Research Article

Lock-In Amplifier Technology in Laser Gyroscope North Finder of Constant Rate Biasing

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This paper presents a new type of strapdown north finder. A simplified configuration is proposed, including three parts: a pendulous force-feedback accelerometer, a ring laser gyroscope, and a single-axis rotating platform. The dynamic northfinding scheme of continuous rotation is adopted to eliminate the lock-in region of the laser gyroscope and to modulate the weak attitude signals; meanwhile the complex leveling process is avoided in this proposed configuration. To suppress the drift noises of the accelerometer and gyroscope, two digital lock-in amplifiers are used to extract the weak attitude signals. Simulation results show that the north finder can determine the heading angle in 72 s, when the rotating rate is $10^\circ/\text{s}$ with the maximum heading error less than $1.0'$; meanwhile the horizontal attitude angles as well as the constant drift noises of the inertial components could also be obtained.

1. Introduction

The equipment mounted in the body of the land-based vehicle would allow the directional reference with respect to geographical true north. This equipment is often referred to as north finder, which plays an important role in both civilian and military fields [1–6]. According to the principles of north finding, there are two kinds of north finders: non-gyro-north finder and gyroscope north finder.

In the non-gyro-north finder, magnetic compass and astronomical navigator are the two major categories. Magnetic compass enables the vehicle's attitude to be determined with respect to the Earth's magnetic field; the astronomical navigator fixed in the body of a vehicle could provide measurements of the remote star's bearing and elevation with respect to the body frame relying on the optical sensors. But these two instruments are highly sensitive to the external environment, such as magnetic field, weather, and sometimes cannot be used any more.

In view of the limitations of both the aforementioned techniques, the gyroscope north finder has the characteristic

of autonomy, which is not impacted by external environment; therefore more attentions have been paid to these greatest inventions. With the development of the inertial components, the gyroscope north finder is mainly divided into two categories: one is pendulous north finder as the representative of mechanical gyroscope, which is based on the principle of the gyrocompass with high accuracy but long alignment time. The basic principle of this equipment is the indication of true north by establishing the equilibrium between the effect of its pendulosity and the angular momentum of the rotation base carrying the mechanical gyroscope; the other is strapdown north finder using the optical gyroscope as a sensor [1, 2]. Usually, the strapdown north finder keeps the sensitive axis of the optical gyroscope in the horizontal plane accomplished by using the outputs of two orthogonal accelerations to drive the motor, respectively. Two measurements of the horizontal components of the Earth's rate are sensed by optical gyroscope in two separations 90° apart. So the heading angle can be calculated from the ratio of these two measurements. The strapdown north finder locates the orientation with a little

lower accuracy to the pendulous north finder but greatly decreases the time.

Through analyzing the dynamic output characteristics of inertial measurement unit in the presence of continuous rotation at a constant rate, this paper designs a new type of strapdown north finder with a simplified configuration and also can avoid the complex leveling process of rotating platform. Digital lock-in amplifier technology is adopted to extract the weak attitude signals submerged in the drift noise of inertial components effectively [7–10]. Three attitude angles are obtained in a short time with certain accuracy, as well as the estimations of the constant drifts of inertial components.

2. The Dynamic Output Characteristics of the Inertial Components at Constant Rate Rotation

2.1. The Measurement Principle of Laser Gyroscope of Constant Rate Biasing. The ring laser gyroscope (RLG) based on the Sagnac effect is an angular rate sensor, being an important inertial component currently. The RLG relies upon the detection of the frequency difference between the two counter-propagating beams of light in the optical resonant cavity, which is also proportional to the rotational motion about an axis perpendicular to the plane containing the light path in inertial space. It appears that the application of RLG to the sensing of angular rate can offer lots of advantages over the mechanical gyroscope, such as rapid response, wide dynamic detecting range, good linearity, little dynamic error, high precision, and high reliability and non- g -dependent bias term. Therefore, RLG gets more and more attentions and now is considered as the ideal inertial component to replace the well-established mechanical gyroscope.

At very low angular rate, due to a variety of nonuniformity factors in the optical resonant cavity of RLG, the two counter propagating beams almost have the same frequency or “lock together,” so there is no output signal, which is usually called the lock-in phenomenon [11, 12], as shown in Figure 1. This phenomenon of frequency synchronization introduces a dead zone of the RLG measurement signal, so how to overcome the lock-in region becomes one of the key questions gradually. Although mechanical dither technology is widely used in RLG, it cannot eliminate the lock-in region completely, the components of which still cause measurement error in the mechanical dither.

Fortunately, the lock-in region could be eliminated completely by employing a so-called constant rate biasing technology which is superior to mechanical dither in principle [13]. RLG is installed on a platform with an appropriate rotating rate, which can make a projection rate on the sensitive axis of RLG, so the output signal can keep away from measurement error induced by the lock-in region. Theoretically, the faster the platform rotates, the smaller the laser measurement error is, but a faster rate will reduce the stability of the rotating platform and gyroscope sampling points in a whole rotation cycle, which will affect the calculation accuracy of the attitude solution. Usually, to keep

FIGURE 1: The RLG input/output characteristic.

the measurement of RLG mounted on the rotating platform away from the lock-in region, the biasing rate (ω_B) provided by the platform should be ten to twenty times that of the lock-in rate (ω_L).

