Research Article

H_∞ Control of Pairwise Distributable Large-Scale TS Fuzzy Systems

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The paper presents new conditions suitable in design of the stabilizing state controller for a class of continuous-time nonlinear systems, which are representable by pairwise distributable Takagi-Sugeno models. Taking into account the affine properties of the TS model structure and applying the pairwise subsystems fuzzy control scheme relating to the parallel distributed output compensators, the extended bounded real lemma form and the sufficient design conditions for pairwise decentralized control are outlined in terms of linear matrix inequalities. The proposed procedure decouples the Lyapunov matrix and the system parameter matrices in the LMIs and, using free tuning parameter, provides the way to obtain global stability of such large-scale TS systems and optimizes subsystems interaction H_∞ norm bounds.

1. Introduction

A number of problems that arise in state control can be reduced to a handful of standard convex and quasiconvex problems that involve matrix inequalities (LMI). It is known that the optimal solution can be computed by using the interior point methods [1], which converge in polynomial time with respect to the problem size. Thus, efficient interior point algorithms have recently been developed and further development of algorithms for these standard problems is an area of active research. For this approach, the stability conditions may be expressed in terms of LMIs, which have a notable practical interest due to the existence of numerical solvers. Some progress review in this field can be found, for example, in [2, 3] and the references therein.

Based on the concept of quadratic stability, the control design problems, in respect of the H_∞ norm of the closed-loop system transfer matrix, are transferred into a standard LMI optimization task, which includes bounded real lemma (BRL) formulation. The first version of BRL presented simple condition under which a transfer function was contractive on the imaginary axis of the complex variable plain. Using

this formulation, it was possible to determine the H_∞ norm of a transfer function, and the BRL has become a significant element to show and prove that the existence of feedback controllers, resulting in a closed-loop transfer matrix having the H_∞ norm less than a given upper bound, is equivalent to the existence of a solution of certain LMIs. Motivated by the underlying ideas, the technique for BRL representation was extended to state feedback control design and stayed preferable for systems with time-varying parameters [4–6]. When used in robust analysis of linear systems with polytopic uncertainties, as the number of polytopes increases, the solution turned out to be very conservative. To reduce conservatism inherent in such use of quadratic methods, the equivalent LMI representations of BRL for continuous-time as well as discrete-time uncertain systems were introduced [7–10]. Moreover, exploiting the sector nonlinearity approach to obtain Takagi-Sugeno (TS) models from the nonlinear system equations [11], suitable BRLs as well as enhanced BRL representation guaranteeing quadratic performances for the closed-loop nonlinear systems are exploited in H_∞ fuzzy control [12–14].

In last years, modern control methods have found their way into design of interconnected systems and led to a wide variety of new concepts and results. In particular, paradigms of LMIs and H_∞ norm have appeared to be very attractive due to their promise of handling systems with relative high dimensions, and design of partly decentralized schemes substantially minimized the information exchange between subsystems of a large scale system. With respect to the existing structure of interconnections in a large-scale system, it is generally impossible to stabilize all subsystems and the whole system simultaneously by using decentralized controllers, since the stability of interconnected systems is not only dependent on the stability degree of subsystems but is also closely dependent on the interconnections [15–17]. Analogous principles were applied in decentralized fuzzy control [18].

Considering the decomposition-based control strategy [19] and including into design step the effects of subsystem pairs interconnections [20], a pairwise decentralized control problem was proposed for linear large-scale systems in [21] and for linear large-scale systems with polytopic uncertainties in [22], respectively. Introducing the results as a pairwise partially decentralized control, in [21] there were formulated design conditions for a linear control, while the design conditions given in [22] reflect the robust linear control design task, both solved in the frames of LMI representations. Since the feedback control for TS models is the so-called parallel distributed compensation (PDC) control, the above results have to be significantly reformulated considering the pairwise distributable large-scale TS fuzzy systems.

Inspired by the enhanced design conditions proposed in [13] in designing the optimal control of TS systems with quadratic optimality criterion, the paper is devoted to studying the partially decentralized control problems from the above given viewpoint for pairwise distributable large-scale TS fuzzy systems. Sufficient stability conditions are stated now as a set of LMIs to encompass the quadratic stability case in respect of the H_∞ approach. Used structures in the presented forms enable us potentially to design systems with an embedded reconfigurable control structure property [21]. To the best of the authors' knowledge, the paper presents a new formulation of the control principle of pairwise distributable large-scale TS fuzzy systems, newly defines the conditions of existence of solutions for the fuzzy control scheme relating to PDCs in such distributable structure, and offers the possibility of new ways to solve the problem of control law synthesis for TS fuzzy systems with a large number of membership functions.

The paper is organized as follows. In Section 2, the basis preliminaries, concerning the TS models and the H_∞ norm problems, are presented with results on BRL and enhanced BRL for TS systems. Formulating the pairwise distributable large-scale TS fuzzy systems structure in Section 3 and continuing with this formalism in Section 4, the equivalent TS BRL design methods are outlined to possess the sufficient conditions for the pairwise decentralized control of given class of TS large-scale systems. Finally, the example is given in Section 5, to illustrate the feasibility and properties of the proposed method and some concluding remarks are stated in Section 6.

