Research Article

A High-Accuracy Linear Conservative Difference Scheme for Rosenau-RLW Equation

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Received 14 July 2013; Revised 15 September 2013; Accepted 16 September 2013

Academic Editor: Chien-Yu Lu

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We study the initial-boundary value problem for Rosenau-RLW equation. We propose a three-level linear finite difference scheme, which has the theoretical accuracy of $O(\tau^2 + h^4)$. The scheme simulates two conservative properties of original problem well. The existence, uniqueness of difference solution, and a priori estimates in infinite norm are obtained. Furthermore, we analyze the convergence and stability of the scheme by energy method. At last, numerical experiments demonstrate the theoretical results.

1. Introduction

In the study of the dynamics of compact discrete systems, wave-wave and wave-wall interactions cannot be described by the well-known KdV equation. To overcome this shortcoming of KdV equation, Rosenau proposed the following Rosenau equation in [1, 2]:

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \quad (1)$$

Rosenau equation (1) is usually used to describe the dense discrete system and simulate the long-chain transmission model through an L-C flow in radio and computer fields. The existence and uniqueness of solution to (1) were proved by Park in [3]. Rosenau equation is also regarded as the transformation of the Regularized Long Wave (RLW) equation (see [4]):

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad (2)$$

which is usually used to simulate the long wave in nonlinear emanative medium. RLW equation plays important role in the study of nonlinear diffusion wave because it could model lots of physical phenomena. As RLW equation and KdV equation have the same approximative order when they are used to describe motivations, RLW equation could simulate almost all of the applications of KdV equation [5]. Therefore,

there are many works about Rosenau equation (1) and RLW equation (2) (see, e.g., [6–23]).

Rosenau-RLW equation is the generalization of Rosenau equation (1) and RLW equation (2). The authors of [24–27] studied the numerical solution of Rosenau-RLW equation. Motivated by the above works, we consider the initial-boundary value problem of the following Rosenau-RLW equation:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + uu_x = 0, \quad (3)$$

$$(x, t) \in (x_L, x_R) \times (0, T],$$

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R], \quad (4)$$

$$\begin{aligned} u(x_L, t) = u(x_R, t) = 0, \quad u_{xx}(x_L, t) = u_{xx}(x_R, t) = 0, \\ t \in [0, T], \end{aligned} \quad (5)$$

where $u_0(x)$ is a smooth function.

As the solitary wave solution of (3) is (see, e.g., [24–26])

$$u(x, t) = \frac{15}{19} \operatorname{sech}^4 \left(\frac{\sqrt{13}}{26} \left(x - \frac{169}{133} t \right) \right), \quad (6)$$

the initial-boundary problem (3)–(5) is in accordance with the Cauchy problem of (3) when $-x_L \gg 0$, $x_R \gg 0$.

It is easy to verify that problem (3)–(5) satisfies the following conservative laws [27]:

$$Q(t) = \int_{x_L}^{x_R} u(x, t) dx = \int_{x_L}^{x_R} u_0(x) dx = Q(0), \quad (7)$$

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0), \quad (8)$$

where $Q(0)$ and $E(0)$ are both constants only depending on initial data.

Li and Vu-Quoc pointed in [28] that in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation. It is said in [29] that conservative difference scheme can simulate the conservative law of initial problem well and it could avoid the nonlinear blowup. Therefore, constructing conservative difference scheme is an important and significant job. To our knowledge, the theoretic accuracy of the existing difference scheme for Rosenau-RLW equation (see [24, 25, 27]) is $O(\tau^2 + h^2)$. Particularly, in [27], the authors proposed a three-level linear conservative difference scheme for problem (3)–(5), whose theoretic accuracy is $O(\tau^2 + h^2)$. One does not need iteration when solving the equation numerically using this scheme. Henceforth, it could save some computing time. Using Richardson extrapolation idea, we will propose a three-level linear difference scheme which has the theoretic accuracy of $O(\tau^2 + h^4)$ without refined mesh in this paper. Our scheme simulates the two conservative laws (7) and (8) well. And we will study the a priori estimate, existence, and uniqueness of the difference solution. Furthermore, we shall analyze the convergence and stability.

The rest of this paper is organized as follows. We propose the conservative difference scheme in Section 2 and prove the existence and uniqueness of solution to difference scheme by mathematical induction in Section 3. Section 4 is devoted to the prior estimate, convergence, and stability of the difference scheme. In Section 5, we verify our theoretical analysis by numerical experiments.

