Research Article

Oscillation on a Class of Differential Equations of Fractional Order

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Based on Riccati transformation and certain inequality technique, some new oscillatory criteria are established for the solutions of a class of sequential differential equations with fractional order defined in the modified Riemann-Liouville derivative. The oscillatory criteria established are of new forms compared with the existing results so far in the literature. For illustrating the validity of the results established, we present some examples for them.

1. Introduction

Recently, research for oscillation of various equations including differential equations, difference equations, and dynamic equations on time scales, has been a hot topic in the literature, and much effort has been done to establish new oscillation criteria for these equations so far (e.g., see [1–21] and the references therein). In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations. Recent results in this direction only include Chen's work [22, 23] and Zheng's work [24].

In this paper, we are concerned with oscillation of a class of fractional differential equations as follows:

$$D_t^\alpha \left[a(t) (D_t^\alpha (r(t) D_t^\alpha x(t)))^\gamma \right] + q(t) f(x(t)) = 0, \quad (1)$$

$$t \geq t_0 > 0, \quad 0 < \alpha < 1,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative [25] with respect to the variable t , γ is the quotient of two odd positive numbers, the functions $a \in C^\alpha([t_0, \infty), R_+)$, $r \in C^{2\alpha}([t_0, \infty), R_+)$, and $q \in C([t_0, \infty), R_+)$, C^α denotes continuous derivative of order α , and the function f is continuous satisfying $f(x)/x^\gamma \geq K$ for some positive constant K and for all $x \neq 0$.

The definition and some important properties for Jumarie's modified Riemann-Liouville derivative of order α are listed as follows (see also in [26–28]):

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \times (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1, \end{cases} \quad (2)$$

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},$$

$$D_t^\alpha (f(t) g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)] D_t^\alpha g(t) = D_g^\alpha f[g(t)] (g'(t))^\alpha. \quad (4)$$

As usual, a solution $x(t)$ of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

We organize this paper as follows. In Section 2, using Riccati transformation, inequality, and integration average

technique, we establish some new oscillatory criteria for (1), while we present some examples for them in Section 3.

2. Oscillatory Criteria for (1)

In the following, we denote $\xi = t^\alpha/\Gamma(1+\alpha)$, $\xi_i = t_i^\alpha/\Gamma(1+\alpha)$, $i = 0, 1, 2, 3, 4, 5$, $a(t) = \bar{a}(\xi)$, $r(t) = \bar{r}(\xi)$, $q(t) = \bar{q}(\xi)$, $R_+ = (0, \infty)$, $\bar{\delta}_1(\xi, \xi_i) = \int_{\xi_i}^{\xi} (1/\bar{a}^{1/\gamma}(s)) ds$, $\delta_1(t, t_i) = \bar{\delta}_1(\xi, \xi_i)$, and $\bar{\delta}_2(\xi, \xi_i) = \int_{\xi_i}^{\xi} (\bar{\delta}_1(s, \xi_i)/\bar{r}(s)) ds$, $\delta_2(t, t_i) = \bar{\delta}_2(\xi, \xi_i)$.

Lemma 1. Assume that $x(t)$ is an eventually positive solution of (1), and

$$\int_{\xi_0}^{\infty} \frac{1}{\bar{a}^{1/\gamma}(s)} ds = \infty, \quad (5)$$

$$\int_{t_0}^{\infty} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt = \infty, \quad (6)$$

$$\int_{\xi_0}^{\infty} \frac{1}{\bar{r}(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{\bar{a}(\tau)} \int_{\tau}^{\infty} \bar{q}(s) ds \right]^{1/\gamma} d\tau d\xi = \infty. \quad (7)$$

Then there exists a sufficiently large T such that $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$ on $[T, \infty)$ and either $D_t^\alpha x(t) > 0$ on $[T, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Let $x(t) = \bar{x}(\xi)$, where $\xi = t^\alpha/\Gamma(1+\alpha)$. Then by use of (2) we obtain $D_t^\alpha \xi(t) = 1$, and furthermore by use of the first equality in (4), we have

$$D_t^\alpha x(t) = D_t^\alpha \bar{x}(\xi) = \bar{x}'(\xi) D_t^\alpha \xi(t) = \bar{x}'(\xi). \quad (8)$$

Similarly we have $D_t^\alpha a(t) = \bar{a}'(\xi)$, $D_t^\alpha r(t) = \bar{r}'(\xi)$. So (1) can be transformed into the following form:

$$\left[\bar{a}(\xi) \left((\bar{r}(\xi) \bar{x}'(\xi))' \right)' \right]' + \bar{q}(\xi) f(\bar{x}(\xi)) = 0, \quad \xi \geq \xi_0 > 0. \quad (9)$$

Since $x(t)$ is an eventually positive solution of (1), then $\bar{x}(\xi)$ is an eventually positive solution of (9), and there exists $\xi_1 > \xi_0$ such that $\bar{x}(\xi) > 0$ on $[\xi_1, \infty)$. Furthermore, we have

$$\begin{aligned} & \left[\bar{a}(\xi) \left((\bar{r}(\xi) \bar{x}'(\xi))' \right)' \right]' \\ &= -\bar{q}(\xi) f(\bar{x}(\xi)) \\ &\leq -K\bar{q}(\xi) \bar{x}^\gamma(\xi) < 0, \quad \xi \geq \xi_1. \end{aligned} \quad (10)$$

Then $\bar{a}(\xi)((\bar{r}(\xi)\bar{x}'(\xi))')^\gamma$ is strictly decreasing on $[\xi_1, \infty)$, and thus $(\bar{r}(\xi)\bar{x}'(\xi))'$ is eventually of one sign. We claim $(\bar{r}(\xi)\bar{x}'(\xi))' > 0$ on $[\xi_2, \infty)$, where $\xi_2 > \xi_1$ is sufficiently large. Otherwise, assume that there exists a sufficiently large $\xi_3 > \xi_2$

such that $(\bar{r}(\xi)\bar{x}'(\xi))' < 0$ on $[\xi_3, \infty)$. Then $\bar{r}(\xi)\bar{x}'(\xi)$ is strictly decreasing on $[\xi_3, \infty)$, and we have

$$\begin{aligned} & \bar{r}(\xi) \bar{x}'(\xi) - \bar{r}(\xi_3) \bar{x}'(\xi_3) \\ &= \int_{\xi_3}^{\xi} \frac{\bar{a}^{1/\gamma}(s) (\bar{r}(s) \bar{x}'(s))'}{\bar{a}^{1/\gamma}(s)} ds \\ &\leq \bar{a}^{1/\gamma}(\xi_3) (\bar{r}(\xi_3) \bar{x}'(\xi_3))' \int_{\xi_3}^{\xi} \frac{1}{\bar{a}^{1/\gamma}(s)} ds. \end{aligned} \quad (11)$$

