Research Article

Fault Detection for a Class of Uncertain Linear Systems

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Design problem of the fault detection filter for a class of uncertain feedback systems is discussed in this paper. The system under consideration is the model with the nonstandard parameter uncertain regions. The fault detection filter design problem is reduced to an optimized filter design problem, which maximizes the sensitivity of the fault in the fault cases and meets the disturbance attenuation performance in both fault-free cases and fault cases, simultaneously. A numerical example is used to demonstrate the effectiveness of the proposed design method.

1. Introduction

In recent years, fault detection technique is given more and more attentions for the higher demands of safety and reliability of the control systems. A considerable sum of results on fault detection, both in the light of the data-driven and the model-based detection techniques, have been achieved (see [1–3]). Among the model-based techniques, a number of results which depend on the linear matrix inequality (LMI) techniques have appeared to deal with the fault detection problem. References [4–6] investigate the estimation problem of the fault by designing a filter. Recently, a so-called H_∞/H_- design approach for the fault detection problem has received considerable attentions. Further, the fault detection problem associated with some certain H_∞/H_- performance indices is discussed in [7–10]. The fault models considered in the above papers are classical and can describe a large class of the fault types. However, it should be pointed out that the actuator stuck fault also needs to be investigated when the amplitude of the faults are not larger enough to be detected.

At the same time, most of the models are obtained by the identification experiments with some limited input/output data. Identification experiment in the prediction error (PE) framework delivers an uncertain region which contains the true system at some (user-chosen) probability level [11]. The uncertain region can result in an identified parametric

ellipsoid set which is regarded as a nonstandard parameter uncertain structure in the related classical papers. The ellipsoidal uncertain structure is studied by some results; see [12–15] which reduce the uncertain region by employing the robust tools. Compared with the stability and the stabilization problems for this particular uncertainty, there are no results that to pay attention to deal with the fault detection problem with the models which have the ellipsoidal uncertainty. Moreover, since the existing approaches about fault detection (FD) are not appropriate for FD problem for this kind of uncertainty, the new FD technique should solve this case.

Motivated by the above reasons, the optimization design problem of the FD filter for uncertain linear feedback system is studied. The design objective is to maximize the sensitivity of the fault in the fault cases, which subjects to the disturbances attenuation performance in both fault-free cases and the fault cases. In practical, the frequency of the disturbance signals and the fault signals are usually in bound. Thus, a notion of the finite-frequency performance indices is introduced to describe the performance indices; LMIs-based sufficient conditions are provided using the finite-frequency approach proposed in [16, 17]. The optimal solution of gain matrices of the FD filter can be obtained.

The main contribution of this paper that the direct design approach of the FD filter proposed to the linear model with elliptical uncertain structure. Unlike other fault detection approaches for the uncertain model, in which the FD filter is achieved not only depending on the known

lower and known upper bounds of the uncertainty but also extracting convex polygons of the uncertain region, a general parametrization for the set of multipliers is introduced to decrease the conservatism. To the best of our knowledge, such a framework for the fault detection with the elliptical parameter uncertainty structure has not been reported in the literature.

The paper is organized as follows. Section 2 introduces the problem under consideration and presents the design objectives. Section 3 illustrates a FD filter design approach in detail. The algorithm is given in Section 4. An illustrative example is given in Section 5 to demonstrate the proposed method. Conclusions to this work are given in the last section.

Notation. For a matrix A , A^T denotes its transpose. For a symmetric matrix, $A > 0$ ($A \geq 0$) and $A < 0$ ($A \leq 0$) denote positive-definiteness (positive semidefinite matrix) and negative-definiteness (negative semi-definite matrix), respectively. The Hermitian part of a square matrix M is denoted by $\text{He}(M) := M + M^T$. The symbol $*$ within a matrix represents the symmetric entries. $\sigma_{\min}(G)$ denotes minimum singular values of the transfer matrix G . The symbol $\otimes$ denotes the Kronecker product, and 0 denotes zero matrix with appropriate dimension. Both I and $\mathcal{I}$ denote identity matrix with appropriate dimension. $\mathcal{I}_0$ and $\mathcal{I}_1$ denote the matrices $[1 \ 0; 0 \ 0]$ and $[1; 0]$ of appropriate dimensions, respectively.

2. Problem Formulation

2.1. System Model. Consider the following uncertain linear systems:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 d(t) + B_2 f(t) + L_1 w(t), \\ y(t) &= Cx(t) + D_1 d(t) + D_2 f(t) + L_2 w(t), \\ w(t) &= \Delta z(t), \\ z(t) &= Hx(t) \quad \Delta \in \Delta, \end{aligned} \quad (1)$$

where $x(t)$ is the state $y(t)$ is the output, $u(t)$ is the input, and $d(t)$ is an energy-bounded disturbance; r_0 is the reference input, $x(t)$ is the state; A , B_1 , B_2 , C , D , L_1 , L_2 , and H are known constant matrices of appropriate dimensions. Consider that $\Delta \in \Delta$, where Δ is defined as

$$\Delta = \text{diag} \left(\underbrace{\theta, \dots, \theta}_{n_0} \right), \quad \theta = [\theta_1^T, \dots, \theta_{n_\theta}^T]^T \in R^{n_\theta}, \quad (2)$$

where n_0 is the number of θ in Δ , then the dimension of the Δ is $n_\theta n_0 \times n_0$, and Δ is the parameter set of Δ . The uncertain parameter θ with a certain use-chosen probability ($\mathcal{X}$) is at the uncertain set $\mathcal{U}$. The uncertain set $\mathcal{U}$ is an ellipsoid in the parameter space:

$$\mathcal{U} = \{\theta \mid \theta^T \mathcal{R} \theta < \mathcal{X}\}, \quad (3)$$

where $\mathcal{R}$ describes the form of the ellipsoid and $\mathcal{X}$ is a constant computed by the desired probability in model identification.

Remark 1. The parameter uncertainty region in (1) is ellipsoidal. This kind of particular uncertainty is derived from classical prediction error identification, and it is a typical parameter uncertainty of the model which is identified. Although the parameter uncertainty can turn into the polytopic uncertainty by extracting convex polygons of the ellipsoidal region, this transformation method enlarges the the uncertain region and increases the conservatism. In this paper, the ellipsoidal uncertainty is disposed directly.

To detect the actuator stuck faults, the FD filter is designed such that the residual $r(t)$ can be obtained as

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t), \\ r(t) &= C_f x_f(t) + D_f y(t), \end{aligned} \quad (4)$$

where x_f is the state of the FD filter. Consider that $A_f \in R^{n_f \times n_f}$, $B_f \in R^{n_f \times n_m}$, $C_f \in R^{n_f}$, $D_f \in R^m$.

