) = (1 - H_j(\mu_1))/2$. Assuming that $\delta_2 = \sup_{j \in \mathbb{Z} \setminus \{0\}} 1/(2 - I_j(\mu_1))$, then $0 < \delta_2 \leq 2/3$.

For $\delta_2 \leq \theta_k < 1$, $k = 0, 1, 2, \dots$, the convergence result can be obtained similarly with (55)–(57). Therefore, for $0 < \theta_k < 1$, the nonoverlapping domain decomposition method is convergent. $\square$

3.3. Discrete Nonoverlapping Alternating Algorithm and Convergence Analysis. Divide the arc Γ_{μ_1} into M parts, and take a finite element subdivision in Ω_1 such that their nodes on Γ_{μ_1} are coincident. That is, we make a regular and quasiuniform triangulation $\mathcal{T}_h$ on Ω_1 , such that

$$\Omega_1 = \bigcup_{K \in \mathcal{T}_h} K, \quad (59)$$

where K is a (curved) triangle and h is the maximal diameter of triangles. Let

$$V_h = \{v_h \in V | v|_K \text{ is a linear polynomial, } \forall K \in \mathcal{T}_h\}. \quad (60)$$

Then, the approximate problem of (39) can be written as follows

$$\text{Find } u_h \in V_h, \text{ such that} \quad (61)$$

$$D(u_h; u_h, v_h) + \widehat{D}(u_h; u_h, v_h) = F(v_h), \quad \forall v_h \in V_h,$$

with

$$D(t_h; u_h, v_h) = \int_{\Omega_1} a(\mathbf{x}, t_h) \left(\alpha \frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial x} + \beta \frac{\partial u_h}{\partial y} \frac{\partial v_h}{\partial y} \right) d\mathbf{x}, \quad (62)$$

$$\begin{aligned} \widehat{D}(t_h; u_h, v_h) &= \sum_{j=1}^{+\infty} \frac{\sqrt{\alpha\beta}}{j\pi} \int_0^{2\pi} \int_0^{2\pi} a_0(t_h(\mu_1, \varphi')) \\ &\quad \times \frac{\partial u_h(\mu_1, \varphi')}{\partial \varphi'} \frac{\partial v_h(\mu_1, \varphi)}{\partial \varphi} \\ &\quad \cdot \cos(j(\varphi' - \varphi)) d\varphi' d\varphi. \end{aligned} \quad (63)$$

In practice, the sum of (63) is truncated to a finite number of terms N . By the hypothesis of $a(\cdot, \cdot)$, it is not difficult to know the following result.

Lemma 4. *There exists a constant $C_2 > 0$ which has a different meaning in a different place and is related to α and β , such that*

$$|D(u; u, v) + \widehat{D}(u; u, v)| \leq C_2 \|u\|_{1, \Omega_1} \|v\|_{1, \Omega_1}, \quad (64)$$

$$D(u; u, u) + \widehat{D}(u; u, u) \geq C_0 \|u\|_{1, \Omega_1}^2, \quad \forall u, v \in V.$$

Proof. From (3) and (40), one can obtain that

$$|D(u; u, v)| \leq C_2' \|u\|_{1, \Omega_1} \|v\|_{1, \Omega_1}, \quad (65)$$

$$D(u; u, u) \geq C_0 \|u\|_{1, \Omega_1}^2, \quad \forall u, v \in V.$$

For the natural integral equation, one can obtain the following result

$$|\widehat{D}(u; u, v)| \leq C_2'' \|u\|_{1, \Omega_1} \|v\|_{1, \Omega_1}, \quad (66)$$

$$\widehat{D}(u; u, u) \geq 0, \quad \forall u, v \in V.$$

For this purpose, for $u, v \in V$, we assume that

$$u(\mu_1, \varphi') = \sum_{j=-\infty}^{+\infty} u_j e^{ij\varphi'}, \quad (67)$$

$$v(\mu_1, \varphi) = \sum_{j=-\infty}^{+\infty} v_j e^{ij\varphi},$$

with $u_j = (1/2\pi) \int_0^{2\pi} u(\mu_1, \varphi') e^{-ij\varphi'} d\varphi'$, $v_j = (1/2\pi) \int_0^{2\pi} v(\mu_1, \varphi) e^{-ij\varphi} d\varphi$. Then, we have

$$\frac{\partial u}{\partial \varphi'}(\mu_1, \varphi') = \sum_{j=-\infty}^{+\infty} i j u_j e^{ij\varphi'}, \quad (68)$$

$$\frac{\partial v}{\partial \varphi}(\mu_1, \varphi) = \sum_{j=-\infty}^{+\infty} i j v_j e^{ij\varphi}.$$