By (5), we have $\lim_{\xi \rightarrow \infty} \bar{r}(\xi)\bar{x}'(\xi) = -\infty$. So there exists a sufficiently large ξ_4 with $\xi_4 > \xi_3$ such that $\bar{x}'(\xi) < 0$, $\xi \in [\xi_4, \infty)$. Furthermore,

$$\begin{aligned} \bar{x}(\xi) - \bar{x}(\xi_4) &= \int_{\xi_4}^{\xi} \bar{x}'(s) ds = \int_{\xi_4}^{\xi} \frac{\bar{r}(s) \bar{x}'(s)}{\bar{r}(s)} ds \\ &\leq \bar{r}(\xi_4) \bar{x}'(\xi_4) \int_{\xi_4}^{\xi} \frac{1}{\bar{r}(s)} ds \\ &= \bar{r}(\xi_4) \bar{x}'(\xi_4) \int_{t_4}^{\infty} \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)r(t)} dt. \end{aligned} \quad (12)$$

By (6) we deduce that $\lim_{\xi \rightarrow \infty} \bar{x}(\xi) = -\infty$, which contradicts the fact that $\bar{x}(\xi)$ is an eventually positive solution of (6). So $(\bar{r}(\xi)\bar{x}'(\xi))' > 0$ on $[\xi_2, \infty)$, and $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$ on $[t_2, \infty)$. Thus $D_t^\alpha x(t) = \bar{x}'(\xi)$ is eventually of one sign. Now we assume that $\bar{x}'(\xi) < 0$, $\xi \in [\xi_5, \infty)$, for some sufficiently large $\xi_5 > \xi_4$. Since $\bar{x}(\xi) > 0$, furthermore we have $\lim_{\xi \rightarrow \infty} \bar{x}(\xi) = \beta \geq 0$. We claim that $\beta = 0$. Otherwise, assume that $\beta > 0$. Then $\bar{x}(\xi) \geq \beta$ on $[\xi_5, \infty)$, and for $\xi \in [\xi_5, \infty)$ by (9) we have

$$\left[\bar{a}(\xi) \left((\bar{r}(\xi) \bar{x}'(\xi))' \right)' \right]' \leq -K\bar{q}(\xi) \bar{x}^\gamma(\xi) \leq -K\bar{q}(\xi) \beta^\gamma. \quad (13)$$

Substituting ξ with s in the inequality previously, an integration with respect to s from ξ to ∞ yields

$$\begin{aligned} & -\bar{a}(\xi) \left((\bar{r}(\xi) \bar{x}'(\xi))' \right)' \\ &\leq -\lim_{\xi \rightarrow \infty} \bar{a}(\xi) \left((\bar{r}(\xi) \bar{x}(\xi))' \right)' \\ &= -K\beta^\gamma \int_{\xi}^{\infty} \bar{q}(s) ds < -K\beta^\gamma \int_{\xi}^{\infty} \bar{q}(s) ds, \end{aligned} \quad (14)$$

which means

$$(\bar{r}(\xi) \bar{x}'(\xi))' > K^{1/\gamma} \beta \left[\frac{1}{\bar{a}(\xi)} \int_{\xi}^{\infty} \bar{q}(s) ds \right]^{1/\gamma}. \quad (15)$$

Substituting ξ with τ in (15), an integration for (15) with respect to τ from ξ to ∞ yields

$$\begin{aligned} -\bar{r}(\xi) \bar{x}'(\xi) &> -\lim_{\xi \rightarrow \infty} \bar{r}(\xi) \bar{x}'(\xi) \\ &+ K^{1/\gamma} \beta \int_{\xi}^{\infty} \left[\frac{1}{\bar{a}(\tau)} \int_{\tau}^{\infty} \bar{q}(s) ds \right]^{1/\gamma} d\tau \quad (16) \\ &> K^{1/\gamma} \beta \int_{\xi}^{\infty} \left[\frac{1}{\bar{a}(\tau)} \int_{\tau}^{\infty} \bar{q}(s) ds \right]^{1/\gamma} d\tau; \end{aligned}$$

that is,

$$\bar{x}'(\xi) < -K^{1/\gamma} \beta \frac{1}{\bar{r}(\xi)} \int_{\xi}^{\infty} \left[\frac{1}{\bar{a}(\tau)} \int_{\tau}^{\infty} \bar{q}(s) ds \right]^{1/\gamma} d\tau. \quad (17)$$

Substituting ξ with ζ in (17), an integration for (17) with respect to ζ from ξ_5 to ξ yields

$$\begin{aligned} \bar{x}(\xi) - \bar{x}(\xi_5) \\ < -K^{1/\gamma} \beta \int_{\xi_5}^{\xi} \frac{1}{\bar{r}(\zeta)} \int_{\zeta}^{\infty} \left[\frac{1}{\bar{a}(\tau)} \int_{\tau}^{\infty} \bar{q}(s) ds \right]^{1/\gamma} d\tau d\zeta. \quad (18) \end{aligned}$$

By (7), one can see $\lim_{t \rightarrow \infty} \bar{x}(t) = -\infty$, which causes a contradiction. So the proof is complete. $\square$

Lemma 2. Assume that x is an eventually positive solution of (1) such that

$$D_t^\alpha(r(t) D_t^\alpha x(t)) > 0, \quad D_t^\alpha x(t) > 0 \quad (19)$$

on $[t_1, \infty)_{\mathbb{T}}$, where $t_1 \geq t_0$ is sufficiently large. Then one has

$$D_t^\alpha x(t) \geq \frac{\delta_1(t, t_1) a^{1/\gamma}(t) D_t^\alpha(r(t) D_t^\alpha x(t))}{r(t)}, \quad (20)$$

$$x(t) \geq \delta_2(t, t_1) a^{1/\gamma}(t) D_t^\alpha(r(t) D_t^\alpha x(t)). \quad (21)$$

Proof. By (10), we obtain that $\bar{a}(\xi)((\bar{r}(\xi)\bar{x}'(\xi))')^\gamma$ is strictly decreasing on $[\xi_1, \infty)$. So

$$\begin{aligned} \bar{r}(\xi) \bar{x}'(\xi) &\geq \bar{r}(\xi) \bar{x}'(\xi) - \bar{r}(\xi_1) \bar{x}'(\xi_1) \\ &= \int_{\xi_1}^{\xi} \frac{\bar{a}^{1/\gamma}(s) (\bar{r}(s) \bar{x}'(s))'}{\bar{a}^{1/\gamma}(s)} ds \quad (22) \\ &\geq \bar{a}^{1/\gamma}(\xi) (\bar{r}(\xi) \bar{x}'(\xi))' \int_{\xi_1}^{\xi} \frac{1}{\bar{a}^{1/\gamma}(s)} ds \\ &= \bar{a}^{1/\gamma}(\xi) (\bar{r}(\xi) \bar{x}'(\xi))' \bar{\delta}_1(\xi, \xi_1); \end{aligned}$$