2.2. The Dynamic Output Characteristics of Inertial Measurement Unit at Constant Rate Rotation. The Cartesian coordinate frames used in this paper are defined as follows [14, 15].

The inertial frame (i -frame) sets its origin (o) at the center of the Earth and axes (ox_i , oy_i , and oz_i) which are invariant in direction with respect to the fixed planets in space. In i -frame, axes ox_i and oy_i are parallel to the equator plane, with axis ox_i coinciding with the spring equinox and axis oz_i coinciding with the Earth's polar axis, respectively.

The Earth frame (e -frame) sets origin (o) at the center of the Earth and axes (ox_e , oy_e , and oz_e) which are in respect to the Earth. In e -frame, axes ox_e and oy_e are parallel to the equator plane, with axis ox_e along the 0° longitude, axis oy_e along the 90° longitude, and axis oz_e coincident with the Earth's polar axis, respectively. The e -frame rotates relative to the i -frame at a rate ω_{ie} about the axis oz_e (oz_i).

The navigation frame (n -frame), the reference frame for the attitude solution, is a local geographic one. Its origin (o) is set at the location of the navigation system. And axes (ox_n , oy_n , and oz_n) lie along the directions of the local east, the local north, and the vertical upturn, respectively.

The body frame (b -frame) is attached to the base of the vehicle with its origin (o) located at the center of vehicle. Axes (ox_b , oy_b , and oz_b) lie along the directions of the pitch (right), roll (front), and yaw (upturn) of the vehicle, respectively.

The gyroscopes frame (g -frame) is defined by the axes (ox_g , oy_g , and oz_g) which are parallel to the sensitive axes of the three gyroscopes, respectively.

The accelerometers frame (a -frame) is defined by the axes (ox_a , oy_a , and oz_a) which are parallel to the sensitive axes of the three accelerometers, respectively.

FIGURE 2: The transformation from n -frame to b -frame.

Moreover, the coordinate transformation is defined as $V^2 = C_1^2 V^1$. V^1 , V^2 represent vectors expressed in the 1-frame, 2-frame, respectively, and C_1^2 is the direction cosine matrix transferring V^1 into V^2 . The ω_{12}^3 represents the angular rate of the 2-frame with respect to the 1-frame expressed in the 3-frame. The positive rotations about each axis are taken to be in a clockwise direction looking along the axis from the origin.

The vehicle's attitude matrix C_n^b is the transform matrix from n -frame to b -frame by three successive rotations about heading axis, pitch axis, and roll axis in turn, as shown in Figure 2:

$$C_n^b = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \cos \gamma \cos \psi + \sin \gamma \sin \theta \sin \psi & -\cos \gamma \sin \psi + \sin \gamma \sin \theta \cos \psi & -\sin \gamma \cos \theta \\ \cos \theta \sin \psi & \cos \theta \cos \psi & \sin \theta \\ \sin \gamma \cos \psi - \cos \gamma \sin \theta \sin \psi & -\sin \gamma \sin \psi - \cos \gamma \sin \theta \cos \psi & \cos \gamma \cos \theta \end{bmatrix}. \quad (4)$$

The purpose of attitude solution is to figure out these three attitude angles [14].

In b -frame, axes ox_b , oy_b are parallel to the plane of the rotating platform with axis oz_b along the rotation axis of the platform. An inertial measurement unit (IMU) generally consists of three mutually perpendicular gyroscopes and accelerometers to measure the angular rates and linear accelerations, respectively. The IMU is mounted on the rotating platform, and the axes ox_a , oy_a in a -frame are parallel to the plane of the rotating platform. The axis oz_a lies along the direction of the rotation axis oz_b , with axis ox_a coinciding with axis ox_b before rotating. In order to provide the three orthogonal gyroscopes with the same biasing rate, the angular rate of the rotating platform should satisfy the following:

$$\begin{aligned} |\omega_x^g|^2 + |\omega_y^g|^2 + |\omega_z^g|^2 &= |\omega_z^r|^2, \\ |\omega_x^g| &= |\omega_y^g| = |\omega_z^g|. \end{aligned} \quad (5)$$

$$\begin{aligned} o-x_n y_n z_n &\xrightarrow{\text{Rot}(\psi, z_n)} o-x_1 y_1 z_1 \xrightarrow{\text{Rot}(\theta, x_1)} o-x_2 y_2 z_2 \\ &\xrightarrow{\text{Rot}(\gamma, y_2)} o-x_b y_b z_b. \end{aligned} \quad (1)$$

Rotate through heading angle ψ about reference $-z_n$ -axis, $\psi \in [0^\circ, 360^\circ]$.

Rotate through pitch angle θ about new x_1 -axis, $\theta \in [-90^\circ, 90^\circ]$.

Rotate through roll angle γ about new y_2 -axis, $\gamma \in [-90^\circ, 90^\circ]$.

where ψ , θ , and γ are referred to as the attitude angles of the vehicle. Usually, θ and γ represent the horizontal attitude angles.

