Research Article

Lie Group Analysis of a Forced KdV Equation

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The Korteweg-de Vries (KdV) equation considered in this work contains a forcing term and is referred to as forced KdV equation in the sequel. This equation has been investigated recently as a mathematical model for waves on shallow water surfaces under the influence of external forcing. We employ the Lie group analysis approach to specify the time-dependent forcing term.

1. Introduction

Many studies in mathematical physics, engineering, life sciences, and in all other sciences use mathematical models to describe certain phenomenon. Such models are represented by the nonlinear evolution equations. The existence of analytical solutions to these equations becomes a basis for a better understanding of the dynamics involved. The KdV equation [1] being a nonlinear evolution equation models the nonlinear wave phenomena on the shallow water surfaces, and its various forms have been proposed depending upon the applications in diverse fields of science and engineering. To date, a lot of solution procedures, both analytical and numerical, have been developed to solve these types of differential equations. However, some of these approaches may involve approximation of solutions. The current work is based upon the Lie group theory approach as a solution procedure.

The Lie point or higher-order symmetries of a differential equation enable one to obtain the solutions invariant under a particular symmetry or a linear combination of symmetries. The invariant solutions are a basis to finding exact solutions or numerical solutions. In most real-life applications, the differential equations, which are used to model a certain situation, contain arbitrary functions of dependent variable or its derivatives and independent variables. Instead of assuming the forms of these model parameters, the method

of group classification can be employed to systematically specify their forms. There are various approaches to solving a group classification problem, namely, the direct analysis or the approach based upon the equivalence transformations.

We consider the forced KdV equation

$$u_t + cu_x + \alpha uu_x + \beta u_{xxx} = F(t), \quad (1)$$

where u represents the elevation of free water surface and α and β are arbitrary constants which depend upon the long wave speed, c . The arbitrary smooth function $F(t)$ is the forcing term. The variables t and x represent time and space, respectively. This equation was proposed in [2] and the Hirota's bilinear approach was used to obtain the multiple soliton solutions. However, it is worth mentioning that $F(t)$ remained unspecified. In [3] the generalized wave transformation was used to transform a forced KdV equation with time-dependent coefficients into a homogeneous equation, and the soliton solutions were obtained by making use of the solitary wave ansatz. Likewise, the time-dependent coefficients remained arbitrary. The symmetry-based approach is given in [4–6]; the investigations in these references include approximate symmetry classification, derivation of some conservation laws, and the construction of the solutions. As mentioned earlier, the arbitrary functions assume their forms in a systematic fashion via the method of group classification. This approach has been utilized on similar equations [7, 8] to the underlying equation.

TABLE 1: Classification results.

No.	F	Condition on consts.	Extension of principal Lie algebra
1.	$F_0(n+t)^{-5/3} + \frac{p}{5\alpha}$	$\alpha, F_0, p \neq 0$	$X_3 = 6\alpha(n+t)\partial_t + \alpha(pt^2 + 2x)\partial_x + 2[pt - 2(c + \alpha u)]\partial_u$
2.	$F_0 + \frac{qt}{3\alpha}$	$\alpha, q \neq 0$	$X_3 = 6\alpha\partial_t + \alpha qt^2\partial_x + 2qt\partial_u$
3.	$F_0(n+t)^{-5/3}$	$F_0 \neq 0$	$X_3 = 3\alpha(n+t)\partial_t + \alpha x\partial_x - 2(c + \alpha u)\partial_u$
4.	F_0	$F_0 \neq 0$	$X_3 = \partial_t,$ $X_4 = 6\alpha\partial_t + \alpha(5\alpha F_0 t^2 + 2x)\partial_x + 2[5\alpha F_0 t - 2(c + \alpha u)]\partial_u$

Here F_0, n, p and q are arbitrary constants.

This work is organized as follows. The next section deals with the generation of the determining equations for Lie point symmetries and includes the classifying relation for the forcing term. The functional forms of the forcing term are specified via the direct analysis of the classifying relation. In Section 3, some results of the Lie group analysis are utilized for symmetry reductions and exact solutions. Finally, we summarize our investigations in Section 4.

2. Lie Group Analysis

In Lie's algorithm (see [9–11] for more details), a vector field

$$X = \xi^1(t, x, u)\partial_t + \xi^2(t, x, u)\partial_x + \eta(t, x, u)\partial_u \quad (2)$$

is a generator of Lie point symmetries of (1) if and only if

$$X^{[3]}(u_t + cu_x + \alpha uu_x + \beta u_{xxx} - F(t))|_{(1)} = 0, \quad (3)$$

where

$$X^{[3]} = X + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{xxx}\partial_{u_{xxx}} \quad (4)$$

is the third prolongation of the vector field X . The variables ζ 's are given by the formulae

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2), \end{aligned} \quad (5)$$

where

$$D_t = \partial_t + u_t\partial_u + \dots, \quad D_x = \partial_x + u_x\partial_u + \dots \quad (6)$$

are the total derivative operators [10].