that is

$$r(t) D_t^\alpha x(t) \geq \delta_1(t, t_1) a^{1/\gamma}(t) D_t^\alpha(r(t) D_t^\alpha x(t)), \quad (23)$$

which admits (20). On the other hand, we have

$$\begin{aligned} \bar{x}(\xi) &\geq \bar{x}(\xi) - \bar{x}(\xi_1) \geq \int_{\xi_1}^{\xi} \frac{\bar{\delta}_1(s, \xi_1) \bar{a}^{1/\gamma}(s) (\bar{r}(s) \bar{x}'(s))'}{\bar{r}(s)} ds \\ &\geq \bar{a}^{1/\gamma}(\xi) (\bar{r}(\xi) \bar{x}'(\xi))' \int_{\xi_1}^{\xi} \frac{\bar{\delta}_1(s, \xi_1)}{\bar{r}(s)} ds \\ &= \bar{\delta}_2(\xi, \xi_1) \bar{a}^{1/\gamma}(\xi) (\bar{r}(\xi) \bar{x}'(\xi))', \quad (24) \end{aligned}$$

which can be rewritten as (21). So the proof is complete. $\square$

Lemma 3 (see [29, Theorem 41]). Assume that A and B are nonnegative real numbers. Then

$$\lambda AB^{\lambda-1} - A^\lambda \leq (\lambda - 1) B^\lambda \quad (25)$$

for all $\lambda > 1$.

Theorem 4. Let $h_1, h_2, \widehat{H} \in C([\xi_0, \infty), \mathbb{R})$ satisfying $\widehat{H}(\xi, \xi) = 0$, $\widehat{H}(\xi, s) > 0$, $\xi > s \geq \xi_0$, and H has continuous partial derivatives $\widehat{H}'_\xi(\xi, s)$ and $\widehat{H}'_s(\xi, s)$ on $[\xi_0, \infty)$. Assume that (5)–(7) hold, and for any sufficiently large $T \geq \xi_0$, there exist $\phi \in C^1([t_0, \infty), \mathbb{R}_+)$ and $\varphi \in C^1([t_0, \infty), [0, \infty))$ and a, b , and c with $T \leq a < c < b$ satisfying

$$\begin{aligned} &\frac{1}{\widehat{H}(b, c)} \int_c^b \widehat{H}(b, s) \left\{ K \bar{\phi}(s) \bar{q}(s) - \bar{\phi}(s) \bar{\phi}'(s) \right. \\ &\quad \left. + \frac{\bar{\phi}(s) \bar{\delta}_1(s, \xi_2) \bar{\varphi}^{1+(1/\gamma)}(s)}{\bar{r}(s)} \right\} ds \\ &+ \frac{1}{\widehat{H}(c, a)} \int_a^c \widehat{H}(s, a) \left\{ K \bar{\phi}(s) \bar{q}(s) - \bar{\phi}(s) \bar{\phi}'(s) \right. \\ &\quad \left. + \frac{\bar{\phi}(s) \bar{\delta}_1(s, \xi_2) \bar{\varphi}^{1+(1/\gamma)}(s)}{\bar{r}(s)} \right\} ds \\ &> \frac{1}{\widehat{H}(b, c)} \int_c^b \widehat{H}(b, s) \left\{ \left([(\gamma + 1) \bar{\varphi}^{1/\gamma}(s) \bar{\phi}(s) \bar{\delta}_1(s, \xi_2) \right. \right. \\ &\quad \left. \left. + \bar{r}(s) \bar{\phi}'(s) \right. \right. \\ &\quad \left. \left. + \bar{r}(s) \bar{\phi}(s) \widehat{H}'_s(b, s) \right)^{\gamma+1} \right. \\ &\quad \left. \times \left((\gamma + 1)^{\gamma+1} \right. \right. \\ &\quad \left. \left. \times [\bar{\phi}(s) \bar{\delta}_1(s, \xi_2)]^\gamma \bar{r}(s) \right)^{-1} \right\} ds \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\tilde{H}(c, a)} \int_a^c \tilde{H}(s, a) \left\{ \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(s, a) \right]^{\gamma+1} \right) \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds,
\end{aligned} \tag{26}$$

where $\tilde{\phi}(\xi) = \phi(t)$, $\tilde{\varphi}(\xi) = \varphi(t)$. Then every solution of (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. Now we assume that $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$.

Define the generalized Riccati function

$$\omega(t) = \phi(t) \left\{ \frac{a(t) (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma}{x^\gamma(t)} + \varphi(t) \right\}. \tag{27}$$

Then for $t \in [t_2, \infty)$, we have

$$\begin{aligned}
D_t^\alpha \omega(t) &= D_t^\alpha \phi(t) \frac{a(t) (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma}{x^\gamma(t)} \\
&\quad + \phi(t) D_t^\alpha \left\{ \frac{a(t) (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma}{x^\gamma(t)} \right\} \\
&\quad + D_t^\alpha \phi(t) \varphi(t) + \phi(t) D_t^\alpha \varphi(t) \\
&= \phi(t) \left\{ \left(x^\gamma(t) D_t^\alpha \left(a(t) (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma \right) \right. \right. \\
&\quad \left. \left. - \gamma x^{\gamma-1}(t) D_t^\alpha x(t) a(t) \right. \right. \\
&\quad \left. \left. \times (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma \right. \right. \\
&\quad \left. \left. \times (x^{2\gamma}(t))^{-1} \right\} \\
&\quad + \frac{D_t^\alpha \phi(t)}{\phi(t)} \omega(t) + \phi(t) D_t^\alpha \varphi(t) \\
&= - \frac{\phi(t) q(t) f(x(t))}{x^\gamma(t)} \\
&\quad - \frac{\gamma \phi(t) D_t^\alpha x(t) a(t) (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma}{x^{\gamma+1}(t)} \\
&\quad + \frac{D_t^\alpha \phi(t)}{\phi(t)} \omega(t) + \phi(t) D_t^\alpha \varphi(t).
\end{aligned} \tag{28}$$

By Lemma 2 and the definition of f we get that

$$\begin{aligned}
D_t^\alpha \omega(t) &\leq -K\phi(t) q(t) \\
&\quad - \left((\gamma \phi(t) \delta_1(t, t_2) a^{1/\gamma}(t) D_t^\alpha(r(t)D_t^\alpha x(t)) a(t) \right. \\
&\quad \left. \times (D_t^\alpha(r(t)D_t^\alpha x(t)))^\gamma \right. \\
&\quad \left. \times (r(t)x^{\gamma+1}(t))^{-1} \right) \\
&\quad + \frac{D_t^\alpha \phi(t)}{\phi(t)} \omega(t) + \phi(t) D_t^\alpha \varphi(t) \\
&= -K\phi(t) q(t) - \frac{\gamma \phi(t) \delta_1(t, t_2)}{r(t)} \left[\frac{\omega(t)}{\phi(t)} - \varphi(t) \right]^{1+(1/\gamma)} \\
&\quad + \frac{D_t^\alpha \phi(t)}{\phi(t)} \omega(t) + \phi(t) D_t^\alpha \varphi(t).
\end{aligned} \tag{29}$$