Thus the direction cosine matrix C_n^b could be expressed as follows:

$$C_n^b = C_2^b C_1^2 C_n^1, \quad (2)$$

where

$$\begin{aligned} C_2^b &= \begin{bmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{bmatrix}, \quad C_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \\ C_n^1 &= \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (3)$$

Substituting and rearranging the above equation yield

Then the following is obtained:

$$\begin{aligned} |\omega_x^g| &= |\omega_y^g| = |\omega_z^g| = \omega_{\text{bias}} = \frac{1}{\sqrt{3}} |\omega_z^r|, \\ \chi_1 &= \chi_2 = \chi_3 = \arctan\left(\frac{1}{\sqrt{3}}\right) = 54.73^\circ. \end{aligned} \quad (6)$$

So this configuration has each of the three sensitive axes of mutually perpendicular gyroscopes at 54.73° to the rotation axis oz_b (oz_a) and is equally spaced at 120° to each other in the plane of the rotating platform [16]. This system is rotated away from axis oz_b (oz_a), so the three gyroscopes are provided $1/\sqrt{3}$ of the rotating rate ω_0 . Meanwhile ensuring that axes ox_a , ox_g , and oz_b (oz_a) are in the same plane, the orthogonal instrument cluster arrangement structure relationship between b -frame coordinate, g -frame coordinate, and a -frame coordinate is as shown in Figure 3.

FIGURE 3: The schematic diagram of inertial measurement unit.

Fixed to the rotating platform and the zero-angle position of the rotating platform lying along the vehicle longitudinal axis, the IMU provides measurement of angular rates and linear accelerations about three orthogonal axes. In the presence of continuous rotation at constant rate, IMU dynamic output can be represented by [14, 15]

$$\begin{aligned} f^a &= C_b^a C_n^b f^n + n^a(t) + b^a \\ \omega_{ig}^g &= C_a^g [C_b^a C_n^b (\omega_{ie}^n + \omega_{en}^n) + C_b^a \omega_{nb}^b + \omega_{ba}^a] \\ &\quad + \omega_{ag}^g + n^g(t) + b^g, \end{aligned} \quad (7)$$

where f^a is the measured linear acceleration in a -frame; ω_{ig}^g is the measured angular rate in g -frame. $n^a(t)$, b^a are the random drift noise and constant drift noise of the accelerometer, respectively. $n^g(t)$, b^g are the random drift noise and constant drift noise of the gyroscope respectively. It is assumed that the vehicle is stationary relative to the Earth, so ω_{en}^n and ω_{nb}^n are zero. Also there is no relative movement between a -frame and g -frame, which means $\omega_{ag}^g = 0$. Equation (7) could be simplified as

$$\begin{aligned} f^a &= C_b^a C_n^b f^n + n^a(t) + b^a, \\ \omega_{ig}^g &= C_a^g (C_b^a C_n^b \omega_{ie}^n + \omega_{ba}^a) + n^g(t) + b^g. \end{aligned} \quad (8)$$

Ignoring scale-factor error and installation error of the inertial component and substituting the above coefficients, (8) can be further written as (9) and (11).

As discussed above, in the presence of continuous and unidirectional rotation with constant rate, the specific forces acting on the vehicle, measured by three mutually perpendicular accelerometers, are given by

$$\begin{aligned} \begin{bmatrix} F_x^a \\ F_y^a \\ F_z^a \end{bmatrix} &= \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \\ &\quad + \begin{bmatrix} n_x^a(t) \\ n_y^a(t) \\ n_z^a(t) \end{bmatrix} + \begin{bmatrix} b_x^a \\ b_y^a \\ b_z^a \end{bmatrix}. \end{aligned} \quad (9)$$

This equation can be rearranged and expressed as follows:

$$\begin{bmatrix} F_x^a \\ F_y^a \\ F_z^a \end{bmatrix} = \begin{bmatrix} C_{13}g \cos(\omega_0 t) - C_{23}g \sin(\omega_0 t) + n_x^a(t) + b_x^a \\ C_{13}g \sin(\omega_0 t) + C_{23}g \cos(\omega_0 t) + n_y^a(t) + b_y^a \\ C_{33}g + n_z^a(t) + b_z^a \end{bmatrix}. \quad (10)$$

As could be seen from (10), the outputs of the three accelerometers involve C_{13} , C_{23} , and C_{33} of the attitude matrix C_n^b in (4), which are only associated with the horizontal attitude angles γ , θ . It will be seen that the specific forces F_x^a and F_y^a are expressed in the form of sine and cosine waveforms with the same amplitudes but different phases. Similarly, ignoring the random drift noise, the output of the accelerometer F_z^a is approximated as a constant value.