The invariance conditions (3) are separated with respect to the powers of the derivatives of u , and this yields the determining equations, which are a system of linear partial differential equations of homogeneous type in ξ^1, ξ^2 , and η . It is easy but tedious to generate the determining equations manually. Nowadays there are many computer software packages for symbolic computation, which have been developed to find symmetries interactively or automatically. However, we are yet to develop the software package which solves the group classification problem whether complete or partial group classification.

The coefficients of the generator of Lie point symmetries (2), namely, ξ^1, ξ^2 , and η , satisfy the determining equations

$$\begin{aligned} \xi_u^1 &= 0, \quad \xi_u^2 = 0, \quad \eta_{uuu} = 0, \quad \xi_x^1 = 0, \\ \eta_{xu} - \xi_{xx}^2 &= 0, \quad 3\xi_x^2 - \xi_t^1 = 0, \\ \xi^1 F'(t) + (\xi_t^1 - \eta_u)F(t) - (c + \alpha u)\eta_x - \beta\eta_{xxx} + \eta_t &= 0, \\ (c + \alpha u)(\xi_x^2 - \xi_t^1) - \xi_t^2 + \beta\xi_{xxx}^2 - \alpha\eta - 3\beta\eta_{xxu} &= 0, \end{aligned} \quad (7)$$

where the subscripts denote partial differentiation with respect to the indicated variables, and a "prime" represents total derivative with respect to t .

The manipulation of (7) leads to the coefficients of Lie point symmetry generator given by

$$\begin{aligned} \xi^1 &= k_1 t + k_2, \\ \xi^2 &= \frac{1}{3}k_1 x + a(t), \\ \eta &= \frac{1}{3\alpha}[3a'(t) - 2k_1(c + \alpha u)], \end{aligned} \quad (8)$$

where k_1, k_2 are arbitrary constants, and $a(t)$ is an arbitrary function which satisfies the classifying relation

$$5\alpha k_1 F(t) + 3\alpha(k_1 t + k_2)F'(t) = 3a''(t). \quad (9)$$

Assume that $F(t)$ is an arbitrary smooth function of t ; then from the classifying (9) we obtain

$$k_1 = 0, \quad k_2 = 0, \quad a(t) = k_3 t + k_4, \quad (10)$$

where k_3 and k_4 are arbitrary constants. Thus, we have a two-dimensional principal Lie algebra, which is spanned by the operators

$$X_1 = \partial_x, \quad X_2 = \alpha t\partial_x + \partial_u. \quad (11)$$

Our goal is to obtain the functional forms of the forcing term, $F(t)$, for which the principal Lie algebra is extended. Therefore, the analysis of the classifying relation (9) considering the cases $a''(t) = 0$ and $a''(t) \neq 0$ yields the various forms of $F(t)$, and their corresponding extensions of the principal Lie algebra are given in Table 1. It is worth mentioning that some of the obtained symmetry classification results are comparable with those found in [4, 5]. It is, however, noted that in [6] a more general case is considered.

TABLE 2: Table of commutators.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	αX_1
X_2	0	0	$-3\alpha X_1 - 2\alpha X_2$
X_3	$-\alpha X_1$	$3\alpha X_1 + 2\alpha X_2$	0

Here $[X_i, X_j] = X_i(X_j) - X_j(X_i)$; $i, j = 1, 2, 3$ is the commutator operation.

3. Symmetry Reductions and Exact Solutions

It can be seen from Table 1 that the symmetry Lie algebra is three-dimensional for the first three cases and four-dimensional in the last case. We consider Case 3 to illustrate the procedure involved in performing similarity reductions. Since the symmetry Lie algebra is three-dimensional, we look for solutions invariant under the linear combination of the operators

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \alpha t \partial_x + \partial_u, \\ X_3 &= 3\alpha(n+t)\partial_t + \alpha x \partial_x - 2(c + \alpha u)\partial_u, \end{aligned} \quad (12)$$

which are the symmetries of the equation

$$u_t + cu_x + \alpha uu_x + \beta u_{xxx} = F_0(n+t)^{-5/3}. \quad (13)$$

In order to obtain all the possible invariant solutions, the most systematic procedure is to determine the optimal system of one-dimensional subalgebras [10, 11] for (13). We follow the approach given in [10] by firstly computing the commutators of the symmetry Lie algebra (12) and thereafter obtaining the adjoint representations (the calculations are summarized in Tables 2 and 3, resp.).

We use Table 3 to simplify the linear combination of operators (12) given by

$$\Gamma = a_1 X_1 + a_2 X_2 + a_3 X_3 \quad (14)$$

for some constants a_1, a_2 , and a_3 .