Using the following inequality (see [2, Equation (2.17)])

$$(u - v)^{1+(1/\gamma)} \geq u^{1+(1/\gamma)} + \frac{1}{\gamma} v^{1+(1/\gamma)} - \left(1 + \frac{1}{\gamma} \right) v^{1/\gamma} u, \tag{30}$$

we obtain

$$\begin{aligned}
\left[\frac{\omega(t)}{\phi(t)} - \varphi(t) \right]^{1+(1/\gamma)} &\geq \frac{\omega^{1+(1/\gamma)}(t)}{\phi^{1+(1/\gamma)}(t)} + \frac{1}{\gamma} \varphi^{1+(1/\gamma)}(t) \\
&\quad - \left(1 + \frac{1}{\gamma} \right) \frac{\varphi^{1/\gamma}(t) \omega(t)}{\phi(t)}.
\end{aligned} \tag{31}$$

A combination of (29) and (31) yields:

$$\begin{aligned}
D_t^\alpha \omega(t) &\leq -K\phi(t) q(t) + \frac{D_t^\alpha \phi(t)}{\phi(t)} \omega(t) + \phi(t) D_t^\alpha \varphi(t) \\
&\quad - \frac{\gamma \phi(t) \delta_1(t, t_2)}{r(t)} \left[\frac{\omega^{1+(1/\gamma)}(t)}{\phi^{1+(1/\gamma)}(t)} + \frac{1}{\gamma} \varphi^{1+(1/\gamma)}(t) \right. \\
&\quad \left. - \left(1 + \frac{1}{\gamma} \right) \frac{\varphi^{1/\gamma}(t) \omega(t)}{\phi(t)} \right] \\
&= -K\phi(t) q(t) + \phi(t) D_t^\alpha \varphi(t) \\
&\quad - \frac{\phi(t) \delta_1(t, t_2) \varphi^{1+(1/\gamma)}(t)}{r(t)} \\
&\quad - \frac{\gamma \phi(t) \delta_1(t, t_2) \omega^{1+(1/\gamma)}(t)}{r(t) \phi^{1+(1/\gamma)}(t)} \\
&\quad + \frac{(\gamma + 1) \varphi^{1/\gamma}(t) \phi(t) \delta_1(t, t_2) + r(t) D_t^\alpha \phi(t)}{r(t) \phi(t)} \\
&\quad \times \omega(t), \quad t \geq t_2.
\end{aligned} \tag{32}$$

Let $\omega(t) = \bar{\omega}(\xi)$. Then $D_t^\alpha \omega(t) = \bar{\omega}'(\xi)$, $D_t^\alpha \phi(t) = \bar{\phi}'(\xi)$, and $D_t^\alpha \varphi(t) = \bar{\varphi}'(\xi)$, and (32) is transformed into the following form:

$$\begin{aligned} & \bar{\omega}'(\xi) \\ & \leq -K\bar{\phi}(\xi)\bar{q}(\xi) + \bar{\phi}(\xi)\bar{\varphi}'(\xi) \\ & \quad - \frac{\bar{\phi}(\xi)\bar{\delta}_1(\xi, \xi_2)\bar{\varphi}^{1+(1/\gamma)}(\xi)}{r(\xi)} \\ & \quad - \frac{\gamma\bar{\delta}_1(\xi, \xi_2)}{\bar{r}(\xi)\bar{\phi}^{1/\gamma}(\xi)}\bar{\omega}^{1+(1/\gamma)}(\xi) \\ & \quad + \frac{(\gamma+1)\bar{\varphi}^{1/\gamma}(\xi)\bar{\phi}(\xi)\bar{\delta}_1(\xi, \xi_2) + \bar{r}(\xi)\bar{\phi}'(\xi)}{\bar{r}(\xi)\bar{\phi}(\xi)}\bar{\omega}(\xi), \\ & \quad \xi \geq \xi_2. \end{aligned} \quad (33)$$

Select a , b , and c arbitrarily in $[\xi_2, \infty)$ with $b > c > a$. Substituting ξ with s , multiplying both sides of (33) by $\bar{H}(\xi, s)$, and integrating it with respect to s from c to ξ for $\xi \in [c, b)$, we get that

$$\begin{aligned} & \int_c^\xi \bar{H}(\xi, s) \left\{ K\bar{\phi}(s)\bar{q}(s) - \bar{\phi}(s)\bar{\varphi}'(s) \right. \\ & \quad \left. + \frac{\bar{\phi}(s)\bar{\delta}_1(s, \xi_2)\bar{\varphi}^{1+(1/\gamma)}(s)}{\bar{r}(s)} \right\} ds \\ & \leq - \int_c^\xi \bar{H}(\xi, s) \bar{\omega}'(s) ds \\ & \quad + \int_c^\xi \bar{H}(\xi, s) \left\{ - \frac{\gamma\bar{\delta}_1(s, \xi_2)}{\bar{r}(s)\bar{\phi}^{1/\gamma}(s)}\bar{\omega}^{1+(1/\gamma)}(s) \right. \\ & \quad + \left(((\gamma+1)\bar{\varphi}^{1/\gamma}(s)\bar{\phi}(s)\bar{\delta}_1(s, \xi_2) \right. \\ & \quad \left. + \bar{r}(s)\bar{\phi}'(s)) \right. \\ & \quad \left. \times (\bar{r}(s)\bar{\phi}(s))^{-1} \right) \bar{\omega}(s) \Big\} ds \\ & = \bar{H}(\xi, c) \bar{\omega}(c) \\ & \quad + \int_c^\xi \bar{H}(\xi, s) \left\{ - \frac{\gamma\bar{\delta}_1(s, \xi_2)}{\bar{r}(s)\bar{\phi}^{1/\gamma}(s)}\bar{\omega}^{1+(1/\gamma)}(s) \right. \\ & \quad + \left(((\gamma+1)\bar{\varphi}^{1/\gamma}(s)\bar{\phi}(s)\bar{\delta}_1(s, \xi_2) \right. \\ & \quad \left. + \bar{r}(s)\bar{\phi}'(s) + \bar{r}(s)\bar{\phi}(s)\bar{H}'_s(\xi, s)) \right. \\ & \quad \left. \times (\bar{r}(s)\bar{\phi}(s))^{-1} \right) \bar{\omega}(s) \Big\} ds. \end{aligned} \quad (34)$$