Meanwhile, the angular rates sensed by three orthogonal gyroscopes sensitive axes are given by

$$\begin{aligned} \begin{bmatrix} G_x^g \\ G_y^g \\ G_z^g \end{bmatrix} &= \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &\quad \times \left(\begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) & 0 \\ -\sin(\omega_0 t) & \cos(\omega_0 t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \right. \\ &\quad \times \left. \begin{bmatrix} 0 \\ \omega_{ieN}^n \\ \omega_{ieU}^n \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \omega_0 \end{bmatrix} \right) \\ &\quad + \begin{bmatrix} n_x^g(t) \\ n_y^g(t) \\ n_z^g(t) \end{bmatrix} + \begin{bmatrix} b_x^g \\ b_y^g \\ b_z^g \end{bmatrix}. \end{aligned} \quad (11)$$

The above equation can be rearranged and expressed as follows:

$$\begin{bmatrix} G_x^g \\ G_y^g \\ G_z^g \end{bmatrix} = \begin{bmatrix} \frac{2R}{\sqrt{6}} \cos(\omega_0 t) + \frac{2M}{\sqrt{6}} \sin(\omega_0 t) + \frac{N - \omega_0}{\sqrt{3}} + n_x^g(t) + b_x^g \\ \frac{\sqrt{3}M - R}{\sqrt{6}} \cos(\omega_0 t) - \frac{M + \sqrt{3}R}{\sqrt{6}} \sin(\omega_0 t) + \frac{N - \omega_0}{\sqrt{3}} + n_y^g(t) + b_y^g \\ -\frac{\sqrt{3}M + R}{\sqrt{6}} \cos(\omega_0 t) - \frac{M - \sqrt{3}R}{\sqrt{6}} \sin(\omega_0 t) + \frac{N - \omega_0}{\sqrt{3}} + n_z^g(t) + b_z^g \end{bmatrix}, \quad (12)$$

where $R = C_{12}\omega_{ieN}^n + C_{13}\omega_{ieU}^n$, $M = C_{22}\omega_{ieN}^n + C_{23}\omega_{ieU}^n$, and $N = C_{32}\omega_{ieN}^n + C_{33}\omega_{ieU}^n$.

According to (12), the output of a triad of gyroscopes is related to both the heading angle ψ and horizontal attitude angles γ , θ , which comprise sinusoidal oscillation superimposed on the constant parameter $(N - \omega_0)/\sqrt{3}$. Clearly, it biases the measurement signals of the RLG away from the lock-in region, reducing the measurement errors.

As mentioned above, the rotating rate modulates the stationary attitude signals as sinusoidal signals, which facilitates the subsequent process of attitude solution. According to the characteristic of the output signals, the acceleration outputs F_x^a , F_y^a only contain the horizontal attitude signals, and the gyroscope outputs G_x^g , G_y^g , and G_z^g contain all the attitude signals. Since the outputs of the inertial components have similar propagation rule, the attitude angles could be determined from each combination outputs of F_x^a - G_x^g , F_x^a - G_y^g , F_x^a - G_z^g , F_y^a - G_x^g , F_y^a - G_y^g , and F_y^a - G_z^g , which are six simplified configurations in total. The solution of horizontal attitude angles is the premise to calculate the heading angle.

3. The Proposed Algorithm for North Finding

3.1. The Principle of the Lock-In Amplifier. Lock-in amplifier (LIA) was firstly used to suppress 1/f noise and constant drift induced by the thermal effects of the circuits and to extract weak signal from those with strong noises [7–10]. It is also widely employed in resource exploration and precision measurement field, and better results are obtained especially for the sine waves and square waves. According to the characteristic of the measurement signal $x(t)$, the lock-in amplifier generates two orthogonal reference signals $r_I(t)$ and $r_Q(t)$, the frequencies of which are consistent with $x(t)$. These two reference signals have the same amplitude but 90° phase difference. The spectrum of the measurement signal $x(t)$ could be changed by multiplication and accumulation operations of two reference signals $r_I(t)$ and $r_Q(t)$ in the phase sensitive devise, and then the drift noise in the signal is removed through a low-pass filter consequently. Since the output signal of LIA is sensitive to the amplitude and phase of the measurement signal $x(t)$, both amplitude discriminating and phase discriminating are accomplished to extract the feature parameters (amplitude and phase) of the measurement signal, as shown in Figure 4.

In Figure 4, the single frequency measurement signal $x(t)$ without drift noise can be represented by $V_s \sin(\omega_0 t + \eta)$. The two square waves reference signals are $r_I(t)$, $r_Q(t)$ with the same amplitude V_r and 90° phase-difference. After the signal sampling with integral multiple of rotation periods, In-phase (I) and Quadrature (Q) components are calculated:

$$I = \frac{1}{t} \sum_0^t x(t) \cdot r_I(t) = \frac{2}{\pi} V_s V_r \cos \eta, \quad (13)$$

$$Q = \frac{1}{t} \sum_0^t x(t) \cdot r_Q(t) = \frac{2}{\pi} V_s V_r \sin \eta.$$

The following are obtained:

$$V_s = \frac{\pi}{2V_r} \sqrt{I^2 + Q^2}, \quad (14)$$

$$\eta = \arctan\left(\frac{Q}{I}\right).$$

As mentioned above, the feature parameters (amplitude V_s and phase η) of the sinusoidal measurement signal could be identified by the lock-in amplifier [10].

3.2. The Attitude Solution Algorithm Proposed for the North Finder. As a result of the preponderance of trigonometric terms in (10) and (12), the accelerometer signals only contain the horizontal attitude signals. The amplitude and phase of the measurement signal could be extracted by the lock-in amplifier, allowing two horizontal attitude angles to be obtained by resolving the accelerometer measurements. Similarly, the heading angle could be obtained by resolving gyroscope measurement signal through another lock-in amplifier consequently according to (13) and (14).