2. Difference Scheme and Its Conservative Law

Let h and τ be the uniform stepsize in the spatial and temporal directions, respectively. Denote $h = (x_R - x_L)/J$, $x_j = x_L + jh$ ($j = -2, -1, 0, 1, \dots, J, J+1, J+2$), and $t_n = n\tau$ ($n = 0, 1, 2, \dots, N$, where $N = \lceil T/\tau \rceil$). Let u_j^n be the difference approximation of $u(x, t)$ at (x_j, t_n) ; that is, $u_j^n \approx u(x_j, t_n)$. Let

$$\begin{aligned} Z_h^0 &= \{u = (u_j) \mid u_{-2} = u_{-1} = u_0 = u_J = u_{J+1} \\ &= u_{J+2} = 0, j = -2, -1, 0, 1, \dots, J, J+1, J+2\}. \end{aligned} \quad (9)$$

We define the difference operators, inner product, and norms that will be used in this paper as follows:

$$\begin{aligned} (u_j^n)_x &= \frac{u_{j+1}^n - u_j^n}{h}, & (u_j^n)_{\bar{x}} &= \frac{u_j^n - u_{j-1}^n}{h}, \\ (u_j^n)_{\bar{x}} &= \frac{u_{j+1}^n - u_{j-1}^n}{2h}, & (u_j^n)_{\bar{x}} &= \frac{u_{j+2}^n - u_{j-2}^n}{4h}, \end{aligned}$$

$$(u_j^n)_{\bar{t}} = \frac{u_j^{n+1} - u_j^n}{2\tau}, \quad \bar{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2},$$

$$\langle u^n, v^n \rangle = h \sum_{j=1}^{J-1} u_j^n v_j^n, \quad \|u^n\|^2 = \langle u^n, u^n \rangle,$$

$$\|u^n\|_\infty = \max_{1 \leq j \leq J-1} |u_j^n|.$$

(10)

Lemma 1. *It follows from the Cauchy-Schwarz inequality and summation by parts (see [30]) that, for any $u \in Z_h^0$,*

$$\|u_{\bar{x}}\|^2 \leq \|u_{\bar{x}}\|^2 \leq \|u_x\|^2, \quad \|u_{\bar{x}}\|^2 = \|u_x\|^2. \quad (11)$$

In the paper, C denotes a general positive constant which may have a different value in a different occurrence.

Consider the following finite difference scheme for problem (3)–(5):

$$\begin{aligned} & (u_j^n)_{\bar{t}} - \frac{4}{3}(u_j^n)_{x\bar{x}\bar{t}} + \frac{1}{3}(u_j^n)_{\bar{x}\bar{x}\bar{t}} + \frac{5}{3}(u_j^n)_{xx\bar{x}\bar{x}\bar{t}} \\ & - \frac{2}{3}(u_j^n)_{x\bar{x}\bar{x}\bar{x}\bar{t}} + \frac{4}{3}(\bar{u}_j^n)_{\bar{x}} - \frac{1}{3}(\bar{u}_j^n)_{\bar{x}} \\ & + \frac{4}{9}[u_j^n(\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}] \end{aligned} \quad (12)$$

$$- \frac{1}{9}[u_j^n(\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}] = 0,$$

$$(j = 1, 2, \dots, J-1, n = 1, 2, \dots, N-1)$$

$$u_j^0 = u_0(x_j) \quad (j = 1, 2, \dots, J-1), \quad (13)$$

$$u_j^1 - \frac{4}{3}(u_j^1)_{x\bar{x}} + \frac{1}{3}(u_j^1)_{\bar{x}\bar{x}} + \frac{5}{3}(u_j^1)_{xx\bar{x}\bar{x}} - \frac{2}{3}(u_j^1)_{x\bar{x}\bar{x}\bar{x}}$$

$$= u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) + \frac{\partial^4 u_0}{\partial x^4}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) \quad (14)$$

$$- \tau u_0(x_j) \frac{\partial u_0}{\partial x}(x_j) \quad (j = 1, 2, \dots, J-1),$$

$$u^n \in Z_h^0, (u_0^n)_{x\bar{x}} = (u_j^n)_{x\bar{x}} = 0, \quad (15)$$

$$(u_0^n)_{\bar{x}\bar{x}} = (u_j^n)_{\bar{x}\bar{x}} = 0 \quad (n = 0, 1, 2, \dots, N).$$

The discrete boundary condition (15) is reasonable from the boundary condition of (3)–(5). The following theorem shows how the difference scheme (12)–(15) simulates the conservative law numerically.

Theorem 2. *Suppose that $u_0 \in H_0^2[x_L, x_R]$ and $u(x, t) \in C^{5,3}$. Then the scheme (12)–(15) is conservative for discrete energy; that is,*