Setting

$$\begin{aligned} \lambda &= 1 + \frac{1}{\gamma}, \quad A^\lambda = \frac{\gamma\bar{\delta}_1(s, \xi_2)}{\bar{r}(s)\bar{\phi}^{1/\gamma}(s)}\bar{\omega}^{1+(1/\gamma)}(s), \\ B^{\lambda-1} &= \gamma^{1/(\gamma+1)} \\ & \times \left(((\gamma+1)\bar{\varphi}^{1/\gamma}(s)\bar{\phi}(s)\bar{\delta}_1(s, \xi_2) \right. \\ & \quad \left. + \bar{r}(s)\bar{\phi}'(s) + \bar{r}(s)\bar{\phi}(s)\bar{H}'_s(\xi, s)) \right. \\ & \quad \left. \times ((\gamma+1)[\bar{\phi}(s)\bar{\delta}_1(s, \xi_2)]^{\gamma/(\gamma+1)}\bar{r}^{1/(\gamma+1)}(s))^{-1} \right), \end{aligned} \quad (35)$$

by a combination of Lemma 3 and (33), we get that

$$\begin{aligned} & \int_c^s \bar{H}(s, s) \left\{ K\bar{\phi}(s)\bar{q}(s) - \bar{\phi}(s)\bar{\varphi}'(s) \right. \\ & \quad \left. + \frac{\bar{\phi}(s)\bar{\delta}_1(s, \xi_2)\bar{\varphi}^{1+(1/\gamma)}(s)}{\bar{r}(s)} \right\} ds \\ & \leq \bar{H}(s, c) \bar{\omega}(c) \\ & \quad + \int_c^s \bar{H}(s, s) \\ & \quad \times \left\{ \left(((\gamma+1)\bar{\varphi}^{1/\gamma}(s)\bar{\phi}(s)\bar{\delta}_1(s, \xi_2) \right. \right. \\ & \quad \left. \left. + \bar{r}(s)\bar{\phi}'(s) \right. \right. \\ & \quad \left. \left. + \bar{r}(s)\bar{\phi}(s)\bar{H}'_s(\xi, s) \right)^{\gamma+1} \right. \\ & \quad \left. \times ((\gamma+1)^{\gamma+1}[\bar{\phi}(s)\bar{\delta}_1(s, \xi_2)]^\gamma\bar{r}(s))^{-1} \right\} ds. \end{aligned} \quad (36)$$

Dividing both sides of inequality (36) by $\bar{H}(\xi, c)$ and letting $\xi \rightarrow b^-$, we obtain

$$\begin{aligned} & \frac{1}{\bar{H}(b, c)} \int_c^b \bar{H}(b, s) \left\{ K\bar{\phi}(s)\bar{q}(s) \right. \\ & \quad \left. - \bar{\phi}(s)\bar{\varphi}'(s) \right. \\ & \quad \left. + \frac{\bar{\phi}(s)\bar{\delta}_1(s, \xi_2)\bar{\varphi}^{1+(1/\gamma)}(s)}{\bar{r}(s)} \right\} ds \\ & \leq \bar{\omega}(c) \\ & \quad + \frac{1}{\bar{H}(b, c)} \int_c^b \bar{H}(b, s) \\ & \quad \times \left\{ \left(((\gamma+1)\bar{\varphi}^{1/\gamma}(s)\bar{\phi}(s)\bar{\delta}_1(s, \xi_2) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \tilde{r}(s) \tilde{\phi}'(s) \\
& + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(b, s) \Big]^{(\gamma+1)} \\
& \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds. \\
& \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds.
\end{aligned} \tag{37}$$

On the other hand, substituting ξ with s , multiplying both sides of (33) by $\tilde{H}(s, \xi)$, and integrating it with respect to s from ξ to c for $\xi \in (a, c]$, similar to (36)-(37), we get that

$$\begin{aligned}
& \int_\xi^c \tilde{H}(s, \xi) \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\phi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \\
& \leq -\tilde{H}(c, \xi) \tilde{w}(c) \\
& \quad + \int_\xi^c \tilde{H}(s, \xi) \left\{ \left([(\gamma+1) \tilde{\phi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(s, \xi) \right]^{(\gamma+1)} \right. \\
& \quad \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds.
\end{aligned} \tag{38}$$

Dividing both sides of inequality (38) by $\tilde{H}(c, \xi)$ and letting $\xi \rightarrow a^+$, we obtain

$$\begin{aligned}
& \frac{1}{\tilde{H}(c, a)} \int_a^c \tilde{H}(s, a) \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\
& \quad \left. - \tilde{\phi}(s) \tilde{\phi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \\
& \leq -\tilde{w}(c) \\
& \quad + \frac{1}{\tilde{H}(c, a)} \int_a^c \tilde{H}(s, a) \\
& \quad \times \left\{ \left([(\gamma+1) \tilde{\phi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(s, a) \right]^{(\gamma+1)} \right. \\
& \quad \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds,
\end{aligned}$$

A combination of (37) and (39) yields

$$\begin{aligned}
& \frac{1}{\tilde{H}(b, c)} \int_c^b \tilde{H}(b, s) \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\
& \quad \left. - \tilde{\phi}(s) \tilde{\phi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \\
& \quad + \frac{1}{\tilde{H}(c, a)} \int_a^c \tilde{H}(s, a) \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\
& \quad \left. - \tilde{\phi}(s) \tilde{\phi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \\
& \leq \frac{1}{\tilde{H}(b, c)} \int_c^b \tilde{H}(b, s) \left\{ \left([(\gamma+1) \tilde{\phi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(b, s) \right]^{(\gamma+1)} \right. \\
& \quad \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \\
& \quad + \frac{1}{\tilde{H}(c, a)} \int_a^c \tilde{H}(s, a) \\
& \quad \times \left\{ \left([(\gamma+1) \tilde{\phi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}(s) \tilde{H}'_s(s, a) \right]^{(\gamma+1)} \right. \\
& \quad \times \left((\gamma+1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds,
\end{aligned} \tag{40}$$

which contradicts (26). So the proof is complete. $\square$

Theorem 5. Under the conditions of Theorem 4, if for any sufficiently large $l \geq \xi_0$,

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \sup \int_l^\xi \widehat{H}(s, l) & \left\{ K\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ & - \left(\left[(\gamma+1)\tilde{\phi}^{1/\gamma}(s)\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}'(s) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}(s)\widehat{H}'_s(s, l) \right]^{(\gamma+1)} \right. \\ & \quad \left. \times \left((\gamma+1)^{\gamma+1} \right. \right. \\ & \quad \left. \left. \times [\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds \\ & > 0, \end{aligned} \quad (41)$$

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \sup \int_l^\xi \widehat{H}(\xi, s) & \left\{ K\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ & - \left(\left[\tilde{\phi}^{1/\gamma}(s)\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}'(s) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}(s)\widehat{H}'_s(\xi, s) \right]^{(\gamma+1)} \right. \\ & \quad \times \left((\gamma+1)^{\gamma+1} \right. \\ & \quad \left. \times [\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \\ & > 0, \end{aligned} \quad (42)$$

then (1) is oscillatory.