In practice, the rotation axis should be kept perpendicular to the vehicle base so that the rolling back and forth motion of accelerometer in the gravitational field could be avoided effectively, and the accelerometer works in linear measurement region.

For the sake of easy installation, a simplified configuration of north finder system is proposed. The system mainly consists of ox_a -axis accelerometer, ox_g -axis laser gyroscope, and a single-axis rotating platform. F_x^a and G_x^g are the ox_a -axis accelerometer output signal and ox_g -axis gyroscope

output signal, respectively, which are also used as the measurement signals of the lock-in amplifiers.

In terms of the components C_{13} and C_{23} of the unknown matrix C_n^b , the amplitude V_s and the phase η of the ox_a -axis accelerometer signal F_x^a could be written as

$$\begin{aligned} F_x^a &= C_{13}g \cos(\omega_0 t) - C_{23}g \sin(\omega_0 t) + n_x^a(t) + b_x^a \\ &= \sqrt{C_{13}^2 + C_{23}^2}g \sin\left(\omega_0 t - \arctan \frac{C_{13}}{C_{23}}\right) + n_x^a(t) + b_x^a. \end{aligned} \quad (15)$$

Due to the presence of the random drift noise, the amplitude and phase calculated by the lock-in amplifier have some deviations from the true values, shown as

$$\begin{aligned} \widehat{V}_s &= \sqrt{C_{13}^2 + C_{23}^2}g, \\ \widehat{\eta} &= -\arctan \frac{C_{13}}{C_{23}}. \end{aligned} \quad (16)$$

Combining (4), (14), and (16), the amplitude and phase of the signal F_x^a from the accelerometer could be calculated, and then C_{13} and C_{23} are determined. Since the inverse-sine function is monotonic in the main interval $[-90^\circ, 90^\circ]$, the horizontal attitude angles θ and γ could be determined solely from C_{13} and C_{23} , as shown in

$$\begin{aligned} \theta &= \theta_{\text{main}} = \sin^{-1}(C_{23}), \\ \gamma &= \gamma_{\text{main}} = \sin^{-1}\left(-\frac{C_{13}}{\cos \theta}\right), \end{aligned} \quad (17)$$

where θ_{main} and γ_{main} are the main values of the θ and γ in the inverse-sine function.

Similarly, The output of the ox_g -axis laser gyroscope signal G_x^g is expressed as

$$\begin{aligned} G_x^g &= \frac{2R}{\sqrt{6}} \cos(\omega_0 t) + \frac{2M}{\sqrt{6}} \sin(\omega_0 t) + \frac{N - \omega_0}{\sqrt{3}} + b_x^g + n_x^g(t) \\ &= \frac{2\sqrt{M^2 + R^2}}{\sqrt{6}} \sin\left(\omega_0 t + \arctan \frac{R}{M}\right) \\ &\quad + \frac{N - \omega_0}{\sqrt{3}} + b_x^g + n_x^g(t), \end{aligned} \quad (18)$$

where $R = C_{12}\omega_{ieN}^n + C_{13}\omega_{ieU}^n$, $M = C_{22}\omega_{ieN}^n + C_{23}\omega_{ieU}^n$, and $N = C_{32}\omega_{ieN}^n + C_{33}\omega_{ieU}^n$.

And $\omega_{ieE}^n = 0$, $\omega_{ieN}^n = \omega_{ie}^n \cdot \cos L$, and $\omega_{ieU}^n = \omega_{ie}^n \cdot \sin L$ are the Earth's rate components which could be calculated from the known local latitude L . Thus, parameters M , R could be expressed as

$$\begin{aligned} M &= C_{22} \cdot \omega_{ieN}^n + C_{23} \cdot \omega_{ieU}^n \\ &= \cos \theta \cos \psi \cdot \omega_{ieN}^n + \sin \theta \cdot \omega_{ieU}^n, \\ R &= C_{12} \cdot \omega_{ieN}^n + C_{13} \cdot \omega_{ieU}^n \\ &= (-\cos \gamma \sin \psi + \sin \gamma \sin \theta \cos \psi) \cdot \omega_{ieN}^n \\ &\quad - \sin \gamma \cos \theta \cdot \omega_{ieU}^n. \end{aligned} \quad (19)$$

TABLE 1: The truth value table of the heading angle ψ .

$\cos \psi$	$\sin \psi$	Heading angle ψ
+	+	ψ_{main}
-	+	$\psi_{\text{main}} + 180^\circ$
-	-	$\psi_{\text{main}} + 180^\circ$
+	-	$\psi_{\text{main}} + 360^\circ$
1	0	0°
0	1	90°
-1	0	180°
0	-1	270°

The main value of heading angle ψ_{main} could be computed as follows:

$$\begin{aligned} \cos \psi &= \frac{(M - \sin \theta \cdot \omega_{ieU}^n)}{(\cos \theta \cdot \omega_{ieN}^n)}, \\ \sin \psi &= \frac{[\sin \gamma \tan \theta \cdot (M - \sin \theta \cdot \omega_{ieU}^n) - \sin \gamma \cos \theta \cdot \omega_{ieU}^n - R]}{(\cos \gamma \cdot \omega_{ieN}^n)}, \\ \psi_{\text{main}} &= \arctan\left(\frac{\sin \psi}{\cos \psi}\right), \quad \text{where } \psi_{\text{main}} \in (-90^\circ, 90^\circ). \end{aligned} \quad (20)$$

As the heading angle ψ lies in the interval $[0^\circ, 360^\circ)$, it could be calculated through ψ_{main} in (20), finishing the whole process of north finding. The truth table of the heading angle ψ is shown in Table 1.