$$Q^n = Q^{n-1} = \dots = Q^0, \quad (16)$$

$$E^n = E^{n-1} = \dots = E^0 \quad (17)$$

($n = 1, 2, \dots, N$), where

$$\begin{aligned} Q^n &= \frac{h}{2} \sum_{j=1}^{J-1} (u_j^{n+1} + u_j^n) + \frac{2h}{9} \tau \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}} \\ &\quad - \frac{h}{18} \tau \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}}, \\ E^n &= \frac{1}{2} \left(\|u^{n+1}\|^2 + \frac{4}{3} \|u_x^{n+1}\|^2 - \frac{1}{3} \|u_{\bar{x}}^{n+1}\|^2 + \frac{5}{3} \|u_{xx}^{n+1}\|^2 \right. \\ &\quad \left. - \frac{2}{3} \|u_{x\bar{x}}^{n+1}\|^2 + \|u^n\|^2 + \frac{4}{3} \|u_x^n\|^2 - \frac{1}{3} \|u_{\bar{x}}^n\|^2 \right. \\ &\quad \left. + \frac{5}{3} \|u_{xx}^n\|^2 - \frac{2}{3} \|u_{x\bar{x}}^n\|^2 \right). \end{aligned} \quad (18)$$

Proof. Multiplying h in the two sides of (12) and taking summation for j , we could obtain from (15) and summation by parts [30] that

$$\begin{aligned} h \sum_{j=1}^{J-1} \frac{(u_j^{n+1} - u_j^{n-1})}{2\tau} + \frac{4}{9} h \sum_{j=1}^{J-1} u_j^n (\bar{u}_j^n)_{\bar{x}} \\ - \frac{1}{9} h \sum_{j=1}^{J-1} u_j^n (\bar{u}_j^n)_{\bar{x}} = 0. \end{aligned} \quad (19)$$

On the other hand,

$$\begin{aligned} h \sum_{j=1}^{J-1} u_j^n (\bar{u}_j^n)_{\bar{x}} &= \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}} + \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n-1})_{\bar{x}} \\ &= \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}} - \frac{h}{2} \sum_{j=1}^{J-1} u_j^{n-1} (u_j^n)_{\bar{x}}, \\ h \sum_{j=1}^{J-1} u_j^n (\bar{u}_j^n)_{\bar{x}} &= \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}} + \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n-1})_{\bar{x}} \\ &= \frac{h}{2} \sum_{j=1}^{J-1} u_j^n (u_j^{n+1})_{\bar{x}} - \frac{h}{2} \sum_{j=1}^{J-1} u_j^{n-1} (u_j^n)_{\bar{x}}. \end{aligned} \quad (20)$$

From the definition of Q^n , we know that (16) could be deduced from (19)–(20).

Taking an inner product of (12) and $2\bar{u}^n$ (i.e., $u^{n+1} + u^{n-1}$), we could obtain from boundary condition (15) and summation by parts [30] that

$$\begin{aligned} \|u^n\|_t^2 + \frac{4}{3} \|u_x^n\|_t^2 - \frac{1}{3} \|u_{\bar{x}}^n\|_t^2 + \frac{5}{3} \|u_{xx}^n\|_t^2 - \frac{2}{3} \|u_{x\bar{x}}^n\|_t^2 \\ + \frac{8}{3} \langle \bar{u}_x^n, \bar{u}^n \rangle - \frac{2}{3} \langle \bar{u}_{\bar{x}}^n, \bar{u}^n \rangle + 2 \langle \phi(u^n, \bar{u}^n), \bar{u}^n \rangle \\ - 2 \langle \kappa(u^n, \bar{u}^n), \bar{u}^n \rangle = 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \phi(u_j^n, \bar{u}_j^n) &= \frac{4}{9} [u_j^n (\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}], \\ \kappa(u_j^n, \bar{u}_j^n) &= \frac{1}{9} [u_j^n (\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}]. \end{aligned} \quad (22)$$

On the other hand,

$$\langle \bar{u}_x^n, \bar{u}^n \rangle = 0, \quad \langle \bar{u}_{\bar{x}}^n, \bar{u}^n \rangle = 0, \quad (23)$$

$$\begin{aligned} \langle \phi(u^n, \bar{u}^n), \bar{u}^n \rangle &= \frac{4}{9} h \sum_{j=1}^{J-1} [u_j^n (\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}] \bar{u}_j^n \\ &= \frac{4}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} + \frac{4}{9} h \sum_{j=1}^{J-1} (u_j^n \bar{u}_j^n)_{\bar{x}} \bar{u}_j^n \\ &= \frac{4}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} - \frac{4}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} \langle \kappa(u^n, \bar{u}^n), \bar{u}^n \rangle &= \frac{1}{9} h \sum_{j=1}^{J-1} [u_j^n (\bar{u}_j^n)_{\bar{x}} + (u_j^n \bar{u}_j^n)_{\bar{x}}] \bar{u}_j^n \\ &= \frac{1}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} + \frac{4}{9} h \sum_{j=1}^{J-1} (u_j^n \bar{u}_j^n)_{\bar{x}} \bar{u}_j^n \\ &= \frac{1}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} - \frac{1}{9} h \sum_{j=1}^{J-1} u_j^n \bar{u}_j^n (\bar{u}_j^n)_{\bar{x}} = 0. \end{aligned} \quad (25)$$