Combining property (3), Cauchy's inequality, and the trace theorem, we have

$$\begin{aligned}
 |\widehat{D}(u; u, v)| &\leq C_2'' \left(\sum_{j=-\infty}^{+\infty} |j| \cdot |u_j|^2 \right)^{1/2} \\
 &\quad \times \left(\sum_{j=-\infty}^{+\infty} |j| \cdot |v_j|^2 \right)^{1/2} \\
 &\leq C_2'' \left(\sum_{j=-\infty}^{+\infty} (1+j^2)^{1/2} \cdot |u_j|^2 \right)^{1/2} \quad (69) \\
 &\quad \times \left(\sum_{j=-\infty}^{+\infty} (1+j^2)^{1/2} \cdot |v_j|^2 \right)^{1/2} \\
 &\leq C_2'' \|u\|_{1/2, \Gamma_{\mu_1}} \|v\|_{1/2, \Gamma_{\mu_1}} \\
 &\leq C_2'' \|u\|_{1, \Omega_1} \|v\|_{1, \Omega_1}, \quad \forall u, v \in V.
 \end{aligned}$$

Next, we show that $\widehat{D}(u; u, u) \geq 0$, for any $u \in V$. For any given $v \in V$, let us consider the following auxiliary problem in Ω_1 :

$$\begin{aligned}
 -\nabla \cdot (a(\mathbf{x}, u) \nabla u) &= 0, \quad \text{in } \Omega_1, \\
 u &= 0, \quad \text{on } \Gamma_0, \\
 u &= v, \quad \text{on } \Gamma_{\mu_1}.
 \end{aligned} \quad (70)$$

From the analysis in Section 2.1, we know that the solution u to problem (70) satisfies

$$a_0(u) \frac{\partial u}{\partial n} \Big|_{\Gamma_{\mu_1}} = \mathcal{K}_1(u(\mu_1, \varphi)) = \mathcal{K}_1(v(\mu_1, \varphi)). \quad (71)$$

Multiplying (70) by u and integrating over Ω_1 , we have

$$\widehat{D}(u; u, u) = \int_{\Omega_1} a_0(u) |\nabla u|^2 d\mathbf{x} \geq 0. \quad (72)$$

Letting $C_2 = C_2' + C_2''$, we can get the desired result. $\square$

We are now in the position to show the existence and uniqueness result for this type of problem (1). We begin with the following estimate.

Lemma 5. Any solution u to problem (39) satisfies

$$\|u\|_{1, \Omega_1} \leq C_3 \|f\|_{L^2(\Omega_1)}, \quad (73)$$

with C_3 a positive constant.

Proof. Taking v equal to u in (39), by Lemma 4, one has

$$C_0 \|u\|_{1, \Omega_1}^2 \leq D(u; u, u) + \widehat{D}(u; u, u) = L(u). \quad (74)$$

Then, the desired result follows from Cauchy-Schwarz and Poincaré-Friedrichs inequalities. $\square$

Theorem 6. Problem (39) is uniquely solvable.

Proof. Since the space V is separable (indeed, it is a closed subspace of the space $H^1(\Omega_1)$ which is separable), there exist increasing sequences of finite-dimensional Hilbert subspaces V^m of V such that

$$V = \bigcup_{m \geq 0} V^m. \quad (75)$$

We define a mapping $\Phi_m : V^m \rightarrow V^m$ by

$$\begin{aligned}
 (\Phi_m(u), v) &= D(u; u, v) + \widehat{D}(u; u, v) \\
 &\quad - L(v), \quad \forall u, v \in V^m,
 \end{aligned} \quad (76)$$

where $(\cdot, \cdot)$ stands for the scalar product on $V^m \times V^m$. Since the function $a(\mathbf{x}, u)$ is bounded and the trace of the function in V on Γ_1 belongs to $H^{1/2}(\Gamma_1)$, hence at least to $L^4(\Gamma_1)$ from the Sobolev embedding, each mapping Φ_m is well defined and continuous on V^m . What's more, by Lemmas 4 and 5, we obtain

$$\begin{aligned}
 (\Phi_m(u), u) &= D(u; u, u) + \widehat{D}(u; u, u) - L(u) \\
 &\geq C_0 \|u\|_{1, \Omega_1}^2 - C_3 \|f\|_{L^2(\Omega_1)} \|u\|_{1, \Omega_1}.
 \end{aligned} \quad (77)$$