By combining (1) and (4), we have the following augmented systems:

$$\begin{aligned} \xi(t) &= \bar{A}\xi(t) + \bar{B}_d d(t) + \bar{L}_1 w(t) + \bar{B}_f f(t), \\ r(t) &= \bar{C}\xi(t) + \bar{D}_d d(t) + \bar{D}_f f(t) + \bar{L}_2 w(t), \\ w(t) &= \Delta z(t) \\ z(t) &= \bar{H}\xi(t) \quad \Delta \in \Delta, \end{aligned} \quad (5)$$

where $\xi(t) = [x(t)^T, x_f(t)^T]^T$,

$$\begin{aligned} &\begin{bmatrix} \bar{A} & \bar{B}_d & \bar{B}_f & \bar{L}_1 \\ \bar{C} & \bar{D}_d & \bar{D}_f & \bar{L}_2 \end{bmatrix} \\ &= \begin{bmatrix} A & 0 & B_1 & B_2 & L_1 \\ B_f C & A_f & B_f D_1 & B_f D_2 & B_f L_2 \\ D_f C & C_f & D_f D_1 & D_f D_2 & D_f L_2 \end{bmatrix} \\ &\bar{H} = [H \ 0]. \end{aligned} \quad (6)$$

2.2. Problem Formulation. The design problem of the FD filter to be addressed in this paper can be expressed as follows.

The design objective: consider a class of uncertain linear systems (1); the FD filter (4) is designed such that the augmented model (5) is stable; the disturbance effects on the residual are minimized in both the fault-free case and the fault cases, while the fault effects on the residual are maximized in the fault cases. To detect actuator faults, our design objective of the FD filter can now be formulated as the following optimization problem:

$$\max \beta \quad (7)$$

$$\text{s.t. } \delta_{\max}(G_{rd}(j\omega)) < \gamma, \quad \gamma > 0, \quad \forall |\omega| \leq \bar{\omega}_1, \quad \Delta \in \Delta \quad (8)$$

$$\delta_{\min}(G_{rf}(j\omega)) > \beta \quad \beta > 0, \quad \forall |\omega| \leq \bar{\omega}_2, \quad \Delta \in \Delta, \quad (9)$$

where $\bar{\omega}_1$ is the upper bounds on the frequency for the disturbance and $\bar{\omega}_2$ is the upper bounds on the frequency for the fault. G_{rd} is the transfer function from the disturbance d to the residual signal r with fault-free case and fault cases. G_{rf} is the transfer function from the fault f to the residual signal r with fault cases.

Remark 2. Condition (8) describes the disturbance attenuation condition in both the fault-free case and fault cases. Condition (9) is formulated as the sensitiveness performance in the fault cases.

Before ending this section, the following lemmas and proposition shall be recalled to prove our main results for the FD filter design.

Lemma 3 (see [18]). Given a symmetric matrix $\Theta \in R^{n \times n}$ and two matrices $\mathcal{M}$ and $\mathcal{H}$ of column dimension n , there exists matrix $\mathcal{F}$ that satisfies $\Theta + \mathcal{M}^T \mathcal{F} \mathcal{H} + \mathcal{H}^T \mathcal{F}^T \mathcal{M} < 0$ if and only if the following two conditions hold:

$$\mathcal{N}_{\mathcal{M}}^T \Theta \mathcal{N}_{\mathcal{M}} < 0 \quad \mathcal{N}_{\mathcal{H}}^T \Theta \mathcal{N}_{\mathcal{H}} < 0, \quad (10)$$

where $\mathcal{N}_{\mathcal{M}}$ and $\mathcal{N}_{\mathcal{H}}$ denote arbitrarily bases of null space of $\mathcal{M}$ and $\mathcal{H}$, respectively.

Lemma 4 (see [19]). Let $\xi \in R^n$, $\mathcal{N} \in R^{n \times n}$, and $\mathcal{M} \in R^{n \times m}$ such that $\text{rank}(\mathcal{N}) < n$. The following statements are equivalent:

- (1) $\xi^T \mathcal{M} \xi < 0$, for $\forall \xi \neq 0$, subject to $\mathcal{N} \xi = 0$;
- (2) $\mathcal{N}^{\perp T} \mathcal{M} \mathcal{N}^{\perp} < 0$, where $\mathcal{N}^{\perp}$ is the kernel of $\mathcal{N}$;
- (3) $\mathcal{M} - \varepsilon \mathcal{N}^T \mathcal{N} < 0$, for some scalar $\varepsilon \in R$;
- (4) $\mathcal{M} + X^T \mathcal{N} + \mathcal{N}^T X < 0$, for some matrix $X \in R^{n \times m}$.

Proposition 5 (see [11]). Consider the uncertainty Δ with $\theta \in \mathcal{U}$ defined in (3), $\Gamma \in \Gamma$ is defined as

$$\Gamma := \left\{ \Gamma = \begin{bmatrix} \Delta \\ I \end{bmatrix} \mid \Delta \in \Delta \right\}. \quad (11)$$

Restrict the parametrization of matrices Λ defined as

$$\Lambda := \left\{ \Lambda \mid \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} \right\} \quad (12)$$

with Λ_{22} as a positive complex Hermitian matrix of dimension $n \times n$. Consider that $\Lambda_{11} = -(\Lambda_{22} \otimes (R/\mathcal{X}) + \Lambda_1) \in R^{n_{\theta} n \times n_{\theta} n}$ and

$$\Lambda_{12} = \begin{pmatrix} 0 & p_{12} & \cdots & p_{1n} \\ -p_{12} & 0 & \cdots & p_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ -p_{1n} & \cdots & \cdots & 0 \end{pmatrix} \in R^{n_{\theta} n \times n}, \quad (13)$$

where $\mathcal{R}$ describes the form of the ellipsoid for uncertainty in (3), and the elements of this parametrization (Λ_a, p_{ij}) can take

any values provided. Consider that $\Lambda_a \in R^{n_{\theta} n \times n_{\theta} n}$ has the following structure that:

$$\Lambda_1 \in \Lambda_1 : \left\{ \begin{bmatrix} 0 & k_{12} & \cdots & k_{1n} \\ -k_{12} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & k_{(n-1)n} \\ -k_{1n} & \cdots & -k_{(n-1)n} & 0 \end{bmatrix} \mid k_{ij} = -k_{ij}^T \in R^{n_{\theta} \times n_{\theta}} \right\} \quad (14)$$

and $p_{ij} \in R^n$, $i = 1, \dots, n$, $j = 1, \dots, n$. For $\Lambda \in \Lambda$, $\Gamma^T \Lambda \Gamma > 0$, $\forall \Gamma \in \Gamma$.

3. Fault Detection Design

In this section, Lemma 6 and Proposition 7 are first given, and inequality conditions for performance indices (8) and (9) are also given. Based on these conditions, an algorithm based on LMIs is presented.

Lemma 6 (see [16]). Consider the following uncertain system:

$$\begin{aligned} \dot{x}(t) &= (A + B_1 \Delta (I - D_{11} \Delta)^{-1} C_1) x(t) + B_2 u(t), \\ y(t) &= (C_2 + D_{21} \Delta (I - D_{11} \Delta)^{-1} C_1) x(t) + D_{22} u(t), \end{aligned} \quad (15)$$

with $\Delta \in \Delta$. Let a real symmetric matrix Π and a positive scalar $\bar{\omega}$ be given. The following robust finite-frequency condition

$$[G_1(j\omega)^T \ I] \Pi [G_1(j\omega)^T \ I]^T < 0 \quad |\omega| \leq \bar{\omega}, \quad \Delta \in \Delta \quad (16)$$

holds if there exist real symmetric matrices $\mathcal{P}$, $\mathcal{Q}$, $\Phi \in \Phi$, and $\Lambda \in \Lambda$ such that

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{F} & 0 \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q} & \mathcal{P} \\ \mathcal{P} & \bar{\omega}^2 \mathcal{Q} \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{F} & 0 \end{bmatrix} < \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} \Phi \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix}, \quad (17)$$

$$\mathcal{Q} \geq \mathcal{C}_1^T \Lambda \mathcal{C}_1, \quad (18)$$

where

$$\Phi := \left\{ \begin{bmatrix} \Psi & 0 \\ 0 & \Pi \end{bmatrix} : \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T \Psi \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \leq 0, \quad \forall \Gamma \in \Gamma \right\},$$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}_1 & \mathcal{D}_1 \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix} := \begin{bmatrix} A & B_1 & B_2 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ C_1 & D_{11} & 0 & 0 \\ C_2 & D_{21} & D_{22} & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix} := \begin{bmatrix} \mathcal{C}_{11} & \mathcal{D}_{11} \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1 & 0 \\ \mathcal{C}_1 \mathcal{A} & \mathcal{C}_1 \mathcal{B} \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix}.$$