Taking (23)–(25) into (21), we have

$$\begin{aligned} \frac{1}{2\tau} (\|u^{n+1}\|^2 - \|u^{n-1}\|^2) + \frac{2}{3\tau} (\|u_x^{n+1}\|^2 - \|u_x^{n-1}\|^2) \\ - \frac{1}{6\tau} (\|u_{\bar{x}}^{n+1}\|^2 - \|u_{\bar{x}}^{n-1}\|^2) + \frac{5}{6\tau} (\|u_{xx}^{n+1}\|^2 - \|u_{xx}^{n-1}\|^2) \\ - \frac{1}{3\tau} (\|u_{x\bar{x}}^{n+1}\|^2 - \|u_{x\bar{x}}^{n-1}\|^2) = 0. \end{aligned} \quad (26)$$

From the definition of E^n , we know that (17) could be obtained by deducing the above equality from 0 to $N-1$. $\square$

3. Solvability

Theorem 3. *The solution of difference scheme (12)–(15) is unique.*

Proof. We will use the mathematical induction to prove our theorem. We first note that u^0 and u^1 are determined uniquely by (13) and (14).

Suppose that $u^0, u^1, \dots, u^{n-1}, u^n$ ($n \leq N-1$) are the unique solution to scheme (12)–(15). Next we prove that there exists unique u^{n+1} which satisfies (12)–(15).

Consider

$$\begin{aligned} \frac{1}{2\tau} u_j^{n+1} - \frac{2}{3\tau} (u_j^{n+1})_{x\bar{x}} + \frac{1}{6\tau} (u_j^{n+1})_{\bar{x}\bar{x}} + \frac{5}{6\tau} (u_j^{n+1})_{xx\bar{x}\bar{x}} \\ - \frac{1}{3\tau} (u_j^{n+1})_{x\bar{x}\bar{x}\bar{x}} + \frac{2}{3} (u_j^{n+1})_{\bar{x}} - \frac{1}{6} (u_j^{n+1})_{\bar{x}} \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{9} \left[u_j^n (u_j^{n+1})_{\hat{x}} + (u_j^n u_j^{n+1})_{\hat{x}} \right] \\
& - \frac{1}{18} \left[u_j^n (u_j^{n+1})_{\ddot{x}} + (u_j^n u_j^{n+1})_{\ddot{x}} \right] = 0 \quad (27) \\
& (j = 1, 2, \dots, J-1, \quad n = 1, 2, \dots, N-1).
\end{aligned}$$

Taking an inner product of (27) and u^{n+1} , we could obtain from boundary condition (15) and summation by parts [30] that

$$\begin{aligned}
& \frac{1}{2\tau} \|u^{n+1}\|^2 + \frac{2}{3\tau} \|u_x^{n+1}\|^2 - \frac{1}{6\tau} \|u_{\hat{x}}^{n+1}\|^2 + \frac{5}{6\tau} \|u_{xx}^{n+1}\|^2 \\
& - \frac{1}{3\tau} \|u_{x\hat{x}}^{n+1}\|^2 + \frac{2}{3} \langle u_{\hat{x}}^{n+1}, u^{n+1} \rangle - \frac{1}{6} \langle u_{\ddot{x}}^{n+1}, u^{n+1} \rangle \\
& + \frac{2h}{9} \sum_{j=1}^{J-1} \left[u_j^n (u_j^{n+1})_{\hat{x}} + (u_j^n u_j^{n+1})_{\hat{x}} \right] u_j^{n+1} \\
& - \frac{h}{18} \sum_{j=1}^{J-1} \left[u_j^n (u_j^{n+1})_{\ddot{x}} + (u_j^n u_j^{n+1})_{\ddot{x}} \right] u_j^{n+1} = 0. \quad (28)
\end{aligned}$$

Note that

$$\begin{aligned}
& \langle u_{\hat{x}}^{n+1}, u^{n+1} \rangle = 0, \quad \langle u_{\ddot{x}}^{n+1}, u^{n+1} \rangle = 0, \\
& h \sum_{j=1}^{J-1} \left[u_j^n (u_j^{n+1})_{\hat{x}} + (u_j^n u_j^{n+1})_{\hat{x}} \right] u_j^{n+1} \\
& = h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\hat{x}} + h \sum_{j=1}^{J-1} (u_j^n u_j^{n+1})_{\hat{x}} u_j^{n+1} \\
& = h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\hat{x}} - h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\hat{x}} = 0, \quad (29) \\
& h \sum_{j=1}^{J-1} \left[u_j^n (u_j^{n+1})_{\ddot{x}} + (u_j^n u_j^{n+1})_{\ddot{x}} \right] u_j^{n+1} \\
& = h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\ddot{x}} + h \sum_{j=1}^{J-1} (u_j^n u_j^{n+1})_{\ddot{x}} u_j^{n+1} \\
& = h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\ddot{x}} - h \sum_{j=1}^{J-1} u_j^n u_j^{n+1} (u_j^{n+1})_{\ddot{x}} = 0.
\end{aligned}$$

Furthermore, from Lemma 1, one can easily obtain that

$$\|u_{\hat{x}}\|^2 \leq \|u_x\|^2, \quad \|u_{x\hat{x}}\|^2 \leq \|u_{xx}\|^2. \quad (30)$$

Henceforth we could have

$$\|u^{n+1}\|^2 + \|u_x^{n+1}\|^2 + \|u_{xx}^{n+1}\|^2 = 0, \quad (31)$$

from (28)–(30).