Proof. For any $T \geq \xi_0$, let $a = T$. In (41) we choose $l = a$. Then there exists $c > a$ such that

$$\begin{aligned} \int_a^c \widehat{H}(s, a) & \left\{ K\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ & - \left(\left[(\gamma+1)\tilde{\phi}^{1/\gamma}(s)\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}'(s) + \tilde{r}(s)\tilde{\phi}(s)\widehat{H}'_s(s, a) \right]^{(\gamma+1)} \right\} ds \end{aligned}$$

$$\begin{aligned} & \times \left((\gamma+1)^{\gamma+1} \right. \\ & \quad \left. \times [\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds > 0. \end{aligned} \quad (43)$$

In (42) we choose $l = c > a$. Then there exists $b > c$ such that

$$\begin{aligned} \int_c^b \widehat{H}(b, s) & \left\{ K\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ & - \left(\left[(\gamma+1)\tilde{\phi}^{1/\gamma}(s)\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2) \right. \right. \\ & \quad \left. \left. + \tilde{r}(s)\tilde{\phi}'(s) + \tilde{r}(s)\tilde{\phi}(s)\widehat{H}'_s(b, s) \right]^{(\gamma+1)} \right. \\ & \quad \times \left((\gamma+1)^{\gamma+1} \right. \\ & \quad \left. \times [\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds > 0. \end{aligned} \quad (44)$$

Combining (43) and (44) we obtain (26). The conclusion thus comes from Theorem 4, and the proof is complete. $\square$

In Theorems 4 and 5, if we choose $\widehat{H}(\xi, s) = (\xi - s)^\lambda$, $\xi \geq s \geq \xi_0$, where $\lambda > 1$ is a constant, then we obtain the following two corollaries.

Corollary 6. Under the conditions of Theorem 4, if for any sufficiently large $T \geq \xi_0$, there exist a, b, c with $T \leq a < c < b$ satisfying

$$\begin{aligned} \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^\lambda & \left\{ K\tilde{\phi}(s)\tilde{q}(s) \right. \\ & \quad \left. - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & \quad \left. + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \\ & + \frac{1}{(b-c)^\lambda} \int_c^b (b-s)^\lambda \\ & \quad \times \left\{ K\tilde{\phi}(s)\tilde{q}(s) \right. \\ & \quad \left. - \tilde{\phi}(s)\tilde{\phi}'(s) \right. \\ & \quad \left. + \frac{\tilde{\phi}(s)\tilde{\delta}_1(s, \xi_2)\tilde{\phi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\} ds \end{aligned}$$

$$\begin{aligned}
&> \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^\lambda \\
&\quad \times \left\{ \left(\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \right. \\
&\quad \left. \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \right. \\
&\quad \left. \left. \left. + \lambda(s-a)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right]^{\gamma+1} \right) \right. \\
&\quad \times \left((\gamma+1)^{\gamma+1} \right. \\
&\quad \left. \left. \times \left[\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds \\
&+ \frac{1}{(b-c)^\lambda} \\
&\quad \times \int_c^b (b-s)^\lambda \left\{ \left(\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \right. \\
&\quad \left. \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \right. \\
&\quad \left. \left. \left. - \lambda(b-s)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right]^{\gamma+1} \right) \right. \\
&\quad \times \left((\gamma+1)^{\gamma+1} \right. \\
&\quad \left. \left. \times \left[\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds, \tag{45}
\end{aligned}$$

then (1) is oscillatory.

Corollary 7. Under the conditions of Theorem 5, if for any sufficiently large $l \geq \xi_0$,

$$\begin{aligned}
&\lim_{\xi \rightarrow \infty} \sup \int_l^\xi (s-l)^\lambda \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\
&\quad - \tilde{\phi}(s) \tilde{\phi}'(s) \\
&\quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\
&\quad - \left(\left(\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \right. \\
&\quad \left. \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \right. \\
&\quad \left. \left. \left. + \lambda(s-l)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right]^{\gamma+1} \right) \right. \\
&\quad \times \left((\gamma+1)^{\gamma+1} \right. \\
&\quad \left. \left. \times \left[\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds \\
&> 0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\xi \rightarrow \infty} \sup \int_l^\xi (\xi-s)^\lambda \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\
&\quad - \tilde{\phi}(s) \tilde{\phi}'(s) \\
&\quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\
&\quad - \left(\left(\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \right. \\
&\quad \left. \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right. \right. \right. \\
&\quad \left. \left. \left. - \lambda(\xi-s)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right]^{\gamma+1} \right) \right. \\
&\quad \times \left((\gamma+1)^{\gamma+1} \right. \\
&\quad \left. \left. \times \left[\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds \\
&> 0, \tag{46}
\end{aligned}$$

then (1) is oscillatory.

Theorem 8. Assume (5)–(7) hold, and there exists two functions $\phi \in C^1([t_0, \infty), \mathbb{R}_+)$ and $\varphi \in C^1([t_0, \infty), [0, \infty))$ such that

$$\begin{aligned}
&\int_{\xi_0}^\infty \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\phi}'(s) \right. \\
&\quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\
&\quad \left. - \frac{\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right]^\gamma \tilde{r}(s)} \right\} ds \\
&= \infty, \tag{47}
\end{aligned}$$

where $\tilde{\phi}, \tilde{\varphi}$ are defined as in Theorem 4. Then every solution of (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. Now we assume that $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. Let $\omega(t), \tilde{\omega}(\xi)$ be defined as in Theorem 4. Then we obtain (32).

Setting

$$\begin{aligned}\lambda &= 1 + \frac{1}{\gamma}, \quad A^\lambda = \frac{\gamma \phi(t) \delta_1(t, t_2) \omega^{1+(1/\gamma)}(t)}{r(t) \phi^{1+(1/\gamma)}(t)}, \\ B^{\lambda-1} &= \gamma^{1/(\gamma+1)} \\ &\times \frac{(\gamma+1) \phi^{1/\gamma}(t) \phi(t) \delta_1(t, t_2) + r(t) D_t^\alpha \phi(t)}{(\gamma+1) [\phi(t) \delta_1(t, t_2)]^{\gamma/(\gamma+1)} r^{1/(\gamma+1)}(t)}.\end{aligned}\quad (48)$$