Proposition 7. Consider the uncertainty Δ with $\theta \in \mathcal{U}$ defined in (3) and $\Gamma \in \Gamma$ defined in (11). Restrict the parametrization of matrices Ψ defined as

$$\Psi = \left[\begin{array}{c|c} \Psi_{11} & \Psi_{12} \\ \hline \Psi_{12}^T & \Psi_{22} \end{array} \right] = \left[\begin{array}{cc|cc} \Psi_{111} & \Psi_{112} & \Psi_{121} & \Psi_{122} \\ \Psi_{112}^T & \Psi_{113} & \Psi_{122}^T & \Psi_{123} \\ \hline \Psi_{121}^T & \Psi_{122} & \Psi_{221} & \Psi_{222} \\ \Psi_{122}^T & \Psi_{123} & \Psi_{222}^T & \Psi_{223} \end{array} \right], \quad (20)$$

where $\begin{bmatrix} \Psi_{113} & \Psi_{123} \\ * & \Psi_{223} \end{bmatrix}$ are positive complex Hermitian matrices of dimension $2n \times 2n$; Ψ_{111} , Ψ_{121} , and Ψ_{221} of the dimension $n_\theta n \times n_\theta n$ have the structure with $-(\Psi_{113} \otimes (R/\mathcal{X}) + \Psi_1)$, $-(\Psi_{123} \otimes (R/\mathcal{X}) + \Psi_2)$, and $-(\Psi_{223} \otimes (R/\mathcal{X}) + \Psi_3)$, respectively. Moreover

$$\Psi_{p2} = \begin{pmatrix} 0 & p_{p2} & \cdots & p_{pn} \\ -p_{p2} & 0 & \cdots & p_{pn} \\ \vdots & \ddots & \ddots & \vdots \\ -p_{p\bar{n}} & \cdots & \cdots & 0 \end{pmatrix} \in R^{n_\theta n \times n}, \quad p = 11, 12, 22, \quad (21)$$

where $\mathcal{R}$ describes the form of the ellipsoid for uncertainty in (3), and the elements of this parametrization (Ψ_1 , Ψ_2 , Ψ_3 , p_{1ij} , p_{2ij} , p_{3ij} , $i = 1, \dots, n$, $j = 1, \dots, n$) can take any values

provided. For $t = 1, 2, 3$, $i = 1, \dots, n$, and $j = 1, \dots, n$, $\Psi_t \in R^{n_\theta n \times n_\theta n}$ has the following structure:

$$\Psi_t \in \Psi_t : \left\{ \begin{array}{l} \left[\begin{array}{cccc} 0 & k_{t12} & \cdots & k_{t1n} \\ -k_{t12} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & k_{t(n-1)n} \\ -k_{t1n} & \cdots & -k_{t(n-1)n} & 0 \end{array} \right] \\ k_{tij} = -k_{tij}^T \in R^{n_\theta \times n_\theta} \end{array} \right\}, \quad (22)$$

and $p_{tij} \in R^n$. Then, for $\Psi \in \Psi$, it has

$$\begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T \Psi \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} > 0, \quad \forall \Gamma \in \Gamma, \quad (23)$$

where Γ is defined in (11).

Proof. Let all Ψ_{p2} , Ψ_t , $p \in \{11, 12, 22\}$, $t \in \{1, 2, 3\}$ have the structure in the statement of the proposition:

$$\Psi_{p2}^T \Delta + \Delta^T \Psi_{p2} = 0, \quad \Delta^T \Psi_t \Delta = 0. \quad (24)$$

Therefore, for every $\Psi \in \Psi$, we can have

$$\begin{aligned} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T \Psi \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} &= \begin{bmatrix} \Delta & 0 \\ 0 & I \\ \Delta & 0 \\ 0 & I \end{bmatrix}^T \Psi \begin{bmatrix} \Delta & 0 \\ 0 & I \\ \Delta & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \Delta^T \Psi_{111} \Delta^T + \Psi_{112}^T \Delta + \Delta^T \Psi_{112} + \Psi_{113} & \Delta^T \Psi_{121} \Delta^T + \Psi_{122}^T \Delta + \Delta^T \Psi_{122} + \Psi_{123} \\ * & \Delta^T \Psi_{221} \Delta^T + \Psi_{222}^T \Delta + \Delta^T \Psi_{222} + \Psi_{223} \end{bmatrix} \\ &= \begin{bmatrix} \left(1 - \theta^T \frac{R}{\mathcal{X}} \theta\right) \Psi_{111} & \left(1 - \theta^T \frac{R}{\mathcal{X}} \theta\right) \Psi_{121} \\ * & \left(1 - \theta^T \frac{R}{\mathcal{X}} \theta\right) \Psi_{221} \end{bmatrix}. \end{aligned} \quad (25)$$

When $\theta \in \mathcal{U}$ and $\begin{bmatrix} \Psi_{113} & \Psi_{123} \\ * & \Psi_{223} \end{bmatrix} > 0$, we can have

$$\begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}^T \Psi \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} > 0, \quad \forall \Gamma \in \Gamma. \quad (26)$$

□

3.1. The Disturbance Attenuation Condition

Theorem 8. Consider the system in (5) with $\Delta \in \Delta$. A real symmetric matrix $\Pi_a = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$ is given. Condition

$$\delta_{\max}(G_{rd}(j\omega)) < \gamma, \quad |\omega| < \bar{\omega}_2, \quad \Delta \in \Delta \quad (27)$$

holds if there exist matrix variables $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$, M , R , Y_a , E_a , L_a , N_a , W_a , S_a ,

$$P_a = \begin{bmatrix} P_{a11} & P_{a12} & P_{a13} \\ P_{a21} & P_{a22} & P_{a23} \\ P_{a31} & P_{a32} & P_{a33} \end{bmatrix},$$

$$Q_a = \begin{bmatrix} Q_{a11} & Q_{a12} & Q_{a13} \\ Q_{a21} & Q_{a22} & Q_{a23} \\ Q_{a31} & Q_{a32} & Q_{a33} \end{bmatrix},$$

$$\Psi_a = \begin{bmatrix} \Psi_{a111} & \Psi_{a112} & \Psi_{a121} & \Psi_{a122} \\ \Psi_{a112}^T & \Psi_{a113} & \Psi_{a122}^T & \Psi_{a123} \\ \Psi_{a121}^T & \Psi_{a122} & \Psi_{a221} & \Psi_{a222} \\ \Psi_{a122}^T & \Psi_{a123}^T & \Psi_{a222}^T & \Psi_{a223} \end{bmatrix},$$

$$\Lambda_a = \begin{bmatrix} \Lambda_{a11} & \Lambda_{a12} \\ \Lambda_{a12}^T & \Lambda_{a22} \end{bmatrix}$$
(28)

of appropriate dimensions satisfying $Q_a > 0$, $\Psi_a \in \Psi$, and $\Lambda_a \in \Lambda$ such that