It is shown that the range of heading angle ψ is $\psi \in [0^\circ, 360^\circ)$, and the range of main value ψ_{main} is $\psi_{\text{main}} \in (-90^\circ, 90^\circ)$. There are totally eight cases for the heading angle ψ , according to the signs of $\cos \psi$, $\sin \psi$.

3.3. Determination of the Inertial Components Constant Drift Noise. As shown in Figure 5, the outputs of the accelerometer F_x^a and gyroscope G_x^g are sinusoidal oscillatory motions with equal frequency summed by a constant parameter. The constant drift noises of the inertial components are estimated by averaging the data in an entire rotation period, which can eliminate the influence of sinusoidal oscillation. The estimations of the constant drift noises of the inertial components are mainly affected by the intensity of random noise and the stability of the rotating rate. Combining (10), (12) and ignoring the random drift noise, the expressions for constant drift noises of accelerometer and gyroscope are given as

$$\begin{aligned} b_x^a &\approx \frac{1}{t} \sum_0^t [C_{13}g \cos(\omega_0 t) - C_{23}g \sin(\omega_0 t) + n_x^a(t)] \\ &= \frac{1}{t} \sum_0^t F_x^a, \\ b_x^g &\approx \frac{1}{t} \sum_0^t \left[\frac{2M}{\sqrt{6}} \sin(\omega_0 t) + \frac{2R}{\sqrt{6}} \cos(\omega_0 t) + n_x^g(t) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t} \sum_0^t G_x^g - \frac{N - \omega_0}{\sqrt{3}}, \\
N &= C_{32} \omega_{ieN}^n + C_{33} \omega_{ieU}^n \\
&= -\sin \gamma \sin \psi \cdot \omega_{ieN}^n - \cos \gamma \sin \theta \cos \psi \cdot \omega_{ieN}^n \\
&\quad + \cos \gamma \sin \theta \cdot \omega_{ieU}^n,
\end{aligned} \tag{21}$$

where $t = nT_0$, $n \in N^+$. T_0 is the time period for rotating one circle (360°), and t is the time for n circles of the platform rotation.

4. Configuration and Experiment of the North Finder

4.1. Configuration of the North Finder System. All of the components of the north finder system are fixed on the base of the vehicle, and the plane of rotating platform is parallel to the plane of the vehicle base. Make sure that the sensitive axis of the accelerometer, the sensitive axis of the laser gyroscope, and the rotation axis lie in the same plane, and the sensitive axis of the gyroscope forms an angle χ_1 with the rotation-axis for the equipment configuration considered above. The skewed sensor configuration ensures that the rotating platform provides the gyroscope with a biasing frequency to overcome the lock-in phenomenon. The zero-angle position of the rotating platform is usually arranged to be aligned with the longitudinal axis of the vehicle; then the heading angle could be determined by combining the outputs from the inertial components and angle encoders [14]. The structure diagram of the north finder configuration system is shown in Figure 5.

Since the rotating platform continuously rotates at a constant rate, a conductive ring is required to connect the signal line and power line. An accurate controller of rotating rate with the compensation for friction torque is critical to the accurate estimation of heading angle [17]. To guarantee stability of rotating rate, a bipolar PWM control mode is employed in the servo system, and Phase Locked Loop (PLL) technology is also used, which comprises four parts: the control algorithm, a frequency discriminator, a low-pass filter, and power driver [18–20]. The angle between rotating platform and the base is measured by an angle encoder with arc-second accuracy, the output of which would be feed back via torque motor to rotate the platform at the reference rate provided. The rotating platform can be also oriented at the zero-angle position at initial time automatically.

Two orthogonal square waves with 90° out of phase are adopted as the reference signals. When the platform rotates at the instant that the output of angle encoder is an integer multiple of 90° , interrupts are triggered to switch to both polarities of the reference signals, respectively. The data from inertial components are not collected until a steady motor rate is reached. After data collection, the two horizontal attitude angles could be calculated from the accelerometer output F_x^a by the LIA1; then the heading angle is determined

TABLE 2: Typical performance parameters for the inertial components.

Sources of errors	Values
The constant drift noise of the gyroscope (b^g)	$0.02^\circ/\text{h}$
The random drift noise of the gyroscope ($n^g(t)$)	$0.01^\circ/\text{h}$
The installation error of the gyroscope (c^g)	$10''$
The scale-factor error of the gyroscope (s^g)	10 ppm
The constant drift noise of the accelerometer (b^a)	$100 \mu\text{g}$
The random drift noise of the accelerometer ($n^a(t)$)	$50 \mu\text{g}$
The installation error of the accelerometer (c^a)	$10''$
The scale-factor error of the accelerometer (s^a)	10 ppm

TABLE 3: Experimental data processing results of ten groups repeated north finding.