Firstly, we let $a_3 \neq 0$ (take $a_3 = 1$). The operator (14) becomes

$$\Gamma = a_1 X_1 + a_2 X_2 + X_3. \quad (15)$$

We eliminate $a_2 X_2$ by acting $\text{Ad}(e^{a_2 X_2})$ on (15) and obtain

$$\Gamma' = a_1' X_1 + X_3 \quad (16)$$

for some constant a_1' . Likewise, in order to eliminate $a_1' X_1$, we act on (16) by $\text{Ad}(e^{(a_1'/\alpha)X_1})$ to get $\Gamma'' = X_3$.

Next we let $a_3 = 0$ ($a_2 \neq 0$). We take $a_2 = 1$, and from (14) we have

$$\Gamma = a_1 X_1 + X_2. \quad (17)$$

If $a_1 \neq 0$ (i.e., $a_1 > 0$ or $a_1 < 0$), then we obtain $\Gamma = \lambda X_1 + X_2$, where $\lambda = \pm 1$.

Finally, we let $a_2 = a_3 = 0$. The operator (14) reduces to $\Gamma = X_1$ for $a_1 = 1$.

Therefore, an optimal system of one-dimensional subalgebras is given by $\{X_1, X_2, \lambda X_1 + X_2, X_3\}$.

Next we utilize the optimal system to construct the invariant solutions of (13). However, the invariance under space translation, that is, $X_1 = \partial_x$, is trivial hence, it is not considered. The other cases are presented as follows.

Case 1. Invariance under X_2 : the corresponding characteristic system is given by

$$\frac{dt}{0} = \frac{dx}{\alpha t} = \frac{du}{1}, \quad (18)$$

the solution of which leads to the invariants

$$C_1 = t, \quad C_2 = u - \frac{x}{\alpha t}. \quad (19)$$

Therefore, the invariant solution takes the form

$$u(t, x) = f(t) + \frac{x}{\alpha t}, \quad (20)$$

where $f(t)$ satisfies the reduced equation

$$\frac{c}{\alpha t} - \frac{F_0}{(n+t)^{5/3}} + \frac{f}{t} + \frac{df}{dt} = 0. \quad (21)$$

Now solving (21) for $f(t)$ and substituting into (20), we obtain the exact solution

$$u(t, x) = \frac{1}{t} \left[f_0 + F_0 \left\{ 3 + \frac{3n}{2(n+t)} \right\} (n+t)^{1/3} + \frac{x-c}{\alpha} \right], \quad (22)$$

where f_0 is an arbitrary constant.

Case 2. The invariance under $\lambda X_1 + X_2$ ($\lambda \neq 0$) yields the invariant solution

$$u(t, x) = f(t) - \frac{x}{\alpha t + \lambda}, \quad (23)$$

where $f(t)$ is an arbitrary smooth function of its argument. Upon substitution of (23) into (13) and solving the resulting ordinary differential equation (ODE), we have

$$f(t) = \frac{1}{\alpha t + \lambda} \left\{ f_0 - ct + \frac{3F_0(\lambda - 3n\alpha - 2t\alpha)}{2(n+t)^{2/3}} \right\}, \quad (24)$$

for an arbitrary constant f_0 . Therefore, we obtain the exact solution

$$u(t, x) = \frac{1}{\alpha t + \lambda} \left\{ f_0 - ct - x + \frac{3F_0(\lambda - 3n\alpha - 2t\alpha)}{2(n+t)^{2/3}} \right\}. \quad (25)$$

Case 3. Invariance under X_3 : in this case, the invariant solution assumes the form

$$u(t, x) = \frac{f(z)}{(n+t)^{2/3}} - \frac{c}{\alpha}, \quad (26)$$

where $z = x(n+t)^{-1/3}$ is the similarity variable. The function $f(z)$ is an arbitrary function which satisfies the third-order ODE

$$3\beta \frac{d^3 f}{dz^3} + \left(3\alpha \frac{df}{dz} + 2 \right) f + z \frac{df}{dz} - 3F_0 = 0. \quad (27)$$

TABLE 3: Table of adjoint representations.

$\text{Ad}(e^{\epsilon X_i})X_j$	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \alpha\epsilon X_1$
X_2	X_1	X_2	$X_3 + \epsilon(3n\alpha^2 X_1 + 2\alpha X_2)$
X_3	$e^{\epsilon\alpha} X_1$	$e^{-2\alpha\epsilon} X_2 + 3n\alpha(e^{-\epsilon\alpha} - 1)X_1$	X_3

Here $\text{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + (1/2!)\epsilon^2[X_i, [X_i, X_j]] - \dots$ is the adjoint representation where ϵ is a real number.

4. Conclusion

In this work, the KdV equation with a forcing term was investigated using the Lie symmetry approach. The direct analysis of the classifying equation was employed to obtain the functional forms of the forcing term, which include power law and linear and constant time dependence. The three- and four-dimensional symmetry Lie algebras were obtained, respectively, for these forms of the forcing term. The optimal system of one-dimensional subalgebras of the Lie algebra of the invariant equation with the forcing term having the power law nonlinearity was obtained. As a result, for the same invariant equation, exact solutions were derived and the symmetry reduction was performed in the case where exact solutions were not obtained.

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