$$\begin{bmatrix} -Q_{a11} & -Q_{a12} & -Q_{a13} & P_{a11} - Y_a^T & P_{a12} + E_{qT}^T & P_{a13} - L_a^T & a_{17} & a_{28} & 0 \\ * & -Q_{a22} & -Q_{a23} & P_{a21} - M^T & P_{a22} - M^T & P_{a23} & a_{27} & a_{28} & 0 \\ * & * & -Q_{a33} & Q_{a31} - W_a^T & P_{a32} - S_a^T & P_{a33} - N_a^T & a_{37} & a_{38} & 0 \\ * & * & * & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & (\widehat{DC})^T \\ * & * & * & * & a_{55} & a_{56} & a_{57} & a_{58} & \widehat{C}^T \\ * & * & * & * & * & a_{66} & a_{67} & a_{68} & (\widehat{DL}_2)^T \\ * & * & * & * & * & * & a_{77} & a_{78} & (\widehat{DD}_1)^T \\ * & * & * & * & * & * & * & a_{88} & 0 \\ * & * & * & * & * & * & * & * & I - \text{He}(R) \end{bmatrix} < 0$$
(29)

$$\begin{bmatrix} Q_{a11} - H^T \Lambda_{a22} H & Q_{a12} & Q_{a13} - H^T \Lambda_{a12}^T \\ * & Q_{a22} & Q_{a23} \\ * & * & Q_{a33} - \Lambda_{a11} \end{bmatrix} > 0,$$
(30)

where

$$\begin{aligned} a_{17} &= -Y_a^T v_1, & a_{27} &= -M^T v_1, & a_{37} &= -W_a^T v_1, \\ a_{18} &= -Y_a^T v_2, & a_{28} &= -M^T v_2, & a_{38} &= -W_a^T v_2, \\ a_{44} &= \bar{\omega}_1^2 Q_{a11} + H^T \Psi_{a113} H + A^T H^T \Psi_{a223} H A \\ &\quad + \text{He}(Y_a A + \widehat{B} C + H^T \Psi_{a123} H A), \\ a_{45} &= \bar{\omega}_1^2 Q_{a12} + \widehat{A} - A^T E_a^T + C^T \widehat{B}^T, \\ a_{46} &= \bar{\omega}_1^2 Q_{a13} + H^T \Psi_{a112}^T + A^T H^T \Psi_{a122}^T + H^T \Psi_{a123} H L_1 \\ &\quad + A^T H^T \Psi_{a223} H L_1 + Y_a L_1 + \widehat{B} L_2 + A^T L_a^T, \\ a_{55} &= \bar{\omega}_1^2 Q_{a22} + \text{He}(\widehat{A}) & a_{56} &= \bar{\omega}_1^2 Q_{a23} - E_a L_1 + \widehat{B} L_2 \\ a_{66} &= \bar{\omega}_1^2 Q_{a33} + \Psi_{a111} + L_1^T H^T \Psi_{a223} H L_1 \\ &\quad + \text{He}(\Psi_{a122} H L_1 + L_a L_1), \\ a_{47} &= H^T \Psi_{a123} H B_1 + A^T H^T \Psi_{a223} H B_1 + Y_a B_1 \\ &\quad + \widehat{B} D_1 + (A^T Y_a^T + C^T \widehat{B}^T) v_1, \\ a_{48} &= H^T \Psi_{a122}^T + A^T H^T \Psi_{a222}^T + W_a + (A^T Y_a^T + C^T \widehat{B}^T) v_2, \\ a_{57} &= -E_a B_1 + \widehat{B} D_1 + \widehat{A}^T v_1, \end{aligned}$$

$$a_{58} = S_a + \widehat{A}^T v_2,$$

$$\begin{aligned} a_{67} &= \Psi_{a122} H B_1 + L_1^T H^T \Psi_{a223} H B_1 + L_a B_1 \\ &\quad + (L_1^T Y_a^T + L_2^T \widehat{B}^T) v_1, \end{aligned}$$

$$a_{68} = \Psi_{a121} + L_1^T H^T \Psi_{a222}^T + N + (L_1^T Y_a^T + L_2^T \widehat{B}^T) v_2,$$

$$\begin{aligned} a_{77} &= -\gamma^2 I + B_1^T H^T \Psi_{a223} H B_1 \\ &\quad + \text{He}((B_1^T Y_a^T + D_1^T \widehat{B}^T) v_1), \end{aligned}$$

$$a_{78} = B_1^T H^T \Psi_{a222}^T + v_1^T W_a + (B_1^T Y_a^T + D_1^T \widehat{B}^T) v_2,$$

$$a_{88} = \Psi_{a221} + \text{He}(v_2^T W_a),$$

$$\Lambda := \{\Lambda_a : \text{satisfy (12) in Proposition 5}\},$$

$$\Psi := \{\Psi_a : \text{satisfy (20) in Proposition 7}\},$$

(31)

Proof. Substituting Π_a into (17), $G_{rd}(j\omega)^T G_{rd}(j\omega) < \gamma^2$ which is equivalent to (27). By applying Lemma 6, (27) holds if

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_a \end{bmatrix}^T \begin{bmatrix} -Q_a & \mathcal{P}_a \\ \mathcal{F} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B}_a \\ \mathcal{F} & 0 \end{bmatrix} \begin{bmatrix} -Q_a & \mathcal{P}_a \\ \mathcal{F} & 0 \end{bmatrix} < 0$$

$$< \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} \begin{bmatrix} \Psi_a & 0 \\ 0 & \Pi_a \end{bmatrix} [\mathcal{C} \ \mathcal{D}] \quad (32)$$

$$\mathcal{Q}_a > \mathcal{C}_1^T \Lambda_a \mathcal{C}_1, \quad (33)$$

where

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_a \\ \mathcal{C}_1 & \mathcal{D}_1 \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix} := \left[\begin{array}{cc|cc} \bar{A} & \bar{L}_1 & \bar{B}_d & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & I & 0 & 0 \\ \hline \bar{H} & 0 & 0 & 0 \\ \hline \bar{C} & \bar{L}_2 & \bar{D}_d & 0 \\ 0 & 0 & I & 0 \end{array} \right], \quad (34)$$

$$[\mathcal{C} \ \mathcal{D}] := \begin{bmatrix} \mathcal{C}_{11} & \mathcal{D}_{11} \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1 & 0 \\ \mathcal{C}_1 \mathcal{A} & \mathcal{C}_1 \mathcal{B}_a \\ \mathcal{C}_2 & \mathcal{D}_2 \end{bmatrix}.$$

Let

$$\Upsilon_a = \begin{bmatrix} -\mathcal{Q}_a & \mathcal{P}_a & 0 \\ \mathcal{P}_a & \bar{\omega}_2 \mathcal{Q}_a + \mathcal{C}_{11}^T \Psi_a \mathcal{C}_{11} + \mathcal{C}_2^T \Pi_a \mathcal{C}_2 & \mathcal{C}_{11}^T \Psi_a \mathcal{D}_{11} + \mathcal{C}_2^T \Pi_a \mathcal{D}_2 \\ 0 & \mathcal{D}_{11}^T \Psi_a \mathcal{C}_{11} + \mathcal{D}_2^T \Pi_a \mathcal{C}_2 & \mathcal{D}_{11}^T \Psi_a \mathcal{D}_{11} + \mathcal{D}_2^T \Pi_a \mathcal{D}_2 \end{bmatrix} \quad (35)$$

(32) is equivalent to

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_a \\ I & 0 \\ 0 & I \end{bmatrix}^T \Upsilon_a \begin{bmatrix} \mathcal{A} & \mathcal{B}_a \\ I & 0 \\ 0 & I \end{bmatrix} < 0. \quad (36)$$