So, the right-hand side is nonnegative on the circle of radius μ which is defined by

$$\mu = \frac{C_3}{C_0} \|f\|_{L^2(\Omega_1)}. \quad (78)$$

Applying Brouwer's fixed point theorem [21] yields the existence of u^m in V^m , with norm less than μ , such that

$$\Phi_m(u^m) = 0. \quad (79)$$

Since the sequence $\{u^m\}_m$ is bounded by μ in V , there exists a subsequence which converges weakly to u in V^m . Using the compactness of the imbedding of $H^{1/2}(\Gamma_1)$ into $L^3(\Gamma_1)$, we obtain that u is a solution to problem (39).

Now, we show the uniqueness of the solution. Let u and $\bar{u}$ be two solutions to this problem. Then, taking v equal to u and $\bar{u}$ in (39), respectively, and combining with Lemma 4, one obtains

$$\begin{aligned}
 C_2 \|u\|_{1, \Omega_1} \|\bar{u}\|_{1, \Omega_1} &\geq D(\bar{u}; \bar{u}, u) + \widehat{D}(\bar{u}; \bar{u}, u) \\
 &= D(u; u, u) + \widehat{D}(u; u, u) \\
 &\geq C_0 \|u\|_{1, \Omega_1}^2,
 \end{aligned} \quad (80)$$

$$\begin{aligned}
 C_2 \|u\|_{1, \Omega_1} \|\bar{u}\|_{1, \Omega_1} &\geq D(u; u, \bar{u}) + \widehat{D}(u; u, \bar{u}) \\
 &= D(\bar{u}; \bar{u}, \bar{u}) + \widehat{D}(\bar{u}; \bar{u}, \bar{u}) \geq C_0 \|\bar{u}\|_{1, \Omega_1}^2.
 \end{aligned} \quad (81)$$

Since $C_2 > C_0$, combining (80) with (81), we deduce the desired result. $\square$

FIGURE 2: (a) The relationship between θ_k and convergence rate ($N = 10$, $m = 4$ and $M = 16$). (b) The relationship between ε and convergence rate ($N = 10$, $m = 4$, $M = 16$ and $\theta_k = 0.58$). (c) The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.65$).

From the discrete problem (61), similar to the discussion for the linear problem in [6, 8], we can get a system of algebraic equations for our quasilinear problem with the following form:

$$\begin{pmatrix} A_{11} + K_h & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix}, \quad (82)$$

where $\mathbf{U}$ is a vector whose components are function values at nodes on Γ_1 , and $\mathbf{V}$ is a vector whose components are function values at interior nodes of Ω_1 . The matrix $A \doteq A(u) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is the stiffness matrix obtained from finite element in Ω_1 , while $K_h \doteq K_h(u|_{\Gamma_1})$ follows from the natural boundary element method on Γ_1 .

Problem (82) can also be rewritten as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} -K_h \mathbf{U} \\ \mathbf{b} \end{pmatrix}. \quad (83)$$

Then, we have the iterative algorithm

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U}_k \\ \mathbf{V}_k \end{pmatrix} = \begin{pmatrix} -K_h \Lambda_k \\ \mathbf{b} \end{pmatrix}, \quad (84)$$

with

$$\Lambda_{k+1} = \theta_k \mathbf{U}_k + (1 - \theta_k) \Lambda_k, \quad k = 0, 1, \dots \quad (85)$$

By condition (3), one obtains that A is a positive definite matrix, so we know that A_{22}^{-1} exists. Now, we let $S_h = S_h^{(1)} + K_h$ be the discrete analogue of the Steklov-Poincaré operator on Γ_1 , with $S_h^{(1)} = A_{11} - A_{12} A_{22}^{-1} A_{21}$ and $\mathbf{B} = -A_{12} A_{22}^{-1} \mathbf{b}$. Then, similar to the proof of Theorems 7.6 and 7.7 in [6], we have that the alternating algorithm (84)-(85) is equivalent to the preconditioned Richardson iteration:

$$S_h^{(1)} (\Lambda^{k+1} - \Lambda^k) = \theta_k (\mathbf{B} - S_h \Lambda^k). \quad (86)$$

TABLE 1: The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.58$, and $\mu_1 = 1.5$).