Using Lemma 3 in (32) we get that

$$\begin{aligned}D_t^\alpha \omega(t) &\leq -K \phi(t) q(t) + \phi(t) D_t^\alpha \varphi(t) \\ &\quad - \frac{\phi(t) \delta_1(t, t_2) \varphi^{1+(1/\gamma)}(t)}{r(t)} \\ &\quad + \frac{[(\gamma+1) \phi^{1/\gamma}(t) \phi(t) \delta_1(t, t_2) + r(t) D_t^\alpha \phi(t)]^{\gamma+1}}{(\gamma+1)^{\gamma+1} [\phi(t) \delta_1(t, t_2)]^\gamma r(t)},\end{aligned}\quad (49)$$

which is rewritten in the following form:

$$\begin{aligned}\tilde{\omega}'(\xi) &\leq -K \tilde{\phi}(\xi) \tilde{q}(\xi) + \tilde{\phi}(\xi) \tilde{\varphi}'(\xi) \\ &\quad - \frac{\tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(\xi)}{\tilde{r}(\xi)} \\ &\quad + \frac{[(\gamma+1) \tilde{\varphi}^{1/\gamma}(\xi) \tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2) + \tilde{r}(\xi) \tilde{\varphi}'(\xi)]^{\gamma+1}}{(\gamma+1)^{\gamma+1} [\tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2)]^\gamma \tilde{r}(\xi)},\end{aligned}\quad \xi \geq \xi_2.\quad (50)$$

Substituting ξ with s in (50), an integration for (50) with respect to s from ξ_2 to ξ yields

$$\begin{aligned}&\int_{\xi_2}^{\xi} \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\ &\quad - \tilde{\phi}(s) \tilde{\varphi}'(s) + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ &\quad \left. - \frac{[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) + \tilde{r}(s) \tilde{\varphi}'(s)]^{\gamma+1}}{(\gamma+1)^{\gamma+1} [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s)} \right\} ds \\ &\leq \omega(\xi_2) - \omega(\xi) \leq \omega(\xi_2) < \infty,\end{aligned}\quad (51)$$

which contradicts (47). So the proof is complete. $\square$

Theorem 9. Assume that (5)–(7) hold, and there exists a function $H \in C([\xi_0, \infty), \mathbb{R})$ such that $H(\xi, \xi) = 0$, for $\xi \geq$

ξ_0 , $H(\xi, s) > 0$, for $\xi > s \geq \xi_0$, and H has a nonpositive continuous partial derivative $H'_s(\xi, s)$. If

$$\begin{aligned}&\limsup_{\xi \rightarrow \infty} \frac{1}{H(\xi, \xi_0)} \\ &\quad \times \left\{ \int_{\xi_0}^{\xi} H(\xi, s) \right. \\ &\quad \times \left\{ K \tilde{\phi}(s) \tilde{q}(s) \right. \\ &\quad \quad - \tilde{\phi}(s) \tilde{\varphi}'(s) \\ &\quad \quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\ &\quad \quad - \left(\left[(\gamma+1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\ &\quad \quad \quad \left. \left. + \tilde{r}(s) \tilde{\varphi}'(s) \right]^{\gamma+1} \right) \\ &\quad \quad \times \left((\gamma+1)^{\gamma+1} \right. \\ &\quad \quad \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \Big\} \\ &= \infty,\end{aligned}\quad (52)$$

where $\tilde{\phi}, \tilde{\varphi}$ are defined as in Theorem 4, then every solution of (1) is oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$, $t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim_{t \rightarrow \infty} x(t) = 0$. Now we assume that $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$.

Let $\omega(t), \tilde{\omega}(\xi)$ be defined as in Theorem 4. By (50) we have

$$\begin{aligned}&K \tilde{\phi}(\xi) \tilde{q}(\xi) - \tilde{\phi}(\xi) \tilde{\varphi}'(\xi) \\ &\quad + \frac{\tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(\xi)}{\tilde{r}(\xi)} \\ &\quad - \frac{[(\gamma+1) \tilde{\varphi}^{1/\gamma}(\xi) \tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2) + \tilde{r}(\xi) \tilde{\varphi}'(\xi)]^{\gamma+1}}{(\gamma+1)^{\gamma+1} [\tilde{\phi}(\xi) \tilde{\delta}_1(\xi, \xi_2)]^\gamma \tilde{r}(\xi)} \\ &\leq \tilde{\omega}'(\xi), \quad \xi \geq \xi_2.\end{aligned}\quad (53)$$

Substituting ξ with s in (53), multiplying both sides by $H(\xi, s)$, and then integrating with respect to s from ξ_2 to ξ yield

$$\begin{aligned}&\int_{\xi_2}^{\xi} H(\xi, s) \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\ &\quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right\}\end{aligned}$$

$$\begin{aligned}
& - \left(\left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \right) \\
& \quad \times \left((\gamma + 1)^{\gamma+1} [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \\
& \leq - \int_{\xi_2}^{\xi} H(\xi, s) \tilde{\omega}'(s) ds \\
& = H(\xi, \xi_2) \omega(\xi_2) \\
& \quad + \int_{\xi_2}^{\xi} H'_s(\xi, s) \omega(s) \Delta s \\
& \leq H(\xi, \xi_2) \omega(\xi_2) \\
& \leq H(\xi, \xi_0) \omega(\xi_2).
\end{aligned} \tag{54}$$

Then

$$\begin{aligned}
& \int_{\xi_0}^{\xi} H(\xi, s) \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right. \\
& \quad \left. - \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \right) \\
& \quad \left. \times \left((\gamma + 1)^{\gamma+1} [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \right\} ds \\
& = \int_{\xi_0}^{\xi_2} H(\xi, s) \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right. \\
& \quad \left. - \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \right) \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \\
& \quad + \int_{\xi_2}^{\xi} H(\xi, s) \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \\
& \quad \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \Big\} ds \\
& \leq H(\xi, \xi_0) \tilde{\omega}(\xi_2) \\
& \quad + H(\xi, \xi_0) \int_{\xi_0}^{\xi_2} \left| K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right. \\
& \quad \left. - \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \right) \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds.
\end{aligned} \tag{55}$$

So

$$\begin{aligned}
& \lim_{\xi \rightarrow \infty} \sup \frac{1}{H(\xi, \xi_0)} \\
& \quad \times \left\{ \int_{\xi_0}^{\xi} H(\xi, s) \right. \\
& \quad \times \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right. \\
& \quad \left. - \left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right]^{\gamma+1} \right) \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^\gamma \tilde{r}(s) \right)^{-1} \Big\} ds \Big\} \\
& \leq \tilde{\omega}(\xi_2)
\end{aligned}$$

$$\begin{aligned}
& + \int_{\xi_0}^{\xi_2} \left| K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad \left. + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right.
\end{aligned}$$

$$\begin{aligned}
& - \left(\left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right)^{\gamma+1} \right. \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^{\gamma} \tilde{r}(s) \right)^{-1} \right) ds \\
& < \infty,
\end{aligned} \tag{56}$$

which contradicts (52). So the proof is complete. $\square$

3. Applications of the Results

Example 10. Consider the following fractional differential equation:

$$\begin{aligned}
& D_t^{1/2} \left[\sqrt{t} (D_t^{1/2} (D_t^{1/2} x(t)))^{5/3} \right] \\
& + \frac{t^{3/2}}{\Gamma(3/2)} [x^{5/3}(t) + x^{11/3}(t)] = 0, \quad t \geq 2.
\end{aligned} \tag{57}$$