And on the other hand,

$$[I \ 0 \ 0] \Upsilon_a [I \ 0 \ 0]^T = -\mathcal{Q}_a < 0. \quad (37)$$

By combining (36) with (37), and applying Lemma 3, (32) holds if and only if

$$\Upsilon_a + \begin{bmatrix} -I \\ \mathcal{A}^T \\ \mathcal{B}_a^T \end{bmatrix} \mathcal{X}_a \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \mathcal{X}_a^T \begin{bmatrix} -I \\ \mathcal{A}^T \\ \mathcal{B}_a^T \end{bmatrix} < 0, \quad (38)$$

where $\mathcal{X}_a$ introduced by Lemma 3 is the variable matrix of appropriate dimensions. Here, $\mathcal{X}_a$ is defined as $\mathcal{X}_a = [\mathcal{X}_{a1} \ \mathcal{X}_{a1} V_1]$ and partition $\mathcal{X}_{a1}$ into $\begin{bmatrix} x_{a11} & -x_{a12} & x_{a13} \\ x_{21} & x_{22} & 0 \\ x_{a31} & x_{a32} & x_{a33} \end{bmatrix}$ where x_{21} and x_{22} are nonsingular matrices variable. V_1 is a given matrix, defined as $V_1 = \begin{bmatrix} v_1^T & 0 & 0 \\ v_2^T & 0 & 0 \end{bmatrix}^T$.

Let $J = \text{diag}\{I, x_{22}^{-1} x_{21}, I\}$, and define the linearizing change of the control variables as follows:

$$\begin{aligned} Q_a &= J^T \mathcal{Q}_a J, \\ X_{a1} &= \begin{bmatrix} Y_a^T & -E_a^T & L_a^T \\ M_a^T & M_a^T & 0 \\ W_a^T & S_a^T & N_a^T \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & -x_{12} x_{22}^{-1} x_{21} & x_{13} \\ x_{21} x_{22}^{-T} x_{21} & x_{21}^T x_{22}^{-T} x_{21} & 0 \\ x_{31} & x_{32} x_{22}^{-1} x_{21} & x_{33} \end{bmatrix}, \end{aligned} \quad (39)$$

$$P_a = J^T \mathcal{P}_a J,$$

$$\begin{bmatrix} \hat{A}_f & \hat{B}_f \\ \hat{C}_f & 0 \end{bmatrix} = \begin{bmatrix} x_{21}^T x_{22} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} \begin{bmatrix} x_{22}^T x_{21} & 0 \\ 0 & I \end{bmatrix}$$

By Pre- and postmultiplying (38) by $\text{diag}\{J^T, J^T, I\}$ and $\text{diag}\{J, J, I\}$ and choos Ψ_a satisfying Proposition 5, (38) is equivalent to $\square$

$$\begin{bmatrix} -Q_a & P_a - X_{a1} & -X_{a1} V_1 \\ * & \bar{\omega}_2 Q_a + \mathcal{C}_{11}^T \Psi_a \mathcal{C}_{11} + \mathcal{C}_2^T \Pi_a \mathcal{C}_2 + \text{He}(\mathcal{A}^T X_{a1}) & \mathcal{C}_{11}^T \Psi_a \mathcal{D}_{11} + \mathcal{C}_2^T \Pi_a \mathcal{D}_2 + \mathcal{A}^T X_{a1} V_1 + X_{a1}^T \mathcal{B}_a \\ * & * & \mathcal{D}_{11}^T \Psi_a \mathcal{D}_{11} + \mathcal{D}_2^T \Pi_a \mathcal{D}_2 + \text{He}(\mathcal{B}_a^T X_{a1} V_1) \end{bmatrix} < 0, \quad (40)$$

where $\mathcal{A}$, $\mathcal{B}_a$, $\mathcal{C}_{11}$, $\mathcal{C}_2$ are the matrices which use the new defined variables $\hat{A}_f$, $\hat{B}_f$, $\hat{C}_f$ in the matrices $\mathcal{A}$, $\mathcal{B}_a$, $\mathcal{C}_{11}$, $\mathcal{C}_2$.

By simple calculation, we can obtain that

$$\begin{bmatrix} \mathcal{C}_2^T \Pi_a \mathcal{C}_2 & \mathcal{C}_2^T \Pi_a \mathcal{D}_2 \\ * & \mathcal{D}_2^T \Pi_a \mathcal{D}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} &+ \begin{bmatrix} \mathfrak{C}^T \\ \bar{L}_2^T \\ \bar{D}_2^T \\ 0 \end{bmatrix} \left[\mathfrak{C} \ \bar{L}_2 \mid \bar{D}_2 \ 0 \right] \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 \mathcal{I}_0 \end{bmatrix} \\ &+ \begin{bmatrix} \mathfrak{C}^T \\ \mathfrak{D}_2^T \end{bmatrix} \left[\mathfrak{C}_1 \mid \mathfrak{D}_2 \right], \end{aligned} \quad (41)$$

where $\mathfrak{C} = [D_f C \ \widehat{C}_f]$, $\mathfrak{C}_1 = [\mathfrak{C} \ \bar{L}_2]$, $\mathfrak{D}_2 = [\bar{D}_2 \ 0]$. By applying the Schur complement formula, the following can be obtained:

$$\begin{bmatrix} -Q_a & P_a - X_{a1} & -X_{a1}V_1 & 0 \\ * & \bar{\omega}_2 Q_a + \mathcal{E}_{11}^T \Psi_a \mathcal{E}_{11} + \text{He}(\mathcal{A}^T X_{a1}) & \mathcal{E}_{11}^T \Psi_a \mathcal{D}_{11} + \mathcal{A}^T X_{a1} V_1 + X_{a1}^T \mathcal{B}_a & \mathfrak{C}_1^T \\ * & * & \mathcal{D}_{11}^T \Psi_a \mathcal{D}_{11} + \text{He}(\mathcal{B}_a^T X_{a1} V_1) - \gamma^2 \mathcal{J}_0 & \mathfrak{D}_2^T \\ * & * & * & -I \end{bmatrix} < 0. \quad (42)$$

Then, pre-and postmultiply $\text{diag}\{I, I, I, R^T\}$ to (42). Due to $-R^T R \leq I - \text{He}(R)$ and defining $\widehat{A} = M\widehat{A}_f$, $\widehat{B} = M\widehat{B}$, $\widehat{C} = R^T \widehat{C}_f$, $\widehat{D} = R^T D_f$, (29) hold. In addition, and pre- and postmultiplying (33) by J_1^T and J_1 , respectively, (33) becomes (30).

Hence, if conditions (29) and (30) holds, the augmented uncertain system (5) is stable and guarantees the H_∞ performance (27), which completes the proof.