(m, M)	Error	Iteration number								
		0	1	2	3	4	5	6	9	
(2,8)	e	4.9212E-01	3.7241E-01	3.0583E-01	2.7007E-01	2.5029E-01	2.3880E-01	2.3175E-01	2.2183E-01	
	e_h	—	1.1971E-01	6.6581E-02	3.5758E-02	1.9781E-02	1.1487E-02	7.0490E-03	2.2318E-03	
	q_h	—	—	1.7980	1.8620	1.8077	1.7220	1.6296	1.3992	
(4,16)	e	3.4870E-01	1.9037E-01	1.1902E-01	8.8599E-02	7.5058E-02	7.0101E-02	6.9010E-02	6.8244E-02	
	e_h	—	1.5833E-01	7.1346E-02	3.0423E-02	1.3541E-02	6.3774E-03	3.2007E-03	6.2995E-04	
	q_h	—	—	2.2192	2.3452	2.2467	2.1233	1.9925	1.5865	

TABLE 2: The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.58$, and $\mu_1 = 1.0$).

(m, M)	Error	0	1	2	3	4	5	6	9
(2,8)	e	3.7701E-01	3.2871E-01	2.9909E-01	2.8081E-01	2.6906E-01	2.6116E-01	2.5562E-01	2.4614E-01
	e_h	—	4.8299E-02	2.9614E-02	1.8287E-02	1.1749E-02	7.8958E-03	5.5417E-03	2.3769E-03
	q_h	—	—	1.6309	1.6194	1.5564	1.4880	1.4248	1.2857
(4,16)	e	3.0912E-01	2.0526E-01	1.4917E-01	1.1931E-01	1.0260E-01	9.2674E-02	8.6414E-02	7.7207E-02
	e_h	—	1.0386E-01	5.6091E-02	2.9857E-02	1.6710E-02	9.9296E-03	6.2607E-03	2.1229E-03
	q_h	—	—	1.8516	1.8787	1.7868	1.6828	1.5860	1.3728

TABLE 3: The relationship between θ_k and convergence rate ($N = 10$, $m = 4$, and $M = 16$).

θ_k	k	0	1	2	3	4	5	6	9
0.18	e	4.4218E-02	3.8978E-02	3.6443E-02	3.4711E-02	3.3392E-02	3.2341E-02	3.1481E-02	2.9668E-02
	e_h	—	5.2395E-03	2.5350E-03	1.7316E-03	1.3185E-03	1.0519E-03	8.5974E-04	5.0325E-04
	q_h	—	—	2.0668	1.4640	1.3133	1.2535	1.2235	1.1860
0.38	e	4.4218E-02	3.6741E-02	3.3193E-02	3.1073E-02	2.9709E-02	2.8795E-02	2.8167E-02	2.7174E-02
	e_h	—	7.4769E-03	3.5473E-03	2.1199E-03	1.3646E-03	9.1335E-04	6.2830E-04	2.3314E-04
	q_h	—	—	2.1078	1.6733	1.5535	1.4941	1.4537	1.3635
0.50	e	4.4218E-02	3.5398E-02	3.1599E-02	2.9601E-02	2.8465E-02	2.7784E-02	2.7357E-02	2.6751E-02
	e_h	—	8.8194E-03	3.7989E-03	1.9979E-03	1.1359E-04	6.8102E-04	4.2741E-04	1.3535E-05
	q_h	—	—	2.3216	1.9014	1.7588	1.6680	1.5934	1.4114
0.58	e	4.4218E-02	3.4503E-02	3.0685E-02	2.8865E-02	2.7913E-02	2.7377E-02	2.7055E-02	2.6609E-02
	e_h	—	9.7143E-03	3.8180E-03	1.8204E-03	9.5190E-04	5.3571E-04	3.2246E-04	9.9192E-05
	q_h	—	—	2.5443	2.0974	1.9124	1.7769	1.6613	1.4090
0.65	e	4.4218E-02	3.3720E-02	2.9983E-02	2.8358E-02	2.7564E-02	2.7135E-02	2.6882E-02	2.6530E-02
	e_h	—	1.0497E-02	3.7371E-03	1.6248E-03	7.9382E-04	4.2892E-04	2.5372E-04	7.9261E-05
	q_h	—	—	2.8089	2.3000	2.04687	1.8507	1.6905	1.3951