In (1), if we set $t_0 = 2$, $\alpha = 1/2$, $\gamma = 5/3$, $a(t) = \sqrt{t}$, $r(t) \equiv 1$, $q(t) = t^{3/2}/\Gamma(3/2)$, then we obtain (57). So $\tilde{a}(\xi) = a(t) = \sqrt{t} = \Gamma(3/2)\xi$, $\tilde{r}(\xi) \equiv 1$, and $\tilde{q}(\xi) = \xi^3$, and $f(x) = x^{5/3} + x^{11/3}$, which implies that $f(x)/x^{5/3} = 1 + x^2 \geq 1 = K$. Furthermore, $\tilde{\delta}_1(\xi, \xi_2) = \int_{\xi_2}^{\xi} (1/\tilde{a}^{1/\gamma}(s)) ds = [\Gamma(3/2)]^{-3/5} \int_{\xi_2}^{\xi} s^{-3/5} ds = (5/2)[\Gamma(3/2)]^{-3/5} (\xi^{2/5} - \xi_2^{2/5})$, which implies that $\lim_{\xi \rightarrow \infty} \tilde{\delta}_1(\xi, \xi_2) = \infty$. So there exists a sufficiently large $T >$

ξ_2 such that $\tilde{\delta}_1(\xi, \xi_2) > 1$ on $[T, \infty)$. Furthermore, one can easily see that (5)–(7) all hold. On the other hand, in (46), after letting $\tilde{\phi}(\xi) \equiv 1$, $\tilde{\varphi}(\xi) = 0$, and $\lambda = 2$, for any sufficiently large l (we may let $l \geq T$ without loss of generality), it holds that

$$\begin{aligned}
& \lim_{\xi \rightarrow \infty} \sup \int_l^{\xi} (s-l)^{\lambda} \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\
& \quad - \left(\left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right)^{\gamma+1} \right. \\
& \quad \left. + \lambda (s-l)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right)^{\gamma+1} \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^{\gamma} \tilde{r}(s) \right)^{-1} \right\} ds
\end{aligned}$$

$$\begin{aligned}
& \geq \lim_{\xi \rightarrow \infty} \sup \int_l^{\xi} (s-l)^2 \left[s^3 - \left(\frac{3}{4} \right)^{8/3} (s-l)^{8/3} \right] ds = \infty, \\
& \lim_{\xi \rightarrow \infty} \sup \int_l^{\xi} (\xi-s)^{\lambda} \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) \right. \\
& \quad + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \\
& \quad - \left(\left([(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \right. \right. \\
& \quad \left. \left. + \tilde{r}(s) \tilde{\phi}'(s) \right)^{\gamma+1} \right. \\
& \quad \left. - \lambda (\xi-s)^{\lambda-1} \tilde{r}(s) \tilde{\phi}(s) \right)^{\gamma+1} \\
& \quad \times \left((\gamma + 1)^{\gamma+1} \right. \\
& \quad \left. \left. \times [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^{\gamma} \tilde{r}(s) \right)^{-1} \right\} ds \\
& \geq \lim_{\xi \rightarrow \infty} \sup \int_l^{\xi} (\xi-s)^2 \left[s^3 - \left(\frac{3}{4} \right)^{8/3} (\xi-s)^{8/3} \right] ds = \infty.
\end{aligned} \tag{58}$$

So according to Corollary 7 we deduce that (57) is oscillatory.

Example 11. Consider the following fractional differential equation:

$$\begin{aligned}
& D_t^{1/2} \left[t^{-1/6} (D_t^{1/2} (D_t^{1/2} x(t)))^{1/3} \right] \\
& + \frac{\Gamma(1/3) (3/2)}{t^{1/6}} x^{1/3}(t) [1 + e^{x^2(t)}] \\
& = 0, \quad t \geq 2.
\end{aligned} \tag{59}$$

In (1), if we set $t_0 = 2$, $\alpha = 1/2$, $\gamma = 1/3$, $a(t) = t^{-1/6}$, $r(t) \equiv 1$, $q(t) = \Gamma(1/3)(3/2)/t^{1/6}$, then we obtain (59). So $\tilde{a}(\xi) = a(t) = t^{-1/6} = [\Gamma(1/3)]^{-1/3} \xi^{-1/3}$, $\tilde{r}(\xi) \equiv 1$, and $\tilde{q}(\xi) = \xi^{-1/3}$, and $f(x) = x^{1/3}[1 + e^{x^2}]$, which implies that $f(x)/x^{1/3} = 1 + e^{x^2} \geq 1 = K$. Furthermore, $\tilde{\delta}_1(\xi, \xi_2) = \int_{\xi_2}^{\xi} (1/\tilde{a}^{1/\gamma}(s)) ds = \int_{\xi_2}^{\xi} \Gamma(1/3) s ds = (1/2)\Gamma(1/3)(\xi^2 - \xi_2^2)$, which implies that $\lim_{\xi \rightarrow \infty} \tilde{\delta}_1(\xi, \xi_2) = \infty$. So there exists a sufficiently large $T > \xi_2$ such that $\tilde{\delta}_1(\xi, \xi_2) > 1$ on $[T, \infty)$. Furthermore, after some computation one can see that (5)–(7) all hold, and in (47), letting $\tilde{\phi}(\xi) = \xi$, $\tilde{\varphi}(\xi) = 0$, we obtain

$$\begin{aligned}
& \int_{\xi_0}^{\infty} \left\{ K \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{\varphi}'(s) + \frac{\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) \tilde{\varphi}^{1+(1/\gamma)}(s)}{\tilde{r}(s)} \right. \\
& \quad \left. - \frac{[(\gamma + 1) \tilde{\varphi}^{1/\gamma}(s) \tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2) + \tilde{r}(s) \tilde{\phi}'(s)]^{\gamma+1}}{(\gamma + 1)^{\gamma+1} [\tilde{\phi}(s) \tilde{\delta}_1(s, \xi_2)]^{\gamma} \tilde{r}(s)} \right\} ds
\end{aligned}$$

$$\begin{aligned}
&= \int_{\xi_0}^T \left\{ s^{2/3} - \frac{1}{(4/3)^{4/3} [s\tilde{\delta}_1(s, \xi_2)]^{1/3}} \right\} ds \\
&\quad + \int_T^\infty \left\{ s^{2/3} - \frac{1}{(4/3)^{4/3} [s\tilde{\delta}_1(s, \xi_2)]^{1/3}} \right\} ds \\
&\geq \int_{\xi_0}^T \left\{ s^{2/3} - \frac{1}{(4/3)^{4/3} [s\tilde{\delta}_1(s, \xi_2)]^{1/3}} \right\} ds \\
&\quad + \int_T^\infty \left\{ s^{2/3} - \frac{1}{(4/3)^{4/3}} s^{-1/3} \right\} ds = \infty.
\end{aligned} \tag{60}$$

Therefore, (59) is oscillatory by Theorem 8.

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