3.2. The Fault Sensitiveness Condition for Faulty Case

Theorem 9. Consider the system in (5) with $\Delta \in \Delta$. A real symmetric matrix $\Pi_b = \begin{bmatrix} -1 & 0 \\ 0 & \beta^2 I \end{bmatrix}$. The following condition

$$\delta_{\min}(G_{rf}(j\omega)) > \beta, \quad |\omega| < \bar{\omega}_2, \quad \Delta \in \Delta \quad (43)$$

holds if there exist matrix variables $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$, M , R , Y_b , E_b , L_b , N_b , W_b , S_b ,

$$P_b = \begin{bmatrix} P_{b11} & P_{b12} & P_{b13} \\ P_{b21} & P_{b22} & P_{b23} \\ P_{b31} & P_{b32} & P_{b33} \end{bmatrix},$$

$$Q_b = \begin{bmatrix} Q_{b11} & Q_{b12} & Q_{b13} \\ Q_{b21} & Q_{b22} & Q_{b23} \\ Q_{b31} & Q_{b32} & Q_{b33} \end{bmatrix}$$

$$\Psi_b = \begin{bmatrix} \Psi_{b111} & \Psi_{b112} & \Psi_{b121} & \Psi_{b122} \\ \Psi_{b112}^T & \Psi_{b113} & \Psi_{b122}^T & \Psi_{b123} \\ \Psi_{b121}^T & \Psi_{b122} & \Psi_{b221} & \Psi_{b222} \\ \Psi_{b122}^T & \Psi_{b123} & \Psi_{b222}^T & \Psi_{b223} \end{bmatrix}, \quad (44)$$

$$\Lambda_b = \begin{bmatrix} \Lambda_{b11} & \Lambda_{b12} \\ \Lambda_{b12}^T & \Lambda_{b22} \end{bmatrix}$$

of appropriate dimensions satisfying $Q_b > 0$, $\Psi_b \in \Psi$, and $\Lambda_b \in \Lambda$ such that

$$\begin{bmatrix} -Q_{b11} & -Q_{b12} & -Q_{b13} & P_{b11} - Y_b^T & P_{b12} - E_b^T & P_{b13} - L_b^T & -Y_b^T v_3 & -Y_b^T v_4 & 0 \\ * & -Q_{b22} & -Q_{b23} & P_{b21} - M^T & P_{b22} - M^T & P_{b23} & -M^T v_3 & -M^T v_4 & 0 \\ * & * & -Q_{b33} & P_{b31} - W_b^T & P_{b32} - S_b^T & P_{b33} - N_b^T & -W_b^T v_3 & -W_b^T v_4 & 0 \\ * & * & * & c_{44} & b_{45} & b_{46} & b_{47} & b_{48} & -\mathcal{J}_1 R^T \\ * & * & * & * & b_{55} & B_{56} & b_{57} & b_{58} & -\mathcal{J}_1 R^T \\ * & * & * & * & * & b_{66} & b_{67} & b_{69} & -\mathcal{J}_1 R^T \\ * & * & * & * & * & * & b_{77} & b_{78} & 0 \\ * & * & * & * & * & * & * & b_{88} & 0 \\ * & * & * & * & * & * & 0 & 0 & -I \end{bmatrix} < 0, \quad (45)$$

$$\begin{bmatrix} Q_{b11} - H^T \Lambda_{b22} H & Q_{b12} & Q_{b13} - H^T \Lambda_{b12}^T \\ * & Q_{b22} & Q_{b23} \\ * & * & Q_{b33} - \Lambda_{b11} \end{bmatrix} > 0, \quad (46)$$

where

$$\begin{aligned} b_{44} &= \bar{\omega}_2^2 Q_{b11} + H^T \Psi_{113} H + A^T H^T \Psi_{b223} H A \\ &+ \text{He}(Y_b A + \widehat{B} C + \mathcal{J}_1 \widehat{D} C + H^T \Psi_{b123} H A), \end{aligned}$$

$$\begin{aligned} b_{45} &= \bar{\omega}_2^2 Q_{b12} + \widehat{A} + A^T E_b^T + C^T \widehat{B}^T + C^T \widehat{B}^T \\ &+ (\mathcal{J}_1 \widehat{D} C)^T + \mathcal{J}_1 \widehat{C}, \\ b_{46} &= \bar{\omega}_2^2 Q_{b13} + H^T \Psi_{b112}^T + A^T H^T \Psi_{b122}^T \end{aligned}$$

$$\begin{aligned}
& + H^T \Psi_{b123} H L_1 + A^T H^T \Psi_{b223} H L_1 \\
& + \widehat{B} L_2 + A^T L_b^T + \mathcal{J}_1 \widehat{D} L_2 + C^T \widehat{D}^T \mathcal{J}_1^T, \\
b_{55} & = \bar{\omega}_2^2 Q_{b22} + \text{He} \left(\widehat{A} + \mathcal{J}_1 \widehat{C} \right), \\
b_{56} & = \bar{\omega}_2^2 Q_{b23} + E L_1 + \widehat{B} L_2 + \mathcal{J}_1 \widehat{D} L_2 + \widehat{C}^T \mathcal{J}_1^T, \\
b_{66} & = \bar{\omega}_2^2 Q_{b33} + \Psi_{b111} + L_1^T H^T \Psi_{b223} H L_1 \\
& + \text{He} \left(\Psi_{b122} H L_1 + L_b L_1 + \mathcal{J}_1 \widehat{D} L_2 \right), \\
b_{47} & = H^T \Psi_{b123} H B_2 + A^T H^T \Psi_{b223} H B_2 + Y_b B_2 \\
& + \widehat{B} D_2 + (Y_b A + \widehat{B} C)^T v_3 + \mathcal{J}_1 \widehat{D} D_2, \\
b_{48} & = H^T \Psi_{b122}^T + A^T H^T \Psi_{b222}^T + W_b + (Y_b A + \widehat{B} C)^T v_4, \\
b_{57} & = E_b B_2 + \widehat{B} D_2 + \widehat{A}^T v_3 + \mathcal{J}_1 \widehat{D} D_2, \\
b_{58} & = S_b + \widehat{A}^T v_4, \\
b_{67} & = \Psi_{b122} H B_2 + L_1^T H^T \Psi_{b223} H B_2 + L_b B_2 \\
& + (Y L_1 + \widehat{B} L_2)^T v_3 + \mathcal{J}_1 \widehat{D} D_2, \\
b_{68} & = \Psi_{b121} + L_1^T H^T \Psi_{b222}^T + N_b + (Y_b L_1 + \widehat{B} L_2)^T v_4, \\
b_{77} & = \beta^2 I + B_2^T H^T \Psi_{a223} H B_2 + \text{He} \left((Y_b B_2 + \widehat{B} D_2)^T v_3 \right), \\
b_{78} & = B_2^T H^T \Psi_{b222}^T + v_3^T W_b + (Y_b B_2 + \widehat{B} D_2)^T v_4, \\
b_{88} & = \Psi_{b221} + \text{He} \left(v_4^T W_b \right), \\
\Lambda & := \{ \Lambda_b : \text{satisfy (12) in Proposition 5} \}, \\
\Psi & := \{ \Psi_b : \text{satisfy (20) in Proposition 7} \},
\end{aligned} \tag{47}$$

Proof. Substituting Π_b into (17), $G_{rr_{oi}}(j\omega)^T G_{rr_{oi}}(j\omega) < \beta^2 I$ which is equivalent to (43). Applying Lemma 6, (43) holds if

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_b \\ \mathcal{F} & 0 \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q}_b & \mathcal{P}_b \\ \mathcal{P}_b & \bar{\omega}_2^2 \mathcal{Q}_b \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B}_b \\ \mathcal{F} & 0 \end{bmatrix} \tag{48}$$

$$\begin{aligned}
& < \begin{bmatrix} \mathcal{E}^T \\ \mathcal{D}^T \end{bmatrix} \begin{bmatrix} \Psi_b & 0 \\ 0 & \Pi_b \end{bmatrix} \begin{bmatrix} \mathcal{E} & \mathcal{D} \end{bmatrix} \\
& \mathcal{Q}_b > \mathcal{E}_1^T \Lambda_b \mathcal{E}_1
\end{aligned} \tag{49}$$

hold.