That is, (27) only admits zero solution. So, there exists unique u_j^{n+1} that satisfies (12). $\square$

4. Convergence and Stability of the Difference Scheme

Suppose that $v(x, t)$ is the solution to problem (3)–(5). Let $v_j^n = v(x_j, t_n)$. Then the truncation error of the difference scheme (12)–(15) is

$$\begin{aligned}
r_j^n &= (v_j^n)_{\hat{t}} - \frac{4}{3} (v_j^n)_{x\hat{x}\hat{t}} + \frac{1}{3} (v_j^n)_{\hat{x}\hat{x}\hat{t}} + \frac{5}{3} (v_j^n)_{xx\hat{x}\hat{t}} \\
& - \frac{2}{3} (v_j^n)_{x\hat{x}\hat{x}\hat{t}} + \frac{4}{3} (\bar{v}_j^n)_{\hat{x}} - \frac{1}{3} (\bar{v}_j^n)_{\ddot{x}} \\
& + \varphi(v_j^n, \bar{v}_j^n) - \kappa(v_j^n, \bar{v}_j^n) \\
& (j = 1, 2, \dots, J-1, \quad n = 1, 2, \dots, N-1), \\
v_j^0 &= u_0(x_j) \quad (j = 1, 2, \dots, J-1), \\
v_j^1 &= \frac{4}{3} (v_j^1)_{x\hat{x}} + \frac{1}{3} (v_j^1)_{\hat{x}\hat{x}} + \frac{5}{3} (v_j^1)_{xx\hat{x}} - \frac{2}{3} (v_j^1)_{x\hat{x}\hat{x}\hat{x}} \\
& = u_0(x_j) - \frac{\partial^2 u_0}{\partial x^2}(x_j) + \frac{\partial^4 u_0}{\partial x^4}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) \\
& - \tau u_0(x_j) \frac{\partial u_0}{\partial x}(x_j) + r_j^0 \\
& (j = 1, 2, \dots, J-1),
\end{aligned}$$

$$\begin{aligned}
v^n &\in Z_h^0, \quad (v_0^n)_{x\hat{x}} = (v_J^n)_{x\hat{x}} = 0, \quad (v_0^n)_{\hat{x}\hat{x}} = (v_J^n)_{\hat{x}\hat{x}} = 0 \\
& (n = 0, 1, 2, \dots, N). \quad (32)
\end{aligned}$$

Making use of the Taylor expansion theorem, we know that $|r_j^n| = O(\tau^2 + h^4)$ as $h \rightarrow 0, \tau \rightarrow 0$.

Lemma 4. Assume that $u_0 \in H_0^2[x_L, x_R]$, $u(x, t) \in C^{5,3}$. Then the solution to problem (3)–(5) satisfies

$$\begin{aligned}
\|u\|_{L_2} &\leq C, \quad \|u_x\|_{L_2} \leq C, \quad \|u_{xx}\|_{L_2} \leq C, \\
\|u\|_{L_\infty} &\leq C, \quad \|u_x\|_{L_\infty} \leq C. \quad (33)
\end{aligned}$$

Proof. From (8), we know that

$$\|u\|_{L_2} \leq C, \quad \|u_x\|_{L_2} \leq C, \quad \|u_{xx}\|_{L_2} \leq C. \quad (34)$$

Then

$$\|u\|_{L_\infty} \leq C, \quad \|u_x\|_{L_\infty} \leq C, \quad (35)$$

from the Sobolev inequality. $\square$

Theorem 5. Suppose that $u_0 \in H^2[x_L, x_R]$, $u(x, t) \in C^{5,3}$. Then the solution to difference scheme (12)–(15) satisfies

$$\|u^n\| \leq C, \quad \|u_x^n\| \leq C, \quad \|u_{xx}^n\| \leq C; \quad (36)$$

thus,

$$\|u^n\|_\infty \leq C, \quad \|u_x^n\|_\infty \leq C \quad (n = 1, 2, \dots, N). \quad (37)$$

Proof. From Theorem 2 and (30) we can get

$$\|u^n\|^2 + \|u_x^n\|^2 + \|u_{xx}^n\|^2 \leq E^n = E^0 = C; \quad (38)$$

that is,

$$\|u^n\| \leq C, \quad \|u_x^n\| \leq C, \quad \|u_{xx}^n\| \leq C. \quad (39)$$

Then the discrete Sobolev inequality (see [30]) shows that

$$\|u^n\|_\infty \leq C, \quad \|u_x^n\|_\infty \leq C. \quad (40)$$

□

Theorem 6. Suppose that $u_0 \in H^2[x_L, x_R]$, $u(x, t) \in C^{5,3}$. Then the solution to difference scheme (12)–(15) u^n converges to the solution of problem (3)–(5) in the sense of norm $\|\cdot\|_\infty$, and the convergent rate is $O(\tau^2 + h^4)$.