Experimental groups	The vehicle's attitudes		
	$[\gamma]$	$[\theta]$	$[\psi]$
Group 1 results	11.9922°	-14.9921°	276.0142°
Group 2 results	11.9923°	-14.9920°	276.0127°
Group 3 results	11.9923°	-14.9922°	276.0163°
Group 4 results	11.9922°	-14.9920°	276.0162°
Group 5 results	11.9924°	-14.9921°	276.0155°
Group 6 results	11.9923°	-14.9920°	276.0152°
Group 7 results	11.9923°	-14.9920°	276.0162°
Group 8 results	11.9922°	-14.9921°	276.0134°
Group 9 results	11.9923°	-14.9920°	276.0145°
Group 10 results	11.9923°	-14.9921°	276.0157°
The mean of the attitude angles	11.9923°	-14.9921°	276.0150°
The maximum of the heading error	$0.0163^\circ (0.978')$		

from the gyroscope output G_x^g by the LIA2, the principle diagram of which is shown in Figure 6.

4.2. Simulation Experiment. Under static condition, the vehicle's attitudes $[\gamma \ \theta \ \psi]$ are $[12^\circ \ -15^\circ \ 276^\circ]$, and the skewed angle χ_1 is 54.73° . Each sensor output is sampled at a rate of 100 Hz, with the rotating rate $\omega_0 = 10^\circ/\text{s}$ and the operation time $t = 72 \text{ s}$. According to the proposed principle, a simulation experiment is designed with the typical performance parameters for the inertial components shown in Table 2.

Ten groups of north finding experiments are executed, and data processing results are shown in Table 3.

It can be seen that the accuracy of horizontal attitude angles γ , θ is higher than that of heading angle ψ from the above ten groups' results. Simulation data results show that the maximum heading error is less than $1.0'$ in 72 s, when the rotating rate is $\omega_0 = 10^\circ/\text{s}$.

4.3. Error Analysis

4.3.1. Effect of Drift Noise on the Accuracy of the North Finding. Uniformity effects on drift noise can be very significant either

FIGURE 4: Lock-in amplifier structure figure.

FIGURE 5: The structure diagram of the north finder.

optically or electronically, which is often modeled as additive Gaussian white noise and constant noise [11, 21]. The output of the inertial component is given by

$$x(t) = V_s \sin(\omega_0 t + \eta) + n(t) + b, \quad (22)$$

where $n(t)$ and b represent the random drift noise and constant drift noise, respectively. Taking the wideband drift noise $n(t)$ into account, (22) can be rewritten as

$$x(t) = V_s \sin(\omega_0 t + \eta) + \sum_{n=1}^{\infty} V_n \sin(\omega_n t + \eta_n) + b, \quad (23)$$

where V_s is the amplitude of the measurement signal, V_n and η_n are the amplitude and phase of the random drift noise $n(t)$ in each frequency, respectively.

One of the reference signals can be expanded into the Fourier series [22]:

$$r(t) = \frac{4V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos[(2n-1)\omega_0 t]. \quad (24)$$

The product of $x(t)$ by $r(t)$ yields

$$\begin{aligned} u_{\text{out}} &= x(t) \times r(t) \\ &= \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(2n-2)\omega_0 t - \eta] \end{aligned}$$

$$\begin{aligned} &+ \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin(2n\omega_0 t + \eta) \\ &+ \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(\omega_n + (2n-1)\omega_0)t + \eta_n] \\ &+ \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(\omega_n - (2n-1)\omega_0)t + \eta_n] \\ &+ \frac{4bV_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos[(2n-1)\omega_0 t]. \end{aligned} \quad (25)$$

As shown in (25), the product u_{out} consists of five components: only $n = 1$ frequency difference has an output after the low-pass filter in the first term, which is a useful information to obtain the amplitude V_s and phase η of the measurement signal $x(t)$. Other frequency components in the first term, the second term, and the third term can be filtered out by the low-pass filter. There is no frequency difference in the fifth term, allowing no output through the low-pass filter, so the output is not impacted by the constant drift noise b ; most of the frequency components of wideband random drift noise $n(t)$ can be filtered out in the fourth term, but when the frequency of the drift noise and the corresponding harmonic frequency of the reference square wave have the relationship $\omega_n = (2n-1)\omega_0$, a direct current output of a phase sensitive is produced, which influences the result to some extent. Therefore, in order to eliminate the effect of the harmonics, a band-pass filter with the center frequency ω_0 is added to filter out the higher harmonic in

FIGURE 6: The principle diagram of the north finder.

the measurement signal channel, using the function of its narrowband characteristic.