Where

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_b \\ \mathcal{E}_1 & \mathcal{D}_1 \\ \mathcal{E}_2 & \mathcal{D}_2 \end{bmatrix} := \begin{bmatrix} \overline{A} & \overline{L}_1 & \overline{B}_f & 0 \\ 0 & 0 & 0 & I \\ \overline{H} & 0 & 0 & 0 \\ \overline{C} & \overline{L}_2 & \overline{D}_f & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \tag{50}$$

$$[\mathcal{E} \ \mathcal{D}] := \begin{bmatrix} \mathcal{E}_{11} & \mathcal{D}_{11} \\ \mathcal{E}_2 & \mathcal{D}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{E}_1 & 0 \\ \mathcal{E}_1 \mathcal{A} & \mathcal{E}_1 \mathcal{B}_b \\ \mathcal{E}_2 & \mathcal{D}_2 \end{bmatrix}.$$

Let

$$Y_b = \begin{bmatrix} -\mathcal{Q}_b & \mathcal{P}_b & 0 \\ \mathcal{P}_b & \bar{\omega}_2 \mathcal{Q}_b + \mathcal{E}_{11}^T \Psi_b \mathcal{E}_{11} + \mathcal{E}_2^T \Pi_b \mathcal{E}_2 & \mathcal{E}_{11}^T \Psi_b \mathcal{D}_{11} + \mathcal{E}_2^T \Pi_b \mathcal{D}_2 \\ 0 & \mathcal{D}_{11}^T \Psi_b \mathcal{E}_{11} + \mathcal{D}_2^T \Pi_b \mathcal{E}_2 & \mathcal{D}_{11}^T \Psi_b \mathcal{D}_{11} + \mathcal{D}_2^T \Pi_b \mathcal{D}_2 \end{bmatrix} \tag{51}$$

(48) is equivalent to

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_b \\ I & 0 \\ 0 & I \end{bmatrix}^T Y_b \begin{bmatrix} \mathcal{A} & \mathcal{B}_b \\ I & 0 \\ 0 & I \end{bmatrix} < 0, \tag{52}$$

and on the other hand,

$$[I \ 0 \ 0] Y_b [I \ 0 \ 0]^T = -\mathcal{Q}_b < 0. \tag{53}$$

Combining (52) with (53) and applying Lemma 3, (48) holds if and only if

$$Y_b + \begin{bmatrix} -I \\ \mathcal{A}^T \\ \mathcal{B}_b^T \end{bmatrix} \mathcal{X}_b \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \mathcal{X}_b^T \begin{bmatrix} -I \\ \mathcal{A}^T \\ \mathcal{B}_b^T \end{bmatrix} < 0, \tag{54}$$

where $\mathcal{X}_b$ introduced by Lemma 3 is the variable matrix of appropriate dimensions. Here, $\mathcal{X}_b$ is defined as $\mathcal{X}_b = [\mathcal{X}_{b1} \ \mathcal{X}_{b1} V_2]$, where $\mathcal{X}_{b1} = \begin{bmatrix} x_{b11} & -x_{b12} & x_{b13} \\ x_{21} & x_{22} & 0 \\ x_{b31} & x_{b32} & x_{b33} \end{bmatrix}$, where x_{21} and x_{22} are the same as in Theorem 8. V_2 is an given matrix, defined as $V_2 = \begin{bmatrix} v_3^T & 0 & 0 \\ v_4^T & 0 & 0 \end{bmatrix}^T$. Pre- and postmultiply (54) by $\text{diag}\{J^T, J^T, I\}$ and $\text{diag}\{J, J, I\}$, where J is defined as Theorem 8 and choose Ψ_b satisfying Proposition 5, (54) is equivalent to

$$\begin{bmatrix} -\mathcal{Q}_b & P_b - X_{b1} & -X_{b1} V_2 \\ * & \bar{\omega}_2 \mathcal{Q}_b - \mathcal{E}_{b2}^T \mathcal{E}_{b2} + \mathcal{E}_{11}^T \Psi_b \mathcal{E}_{11} + \text{He}(X_{b1}^T \overline{\mathcal{A}}_b) & -\mathcal{E}_{b2}^T \mathcal{D}_{b2} + \mathcal{E}_{11}^T \Psi_b \mathcal{D}_{11} + X_{b1}^T \overline{\mathcal{B}}_b + \overline{\mathcal{A}}_b X_{b1} V_2 \\ * & * & -\mathcal{D}_{b2}^T \mathcal{D}_{b2} + \mathcal{D}_{11}^T \Psi_b \mathcal{D}_{11} + \beta^2 \mathcal{J}_0 + \text{He}(\overline{\mathcal{B}}_b^T X_{b1} V_2) \end{bmatrix} < 0, \tag{55}$$

where

$$\begin{bmatrix} \widehat{A}_f & \widehat{B}_f \\ \widehat{C}_f & 0 \end{bmatrix} = \begin{bmatrix} x_{21}^T x_{22} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} \begin{bmatrix} x_{22}^T x_{21} & 0 \\ 0 & I \end{bmatrix},$$

$$\mathcal{X}_{b1} = J^T X_1 J = \begin{bmatrix} Y_b^T & E_b^T & L_b^T \\ M^T & M^T & 0 \\ W_b^T & S_b^T & N_b^T \end{bmatrix},$$

$$\overline{\mathcal{A}}_b = J^{-1} \mathcal{A} J = \begin{bmatrix} A & 0 & L_1 \\ \widehat{B}_f C & \widehat{A}_f & \widehat{B}_f L_2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\mathcal{B}}_b = J^{-1} \mathcal{B}_b = \begin{bmatrix} B_2 & 0 \\ \widehat{B}_f D_2 & 0 \\ 0 & I \end{bmatrix},$$

$$\mathcal{C}_{b2} = [D_f C \quad \widehat{C}_f \quad D_f L_2], \quad \mathcal{D}_{b2} = [D_f D_2 \quad 0], \quad (56)$$

Condition (55) is nonlinear owing to the product terms $-C_{b2}^T C_{b2}$. To solve this problem, condition (55) is equivalent to

$$\Xi_{b2}^T \Theta \Xi_{b2} < 0, \quad (57)$$

where

$$\Xi_{b2} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & \mathcal{C}_{b2} & \mathcal{D}_{b2} \end{bmatrix}, \quad (58)$$

$$\Theta = \begin{bmatrix} -Q_b & P_b - X_1 & -X_1 V_2 & 0 \\ * & \mathcal{C}_{11}^T \Psi_b \mathcal{C}_{11} + \text{He}(X_1^T \overline{\mathcal{A}}_{bi}) & X_1^T \overline{\mathcal{B}}_b + \overline{\mathcal{A}}_{bi} X_1 V_2 + \mathcal{C}_{11}^T \Psi_b \mathcal{D}_{11} & 0 \\ * & * & \mathcal{D}_{11}^T \Psi_b \mathcal{D}_{11} + \beta^2 \mathcal{J}_0 + \text{He}(\overline{\mathcal{B}}_b^T X_{b1} V_2) & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}.$$

Defining $G = [0 \quad \mathcal{C}_{b2} \quad \mathcal{D}_{b2} \quad -I]$ satisfies $G \Xi_{b2} = 0$. Using Lemma 4, and given a matrix T , (57) is equivalent to

$$\Theta + TG + (TG)^T < 0, \quad (59)$$

where one notes T as

$$T = [0 \quad \mathcal{R} \quad 0 \quad 0]^T, \quad \mathcal{R} = [R\mathcal{J}_1^T \quad R\mathcal{J}_1^T \quad R\mathcal{J}_1^T]. \quad (60)$$

Defining $\widehat{A} = M\widehat{A}_f$, $\widehat{B} = M\widehat{B}$, $\widehat{C} = R^T \widehat{C}_f$, and $\widehat{D} = R^T D_f$ and after some matrix manipulation, (59) becomes (45). By pre- and postmultiplying (49) by J^T and J , respectively, (49) becomes (46). $\square$

Remark 10. Theorem 8 considers the attenuation performance for the disturbance $d(t)$, and the conservatism comes from the special structure of $\mathcal{X}_{a1}$ and the same one in $\mathcal{X}_a$. Theorem 9 considers the sensitiveness performance for the fault $f(t)$, and the conservatism comes from the special structure of $\mathcal{X}_{b1}$ and the same one in $\mathcal{X}_b$. Then, the conservatism for Theorem 9 comes as the same as Theorem 8. However, in order to obtain the better detection performance, the sensitiveness performance is considered over the conservatism.