Proof. Subtracting (12)–(15) from (32) and letting $e_j^n = v_j^n - u_j^n$, we have

$$\begin{aligned} r_j^n &= (e_j^n)_{\hat{t}} - \frac{4}{3}(e_j^n)_{x\hat{x}\hat{t}} + \frac{1}{3}(e_j^n)_{\hat{x}\hat{x}\hat{t}} + \frac{5}{3}(e_j^n)_{xx\hat{x}\hat{t}} \\ &\quad - \frac{2}{3}(e_j^n)_{x\hat{x}\hat{x}\hat{t}} + \frac{4}{3}(\bar{e}_j^n)_{\hat{x}} - \frac{1}{3}(\bar{e}_j^n)_{\hat{x}} + \varphi(v_j^n, \bar{v}_j^n) \\ &\quad - \varphi(u_j^n, \bar{u}_j^n) - \kappa(v_j^n, \bar{v}_j^n) + \kappa(u_j^n, \bar{u}_j^n) \\ &\quad (j = 1, 2, \dots, J-1, n = 1, 2, \dots, N-1), \end{aligned} \quad (41)$$

$$e_j^0 = 0 \quad (j = 0, 1, 2, \dots, J), \quad (42)$$

$$\begin{aligned} r_j^0 &= e_j^1 - \frac{4}{3}(e_j^1)_{x\hat{x}} + \frac{1}{3}(e_j^1)_{\hat{x}\hat{x}} + \frac{5}{3}(e_j^1)_{xx\hat{x}} \\ &\quad - \frac{2}{3}(e_j^1)_{x\hat{x}\hat{x}\hat{x}} \quad (j = 1, 2, \dots, J-1), \end{aligned} \quad (43)$$

$$\begin{aligned} e^n &\in Z_h^0, \quad (e_0^n)_{x\hat{x}} = (e_j^n)_{x\hat{x}} = 0, \\ (e_0^n)_{\hat{x}\hat{x}} &= (e_j^n)_{\hat{x}\hat{x}} = 0 \quad (n = 0, 1, 2, \dots, N). \end{aligned} \quad (44)$$

Taking the inner product of (43) and e^1 , and using boundary condition (44), we obtain

$$\begin{aligned} \|e^1\|^2 + \frac{4}{3}\|e_x^1\|^2 - \frac{1}{3}\|e_{\hat{x}}^1\|^2 + \frac{5}{3}\|e_{xx}^1\|^2 - \frac{2}{3}\|e_{x\hat{x}}^1\|^2 \\ = \langle r^0, e^1 \rangle. \end{aligned} \quad (45)$$

Noticing that

$$|r_j^0| = O(\tau^2 + h^4) \quad (j = 1, 2, \dots, J-1), \quad (46)$$

we can obtain that

$$\|e^1\|^2 + \|e_x^1\|^2 + \|e_{xx}^1\|^2 \leq O(\tau^2 + h^4)^2, \quad (47)$$

from the Cauchy-Schwarz inequality and (30), and (44).

Then taking the inner product of (41) and $2\bar{e}^n$, and combining with boundary condition (44) again, one can get

$$\begin{aligned} \langle r^n, 2\bar{e}^n \rangle &= \|e^n\|_{\hat{t}}^2 + \frac{4}{3}\|e_x^n\|_{\hat{t}}^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_{\hat{t}}^2 + \frac{5}{3}\|e_{xx}^n\|_{\hat{t}}^2 \\ &\quad - \frac{2}{3}\|e_{x\hat{x}}^n\|_{\hat{t}}^2 + \frac{8}{3}\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle - \frac{2}{3}\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle \end{aligned}$$

$$\begin{aligned} &+ 2\langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle \\ &- 2\langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle. \end{aligned} \quad (48)$$

Similar to (23), we have

$$\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle = 0, \quad \langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle = 0. \quad (49)$$