4.3.2. Effect of Rate Stability on the Accuracy of the North Finding. Here, the drift noises of the inertial components are ignored, and only the deviation of rotating rate is considered [2]. Thus the constant rate ω_0 has a random frequency component, which randomizes slightly the angular motion of the rotating platform [23]. Therefore, the output signal of the inertial measurement is given by

$$\begin{aligned} x(t) &= V_s \sin[\omega_0 t + \eta + \theta_0 \sin(\alpha t)] \\ &= V_s [\sin(\omega_0 t + \eta) \cos(\theta_0 \sin \alpha t) \\ &\quad + \cos(\omega_0 t + \eta) \sin(\theta_0 \sin \alpha t)]. \end{aligned} \quad (26)$$

Expanding $\cos(\theta_0 \sin \alpha t)$ and $\sin(\theta_0 \sin \alpha t)$ into the form of the first Bessel function yields

$$\begin{aligned} x(t) &= V_s J_0 \sin(\omega_0 t + \eta) + V_s \sum_{m=1}^{\infty} J_m [\sin((\omega_0 + m\alpha)t + \eta)] \\ &\quad + (-1)^m \sin((\omega_0 - m\alpha)t + \eta), \end{aligned} \quad (27)$$

where J_m is the coefficient of the first Bessel function, m is the order, and θ_0 is the amplitude of the frequency error. As θ_0 is very small, $J_1 = \theta_0/2$, $J_2 = J_3 = J_4 \cdots \approx 0$, and (27) could be rewritten as

$$x(t) = V_s \sin(\omega_0 t + \eta) + V_s \theta_0 \cos(\omega_0 t + \eta). \quad (28)$$

One of the reference signals can be expanded into the Fourier series [22]:

$$r(t) = \frac{4V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos[(2n-1)\omega_0 t]. \quad (29)$$

According to (28) and (29), the product of $x(t)$ by $r(t)$ yields:

$$\begin{aligned} u_{\text{out}} &= x(t) \times r(t) \\ &= \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(2n-2)\omega_0 t - \eta] \\ &\quad + \frac{2V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin(2n\omega_0 t + \eta) \\ &\quad + \frac{\theta_0 V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(2n-2)\omega_0 t - \alpha t - \eta] \\ &\quad + \frac{\theta_0 V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin(2n\omega_0 t + \alpha t + \eta) \\ &\quad - \frac{\theta_0 V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin[(2n-2)\omega_0 t + \alpha t - \eta] \\ &\quad - \frac{\theta_0 V_s V_r}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin(2n\omega_0 t - \alpha t + \eta). \end{aligned} \quad (30)$$

As shown in (30), the product u_{out} consists of six components: only $n = 1$ frequency difference has an output after the low-pass filter in the first term, which is considered useful information to obtain the amplitude V_s and phase η of the measurement signal $x(t)$. Other frequency components in the first term, the second term, the fourth term, and the fifth term can be filtered out by the low-pass filter. The errors of the result are from the frequency-difference of $\alpha = (2n-2)\omega_0$ in the third term, the difference component of $\alpha = 2n\omega_0$ in the fifth term and in the sixth term, causing an error output through the lock-in amplifier. Since the amplitude θ_0 of the error is small, its effect on the result is not significant.

4.3.3. Effect on North Finder Accuracy due to Installing Error. In the ideal condition, the gyroscope sensitive axis ox_g is OA-axis with the skewed angle χ_1 to the rotation axis $oz_b(oz_a)$. Two parameters of installing error ∂ , β represent the deviation of the matrix C_a^g from the ideal condition, and the result of the gyroscope sensitive ox'_g -axis is turned into OC-axis. It

FIGURE 7: Installing error of sensitive axis of the inertial component.

is stressed that, although there is no effect on the output of the ox_a -axis accelerometer, the angular rate that the ox'_g -axis gyroscope sensed at this time can be expressed as

$$\begin{aligned} G_x^g = & [R \cos \beta \cos (\chi_1 - \partial) + M \sin \beta] \cos (\omega_0 t) \\ & + [M \cos \beta \cos (\chi_1 - \partial) - R \sin \beta] \sin (\omega_0 t) \quad (31) \\ & + (N - \omega_0) \cos \beta \cos (\chi_1 - \partial) + b_x^g + n_x^g(t). \end{aligned}$$

The horizontal attitude angle solutions will not be affected as the output of the ox_a -axis accelerometer is not changed by the errors. However, the heading angle solution will be affected. Under the conditions $\partial = 12''$ and $\beta = 12''$, $15''$ heading error will be induced, which illustrates that this type of installation error should be reduced as small as possible [24].

In addition to the installing error described in Figure 7, when the sensitive center of the accelerometer does not go through the center of the rotation axis sufficiently, a steady drift bias of acceleration is introduced as a result of size effect [14]. It does not affect the solution of attitude angles, which is similar to the constant drift noise of the accelerometer.

5. Conclusion

This paper focuses on a dynamic strapdown north finding scheme at the constant rate rotation. A simplified configuration of the north finder system consists of three major parts: a pendulous force-feedback accelerometer, an RLG, and a rotating platform, which reduces the cost of equipment. In order to suppress the drift noise of the inertial components, two lock-in amplifiers are employed to extract and amplify the vehicle's attitude signals that are weak and slow-varying. The main advantage of the new scheme keeps the rotating platform away from the process of leveling, which is convenient for engineering applications. The vehicle's attitude angles could be determined in a short time with certain accuracy, as well as the estimations of the constant drift noise of accelerometer and gyroscope.

The only remaining problem of using this scheme arises when the installation errors of gyroscope and accelerometer are significant. These installation errors cause the heading error during north finding, which indicates that this type of installation errors should be reduced as small as possible.

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