TABLE 4: The relationship between ε and convergence rate ($N = 10$, $m = 4$, $M = 16$, and $\theta_k = 0.58$).

ε	k	0	1	2	3	4	5	6	9
0.75	e	2.3433E-01	1.2450E-01	7.4989E-02	5.2822E-02	4.7623E-02	4.6278E-02	4.5525E-02	4.4564E-02
	e_h	—	1.0983E-01	4.9507E-02	2.2168E-02	1.0239E-02	4.9334E-03	2.5048E-03	4.9131E-04
	q_h	—	—	2.2185	2.2333	2.1651	2.0754	1.9696	1.5930
0.5	e	2.3120E-01	1.3855E-01	9.5116E-02	7.4367E-02	6.3994E-02	5.8518E-02	5.6670E-02	5.5578E-02
	e_h	—	9.2654E-02	4.3431E-02	2.0749E-02	1.0373E-02	5.4758E-03	3.0715E-03	7.8777E-04
	q_h	—	—	2.1334	2.0931	2.0003	1.8943	1.7828	1.4837
0.05	e	3.1906E-01	2.6051E-01	2.3278E-01	2.1796E-01	2.0941E-01	2.0414E-01	2.0068E-01	1.9529E-01
	e_h	—	7.1017E-02	2.7730E-02	1.4816E-02	8.5529E-03	5.2765E-03	3.4545E-04	1.2856E-04
	q_h	—	—	2.5610	1.8716	1.7323	1.6209	1.5274	1.3363
0.005	e	3.6584E-01	3.2417E-01	3.0784E-01	2.9940E-01	2.9434E-01	2.9100E-01	2.8867E-01	2.8468E-01
	e_h	—	1.9591E-01	1.6329E-02	1.8640E-02	1.2301E-02	4.9778E-03	2.3344E-03	9.9644E-04
	q_h	—	—	11.9976	0.8760	1.5153	2.4712	2.1324	1.2896
0.0005	e	3.7328E-01	3.4165E-01	3.3061E-01	3.2510E-01	3.2187E-01	3.1978E-01	3.1833E-01	3.1583E-01
	e_h	—	1.1714E-01	3.6606E-02	1.1542E-02	3.2283E-03	2.0883E-03	1.4491E-03	6.3191E-04
	q_h	—	—	3.2001	3.1715	3.5754	1.5459	1.4411	1.2749

TABLE 5: The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.65$, and $\varepsilon = 0.75$).

(m, M)	Error	Iteration number								
		0	1	2	3	4	5	6	9	
(2,8)	e_0	6.6354E-01	3.8411E-01	2.8271E-01	2.3876E-01	2.1771E-01	2.0659E-01	2.0013E-01	1.9132E-01	
	e	3.2474E-01	2.0835E-01	1.6359E-01	1.4308E-01	1.3278E-01	1.2713E-01	1.2375E-01	1.1900E-01	
	e_h	—	1.5452E-01	5.6859E-02	2.5207E-02	1.2187E-02	6.4127E-03	3.6744E-03	1.0834E-03	
	q_h	—	—	2.7176	2.2557	2.0684	1.9004	1.7452	1.3973	
(4,16)	e_0	4.4653E-01	1.7098E-01	8.8097E-02	5.8527E-02	4.7483E-02	4.2925E-02	4.0759E-02	3.8380E-02	
	e	2.5970E-01	9.5295E-02	5.1105E-02	3.9408E-02	3.4740E-02	3.2516E-02	3.1348E-02	2.9955E-02	
	e_h	—	1.7293E-01	5.2160E-02	2.0094E-02	8.4650E-03	3.8880E-03	1.9662E-03	4.5470E-04	
	q_h	—	—	3.3154	2.5958	2.3738	2.1772	1.9774	1.5007	

TABLE 6: The relationship between meshes and convergence rate ($N = 10$, $\theta_k = 0.65$, and $\varepsilon = 0.50$).