By using Lemma 4, Theorem 5, Lemma 1, and the Cauchy-Schwarz inequality, we can get

$$\begin{aligned} &\langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle \\ &= \frac{4h}{9} \sum_{j=1}^{J-1} [v_j^n(\bar{v}_j^n)_{\hat{x}} - u_j^n(\bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &\quad + \frac{4h}{9} \sum_{j=1}^{J-1} [(v_j^n \bar{v}_j^n)_{\hat{x}} - (u_j^n \bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &= \frac{4h}{9} \sum_{j=1}^{J-1} [v_j^n(\bar{v}_j^n)_{\hat{x}} - u_j^n(\bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &\quad - \frac{4h}{9} \sum_{j=1}^{J-1} (v_j^n \bar{v}_j^n - u_j^n \bar{u}_j^n) (\bar{e}_j^n)_{\hat{x}} \\ &= \frac{4h}{9} \sum_{j=1}^{J-1} [e_j^n(\bar{v}_j^n)_{\hat{x}} + u_j^n(\bar{e}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &\quad - \frac{4h}{9} \sum_{j=1}^{J-1} (e_j^n \bar{v}_j^n + u_j^n \bar{e}_j^n) (\bar{e}_j^n)_{\hat{x}} \\ &\leq C(\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_{\hat{x}}^n\|^2) \\ &\leq C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \\ &\quad + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2), \\ &\langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle \\ &= \frac{h}{9} \sum_{j=1}^{J-1} [v_j^n(\bar{v}_j^n)_{\hat{x}} - u_j^n(\bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &\quad + \frac{h}{9} \sum_{j=1}^{J-1} [(v_j^n \bar{v}_j^n)_{\hat{x}} - (u_j^n \bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &= \frac{h}{9} \sum_{j=1}^{J-1} [v_j^n(\bar{v}_j^n)_{\hat{x}} - u_j^n(\bar{u}_j^n)_{\hat{x}}] \bar{e}_j^n \\ &\quad - \frac{h}{9} \sum_{j=1}^{J-1} (v_j^n \bar{v}_j^n - u_j^n \bar{u}_j^n) (\bar{e}_j^n)_{\hat{x}} \\ &= \frac{h}{9} \sum_{j=1}^{J-1} [e_j^n(\bar{v}_j^n)_{\hat{x}} + u_j^n(\bar{e}_j^n)_{\hat{x}}] \bar{e}_j^n \end{aligned}$$

TABLE 1: The errors estimates of numerical solution with various h and τ .

	$\tau = 0.4, h = 0.2$		$\tau = 0.1, h = 0.1$		$\tau = 0.025, h = 0.05$	
	$\ e^n\ $	$\ e^n\ _\infty$	$\ e^n\ $	$\ e^n\ _\infty$	$\ e^n\ $	$\ e^n\ _\infty$
$t = 10$	$9.496041e - 3$	$3.527311e - 3$	$5.870764e - 4$	$2.186688e - 4$	$3.679896e - 5$	$1.371124e - 5$
$t = 20$	$1.558799e - 2$	$5.383055e - 3$	$9.659182e - 4$	$3.320382e - 4$	$6.028058e - 5$	$2.073326e - 5$
$t = 30$	$1.901987e - 2$	$6.232659e - 3$	$1.176163e - 3$	$3.836776e - 4$	$7.326285e - 5$	$2.389046e - 5$
$t = 40$	$2.083318e - 2$	$6.558748e - 3$	$1.286929e - 3$	$4.032652e - 4$	$8.004082e - 5$	$2.503454e - 5$

$$\begin{aligned}
& -\frac{h}{9} \sum_{j=1}^{J-1} (e_j^n v_j^n + u_j^n \bar{e}_j^n) (\bar{e}_j^n)_{\bar{x}} \\
& \leq C (\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_x^n\|^2) \\
& \leq C (\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \\
& \quad + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2), \\
& \langle r^n, 2\bar{e}^n \rangle = \langle r^n, e^{n+1} + e^{n-1} \rangle \\
& \leq \|r^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2.
\end{aligned} \tag{50}$$

Taking (49)-(50) into (48), we obtain

$$\begin{aligned}
& \|e^n\|_t^2 + \frac{4}{3} \|e_x^n\|_t^2 - \frac{1}{3} \|e_{\bar{x}}^n\|_t^2 + \frac{5}{3} \|e_{xx}^n\|_t^2 - \frac{2}{3} \|e_{x\bar{x}}^n\|_t^2 \\
& \leq \|r^n\|^2 + C (\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \\
& \quad + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2).
\end{aligned} \tag{51}$$

Set

$$\begin{aligned}
B^n &= \|e^{n+1}\|^2 + \|e^n\|^2 + \frac{4}{3} \|e_x^{n+1}\|^2 + \frac{4}{3} \|e_x^n\|^2 \\
& - \frac{1}{3} \|e_{\bar{x}}^{n+1}\|^2 - \frac{1}{3} \|e_{\bar{x}}^n\|^2 + \frac{5}{3} \|e_{xx}^{n+1}\|^2 \\
& + \frac{5}{3} \|e_{xx}^n\|^2 - \frac{2}{3} \|e_{x\bar{x}}^{n+1}\|^2 - \frac{2}{3} \|e_{x\bar{x}}^n\|^2.
\end{aligned} \tag{52}$$