3.3. Stability Condition for the Filter. Due to the fault tolerant controller, the closed-loop system is stable both in the fault-free cases and in the fault cases. Then, the condition that the augmented systems (1) is stable can translate into the condition that the FD filter is stable.

Lemma 11 (see [20]). *By considering the FD filter (4), the FD filter is stable, if there exist matrix variables M , $\widehat{A}$ and $P_s > 0$ satisfying*

$$\begin{bmatrix} -(M + M^T) & \widehat{A} + P_s & M \\ * & -P_s & 0 \\ * & * & -P_s \end{bmatrix} < 0. \quad (61)$$

3.4. Algorithm. In the precious sections, Theorems 8 and 9 and Lemma 11 have formulated the inequality conditions for the performance indexes (8) and (9) and the stable condition, respectively. Summarily, we have the following theorem.

Theorem 12. *By considering the uncertain system model (1), there exists a FD filter (4) such that the augmented system model (5) is stable and satisfies the performance indices (8) and (9) if inequality conditions (29), (30), (45), (46), and (61) hold.*

Proof. Combining Theorems 8–9 and Lemma 11, it is obviously that the theorem holds. $\square$

Remark 13. In Theorems 8–9, if the matrices v_i , $i \in \{1, \dots, 4\}$ are given, all conditions (29), (30), (45), and (46) are the LMI conditions. Specially, if the disturbance and the fault are a scalar, respectively, v_i is vector, and it can be easily chosen, which has been further confirmed in the simulink. In addition, Propositions 5 and 7 are used to directly deal with the ellipsoidal uncertainty; the variable structure is complex. However, all the variables in Propositions 5 and 7 is composed of linear matrices, and can be used for LMI conditions.

The gain matrices A_f , B_f , C_f , and D_f defining the FD filter can be derived by the means of a standard procedure. Denote $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and M is the optimal solution of (7) with condition (29), (30), (45), (46), and (61):

- (1) compute the x_{21} , x_{22} by solving the following factorization problem $M = x_{21}^T x_{22}^{-1} x_{21}$;
- (2) compute $\widehat{A}_f$, $\widehat{B}_f$ by $\widehat{A}$, $\widehat{B}$ and M . Compute $\widehat{C}_f$, D_f by $\widehat{C}$, $\widehat{D}$ and R ;
- (3) finally, obtain the gain matrices of the FD filter

$$\begin{bmatrix} \widehat{A}_f & \widehat{B}_f \\ \widehat{C}_f & 0 \end{bmatrix} = \begin{bmatrix} x_{21}^T x_{22} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} \begin{bmatrix} x_{22}^T x_{21} & 0 \\ 0 & I \end{bmatrix}. \quad (62)$$

4. Thresholds Computation

After the parameter matrices of the FD filter A_f , B_f , C_f , and D_f are designed, similar to that proposed in [21], the residual evaluation function $J_r(t)$ can be selected as

$$J_r(t) = \sqrt{\frac{1}{t} \int_0^t r^T(\tau) r(\tau) d\tau}. \quad (63)$$

Under fault-free condition, the residual $r(t)$ becomes

$$r(t) = G_{rd}(j\omega) d(t), \quad (64)$$

and namely, the Parseval Theorem (see [22]), one has that

$$J_r(t) \leq \|G_{rd_0}\|_{\infty} \|d\|_{\text{rms}} \leq \|G_{rd_0}\|_{\infty} \bar{d}, \quad (65)$$

where $\bar{d}$ is the upper bound on the disturbance. It is assumed that $\bar{d}$ is the known. As a consequence, the following threshold results:

$$J_{\text{th}} = \|G_{rd_0}\|_{\infty} \bar{d}. \quad (66)$$

Based on this, the occurrence of faults can be detected by the following logic rule.

$$\begin{aligned} \|J_r\| &\leq J_{\text{th}}, & \text{no alarm,} \\ \|J_r\| &> J_{\text{th}}, & \text{with alarm.} \end{aligned} \quad (67)$$

FIGURE 1: Ellipsoidal uncertainty.

5. Example

In this section, an example is given to illustrate the effectiveness of our design method. Consider the uncertain linear system by

$$\begin{aligned} A &= \begin{bmatrix} 0.5 & 1 \\ -3 & -3 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0.2 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, & C &= [1 \quad 1], \\ D_1 &= 0.5, & D_2 &= 0.5, \\ L_2 &= [0 \quad 0 \quad 0 \quad 0], & H &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \Delta &= \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, & \theta &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \end{aligned} \quad (68)$$

and the uncertain parameters $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \in \mathcal{U}$ satisfy $\{\theta \mid \theta^T \mathcal{R} \theta < 1\}$, as shown in Figure 1, where $\mathcal{R} = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 1.25 \end{bmatrix}$. It is worth pointing out that if the method in this paper is adopted, the uncertain region is only in the ellipsoid (solid lines). Otherwise, the ellipsoid has to translate into the convex bounded polyhedral domain (dashed line) for the existing method to deal with the uncertainty, which enlarges the uncertain region and results in increasing the conservatism.

Given the matrices

$$\begin{aligned} v_1 &= \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, & v_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ v_3 &= \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, & v_4 &= \begin{bmatrix} 1.5 & 1.5 & 1.5 & -1.5 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \end{aligned} \quad (69)$$

the disturbance is $d(t) = 0.8 \sin(0.1t) \exp(-0.05t)$. Then, the upper bounds on the frequency for the disturbance and the reference input can choose $\bar{\omega}_1 = 0.5$ and $\bar{\omega}_2 = 0$, respectively.

FIGURE 2: The residual signal and the residual evaluation function J_r .

Given $\gamma = 1$, for the practical needs, we obtain the sensitivity performance $\beta = 0.4234$ by solving the optimization problem (16) with the conditions in Theorem 12 and the FD filter with gain matrices as

$$\begin{aligned} A_f &= \begin{bmatrix} -3.5304 & 0.0850 \\ -6.0843 & -0.2812 \end{bmatrix}, & B_f &= \begin{bmatrix} 0.0696 \\ 2.5117 \end{bmatrix}, \\ C_f &= \begin{bmatrix} -0.0539 & -0.2526 \end{bmatrix}, & D_f &= -1.3339. \end{aligned} \quad (70)$$

Choose $\theta_1 = 0.2$ and $\theta_1 = -0.6$. Considering the case that the fault occurs between 70 sec and 150 sec, the residual signal and the residual evaluation function $J_r(t)$ and the threshold $J_{th}(t)$ are shown in Figure 2. Consider that $J_r(t) > J_{th}(t)$ at 92.5 sec, which means that the fault alarms when the outage fault occurs. Hence, the fault can be detected.

6. Conclusions

In this paper, the design problem of the FD filter for uncertain linear systems has been addressed. A method for the design problem is given in terms of the linear matrix inequality (LMI) optimization techniques. The new proposed method has two main advantages. The first one is to resolve the problem of designing FD filters subjected to the model with ellipsoidal parametric uncertainty region, while this problem cannot be directly resolved using the existing method. The second one is that the finite-frequency performances of the fault and the disturbance have been detected.

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