(m, M)	Error	Iteration number								
		0	1	2	3	4	5	6	9	
(2,8)	e_0	8.2535E-01	4.8503E-01	3.7829E-01	3.3445E-01	3.1417E-01	3.0370E-01	2.9771E-01	2.8963E-01	
	e	4.0734E-01	2.6879E-01	2.2125E-01	2.0008E-01	1.8968E-01	1.8407E-01	1.8075E-01	1.7609E-01	
	e_h	—	2.0397E-01	6.6268E-02	2.7907E-02	1.2785E-02	6.3537E-03	3.4410E-03	1.0625E-03	
	q_h	—	—	3.0780	2.3746	2.1829	2.0122	1.8465	1.3787	
(4,16)	e_0	5.3524E-01	1.8866E-01	1.1017E-01	8.6488E-02	7.8123E-02	7.4538E-02	7.2733E-02	7.0595E-02	
	e	3.0655E-01	1.1062E-01	7.6720E-02	6.5692E-02	6.0842E-02	5.8486E-02	5.7219E-02	5.5639E-02	
	e_h	—	2.3531E-01	5.4156E-02	1.7431E-02	6.7638E-03	2.9679E-03	1.4447E-03	3.7294E-04	
	q_h	—	—	4.3449	3.1069	2.5771	2.2790	2.0543	1.4481	

And we also have the following convergence result.

Theorem 7. *If $0 < \min \theta_k \leq \max \theta_k < 1$, then the discrete nonoverlapping alternating method (84)-(85) is convergent, and both the convergence rate and the condition number of $[S_h^{(1)}]^{-1}S_h$ are independent of the mesh size h .*

4. Numerical Examples

In this section, we will give some examples to confirm our theoretical results. In the following, we choose the finite element space as given in (60). For simplicity, we let

$$\begin{aligned} \Delta\mu &= \frac{\mu_1 - \mu_0}{m}, & \Delta\varphi &= \frac{2\pi}{M}, \\ e_0(k) &= \|u - u_{h,N}^k\|_{L^2(\Omega_i)}, \\ e(k) &= \|u - u_{h,N}^k\|_{L^\infty(\Omega_i)}. \end{aligned} \quad (87)$$

And we let $e_h(k)$ denote the maximal error between the iteration $k-1$ and k , that is, $e_h(k) = \|u_{h,N}^k - u_{h,N}^{k-1}\|_{L^\infty(\Omega_i)}$ and $q_h(k) = e_h(k-1)/e_h(k)$ simulate the convergence rate.

We give the following four examples. Examples 8 and 11 show the relationship between meshes and convergence rate for the cases $\alpha = \beta = 1$ and $\beta > \alpha > 0$, respectively. Example 9 focuses on the effect of the relaxation factor θ_k to the convergence rate. And Example 10 wants to show the relationship between the coefficients α, β and the convergence rate.

Example 8. We assume the exterior domain Ω^c with elliptical boundary $\Gamma_0 = \{(\mu_0, \varphi) | \mu_0 = 0.8, 0 \leq \varphi \leq 2\pi\}$, $\Gamma_{\mu_1} = \{(\mu_1, \varphi) | \mu_1 > \mu_0, 0 \leq \varphi \leq 2\pi\}$. Now, we consider the problem

$$\begin{aligned} -\nabla \cdot (a(\mathbf{x}, u) \nabla u) &= f, & \text{in } \Omega_1, \\ u &= 0, & \text{on } \Gamma_0, \\ a_0(u) \frac{\partial u}{\partial n} &= \mathcal{K}_1(u(\mu_1, \varphi)), & \text{on } \Gamma_{\mu_1}, \end{aligned} \quad (88)$$

when $a(\mathbf{x}, u) = 1/(1 + u^2)$, $f = 0$, and $f_0 = 1.35$.

The exact solution of Example 8 is $u = \tan(2 \sinh \mu \sin \varphi / f_0 (\cosh 2\mu + \cos 2\varphi))$. For different meshes and μ_1 , we have the following results ranked in Tables 1 and 2.