Summarizing (51) from 1 to n , we get

$$B^n \leq B^0 + 2\tau \sum_{l=1}^n \|r^l\|^2 + C\tau \sum_{l=0}^{n+1} (\|e^l\|^2 + \|e_x^l\|^2). \tag{53}$$

From (42) and (47), we know that

$$B^0 \leq O(\tau^2 + h^4)^2. \tag{54}$$

Furthermore,

$$\tau \sum_{l=1}^n \|r^l\|^2 \leq n\tau \max_{1 \leq l \leq n} \|r^l\|^2 \leq T \cdot O(\tau^2 + h^4)^2. \tag{55}$$

Therefore,

$$\begin{aligned}
& \|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e_{xx}^{n+1}\|^2 + \|e_{xx}^n\|^2 \\
& \leq B^n \leq O(\tau^2 + h^4)^2 \\
& + C\tau \sum_{l=0}^{n+1} (\|e^l\|^2 + \|e_x^l\|^2 + \|e_{xx}^l\|^2).
\end{aligned} \tag{56}$$

The discrete Gronwall inequality (see [30]) implies that

$$\begin{aligned}
& \|e^n\| \leq O(\tau^2 + h^4), \quad \|e_x^n\| \leq O(\tau^2 + h^4), \\
& \|e_{xx}^n\| \leq O(\tau^2 + h^4).
\end{aligned} \tag{57}$$

Then we can obtain

$$\|e^n\|_\infty \leq O(\tau^2 + h^4), \tag{58}$$

from the discrete Sobolev inequality. $\square$

We could prove the following theorems in a similar way of Theorem 6.

Theorem 7. Suppose that $u_0 \in H^2[x_L, x_R]$ and $u(x, t) \in C^{5,3}$. The solution to difference scheme (12)–(15) u^n is stable in the sense of norm $\|\cdot\|_\infty$.

5. Numerical Simulations

As the difference scheme (12)–(15) is a linear system about u_j^{n+1} , it does not need iteration when we solve it numerically.

Let $x_L = -50$, $x_R = 100$, $T = 40$, and

$$u_0(x) = \frac{15}{19} \operatorname{sech}^4 \frac{x}{2\sqrt{13}}. \tag{59}$$

For some different values of τ and h , we list errors at several times in Table 1 and verify the accuracy of the difference scheme in Table 2. The numerical simulation of two conservative quantities (7) and (8) is listed in Table 3.

The stability and convergence of the scheme are verified by these numerical experiments. And it shows that our proposed algorithm is effective and reliable.

TABLE 2: The numerical verification of theoretical accuracy $O(\tau^2 + h^4)$.

	$\ e^n(h, \tau)\ /\ e^{4n}(h/2, \tau/4)\ $			$\ e^n(h, \tau)\ _\infty/\ e^{4n}(h/2, \tau/4)\ _\infty$		
	$\tau = 0.4$ $h = 0.2$	$\tau = 0.1$ $h = 0.1$	$\tau = 0.025$ $h = 0.05$	$\tau = 0.4$ $h = 0.2$	$\tau = 0.1$ $h = 0.1$	$\tau = 0.025$ $h = 0.05$
$t = 10$	—	16.13800	15.95361	—	16.13083	15.94813
$t = 20$	—	16.17513	16.02370	—	16.21215	16.01475
$t = 30$	—	16.17110	16.05402	—	16.24451	16.05986
$t = 40$	—	16.18828	16.07841	—	16.26410	16.10835

TABLE 3: The numerical simulation for conservative quantities (7) and (8).

	$\tau = 0.4, h = 0.2$		$\tau = 0.1, h = 0.1$		$\tau = 0.025, h = 0.05$	
	Q^n	E^n	Q^n	E^n	Q^n	E^n
$t = 0$	7.59537171309	4.26542015423	7.59093169500	4.26542024995	7.59065285879	4.26542025595
$t = 10$	7.59537171275	4.26558683637	7.59093168588	4.26542086750	7.59065230880	4.26541966402
$t = 20$	7.59537171886	4.26577432049	7.59093167701	4.26542158265	7.59065176013	4.26541907347
$t = 30$	7.59537132913	4.26590472629	7.59093166975	4.26542206996	7.59065121035	4.26541848035
$t = 40$	7.59537390280	4.26598379748	7.59093166785	4.26542236189	7.59065066159	4.26541788802

Acknowledgments

The authors are very grateful to the anonymous referees for their careful reading and useful suggestions, which greatly improved the presentation of the paper. This work is supported in part by Scientific Research Foundation of Sichuan Provincial Education Department (no. 11ZB009), the Key Scientific Research Foundation of Xihua University (no. Z0912611), and the fund of Key Disciplinary of Computer Software and Theory, Sichuan, China, Grant no. SZD0802-09-1.

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