Example 9. Similar with Example 8, Γ_0 and $a(\mathbf{x}, u)$ are replaced by $\Gamma_0 = \{(\mu_0, \varphi) | \mu_0 = 0.5, 0 \leq \varphi \leq 2\pi\}$, $\Gamma_1 = \{(\mu_1, \varphi) | \mu_1 = 1.0, 0 \leq \varphi \leq 2\pi\}$, and $a(\mathbf{x}, u) = 1/\sqrt{1 - u^2}$, respectively.

The exact solution of Example 9 is $u = \sin(2 \cosh \mu \cos \varphi / f_0 (\cosh 2\mu + \cos 2\varphi))$. The numerical results are given in Table 3 and Figure 2(a).

Example 10. We assume the exterior domain Ω^c with elliptical boundary $\Gamma_0 = \{(\mu_0, \varphi) | \mu_0 = 0.8, 0 \leq \varphi \leq 2\pi\}$, $\Gamma_{\mu_1} = \{(\mu_1, \varphi) | \mu_1 = 2.0, 0 \leq \varphi \leq 2\pi\}$. Now, we consider the problem

$$\begin{aligned} -\left(\frac{\partial}{\partial x} \left(\varepsilon a(\mathbf{x}, u) \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y} \left(a(\mathbf{x}, u) \frac{\partial u}{\partial y}\right)\right) &= f(\mathbf{x}), \\ &\text{in } \Omega_i, \\ u &= 0, & \text{on } \Gamma_0, \end{aligned} \quad (89)$$

$$\begin{aligned} \varepsilon n_x a_0(u) \frac{\partial u}{\partial x} + n_y a_0(u) \frac{\partial u}{\partial y} \\ = \mathcal{K}_1(u(\mu_1, \varphi)), & \text{on } \Gamma_{\mu_1}, \end{aligned}$$

when $a(\mathbf{x}, u) = 1/(1 + u^2)$, $f_0 = 1.25$ and

$$f = \frac{2(1 - \varepsilon) \sinh \mu \sin \varphi (3 \cosh^2 \mu \cos^2 \varphi - \sinh^2 \mu \sin^2 \varphi)}{(\cosh 2\mu + \cos 2\varphi)^3}. \quad (90)$$

The exact solution of Example 10 is $u = \tan(2 \sinh \mu \sin \varphi / f_0 (\cosh 2\mu + \cos 2\varphi))$. The numerical results are given in Table 4 and Figure 2(b).

Example 11. Similar with Example 10, Γ_0 , Γ_1 , and $a(\mathbf{x}, u)$ are replaced by $\Gamma_0 = \{(\mu_0, \varphi) | \mu_0 = 0.5, 0 \leq \varphi \leq 2\pi\}$, $\Gamma_1 = \{(\mu_1, \varphi) | \mu_1 = 1.5, 0 \leq \varphi \leq 2\pi\}$, and $a(\mathbf{x}, u) = 1/\sqrt{1 - u^2}$, respectively. And we take

$$f = \frac{2(1 - \varepsilon) \cosh \mu \cos \varphi (\cosh^2 \mu \cos^2 \varphi - 3 \sinh^2 \mu \sin^2 \varphi)}{(\cosh 2\mu + \cos 2\varphi)^3}. \quad (91)$$

The exact solution of Example 11 is $u = \sin(2 \cosh \mu \cos \varphi / f_0 (\cosh 2\mu + \cos 2\varphi))$. The numerical results are given in Tables 5 and 6 and Figure 2(c).

From the numerical results, one obtains that the numerical errors can be affected by the choice of relaxation factor θ_k , the coefficient ε , and the location of the artificial boundary. In Tables 1, 2, 5, and 6 and Figure 2(c), the relationship between the meshes and the convergence rate is presented. We obtain that the convergence rate is independent of the finite element mesh size. In Table 3 and Figure 2(c), the convergence rates for different relaxation factor θ_k are compared. The results indicate that the choice of the relaxation factor is very important for the performance of the nonoverlapping alternating method. On the other hand, the convergence rate is not sensitive to the relaxation factor θ_k between the interval (0.5, 0.67). What's more, as it is shown in Table 4 and Figure 2(c), for the anisotropic problems, the choice of ε can also affect the convergence rate.

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