Throughout the paper, the following notations are used: $\mathbf{x}^T$, $\mathbf{X}^T$ denote the transposes of the vector $\mathbf{x}$ and matrix $\mathbf{X}$, respectively, $\text{diag}[\mathbf{X}_i]$, $i = 1, 2, \dots, p$ denotes a block diagonal matrix with p blocks, $\text{blk}[\mathbf{X}_{il}]$, $i, l = 1, 2, \dots, p$ entails a row- and column-wise partitioned matrix, broken into blocks by partitioning its rows and columns into p collection of row-groups and p collection of column-groups, for a square matrix, $\mathbf{X} < 0$ means that $\mathbf{X}$ is a symmetric negative definite matrix, the symbol $\mathbf{I}_n$ indicates the n th order unit matrix, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^{n \times r}$ refers to the set of $n \times r$ real matrices.

2. Preliminaries

2.1. System Model. The class of TS systems, considered in the paper, is formed as follows:

$$\dot{\mathbf{q}}(t) = \sum_{j=1}^s h_j(\boldsymbol{\theta}(t)) (\mathbf{A}_j \mathbf{q}(t) + \mathbf{B}_j \mathbf{u}(t)), \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C} \mathbf{q}(t), \quad (2)$$

where $\mathbf{q}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^r$, and $\mathbf{y}(t) \in \mathbb{R}^m$ are vectors of the state, input, and measurable output variables, respectively, matrices $\mathbf{A}_j \in \mathbb{R}^{n \times n}$, $\mathbf{B}_j \in \mathbb{R}^{n \times r}$, and $\mathbf{C} \in \mathbb{R}^{m \times n}$ are real matrices, $t \in \mathbb{R}$ is the time variable, and $\mu_j(\boldsymbol{\theta}(t))$ is the weight for j th fuzzy rule. Satisfying the following, by definition, the property

$$0 \leq \mu_j(\boldsymbol{\theta}(t)) \leq 1, \quad \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) = 1 \quad \forall j \in \{1, 2, \dots, s\},$$

$$\boldsymbol{\theta}(t) = [\theta_1(t) \quad \theta_2(t) \quad \dots \quad \theta_v(t)] \quad (3)$$

is the vector of the premise variables, where s , v are the numbers of fuzzy rules and premise variables, respectively. It is supposed next that all premise variables are measurable and independent on $\mathbf{u}(t)$ (more details can be found, e.g., in [11, 13]).

2.2. LMI Formulations for H_∞ Performance. Let the TS systems model (1), (2) be considered (in this subsection only) in the next extended form

$$\dot{\mathbf{q}}(t) = \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) (\mathbf{A}_j \mathbf{q}(t) + \mathbf{B}_j \mathbf{u}(t) + \mathbf{G}_j \mathbf{w}(t)), \quad (4)$$

$$\mathbf{y}(t) = \mathbf{C} \mathbf{q}(t) + \mathbf{D} \mathbf{w}(t), \quad (5)$$

where $\mathbf{G}_j \in \mathbb{R}^{n \times r}$, $\mathbf{D} \in \mathbb{R}^{m \times r}$, and $\mathbf{w}(t) \in \mathbb{R}^r$ is the disturbance input that belongs to $L_2(0, +\infty)$ and, using the same set of membership functions, the fuzzy state control law is defined as

$$\mathbf{u}(t) = - \sum_{g=1}^s \mu_g(\boldsymbol{\theta}(t)) \mathbf{K}_g \mathbf{q}(t). \quad (6)$$

Evidently, the closed-loop system state variable dynamics is described by the next equation

$$\dot{\mathbf{q}}(t) = \sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) (\mathbf{H}_{jg} \mathbf{q}(t) + \mathbf{G}_j \mathbf{w}(t)) \quad (7)$$

and also, owing to the symmetry in summations, by the symmetric equation

$$\dot{\mathbf{q}}(t) = \sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) (\mathbf{H}_{gj} \mathbf{q}(t) + \mathbf{G}_g \mathbf{w}(t)), \quad (8)$$

where

$$\mathbf{H}_{jg} = \mathbf{A}_j - \mathbf{B}_j \mathbf{K}_g, \quad \mathbf{H}_{gj} = \mathbf{A}_g - \mathbf{B}_g \mathbf{K}_j. \quad (9)$$

Thus, adding (8), (9) gives

$$\begin{aligned} 2\dot{\mathbf{q}}(t) &= \sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\ &\quad \times ((\mathbf{H}_{jg} + \mathbf{H}_{gj}) \mathbf{q}(t) + (\mathbf{G}_j + \mathbf{G}_g) \mathbf{w}(t)). \end{aligned} \quad (10)$$

Rearranging the computation, (10) can be written as

$$\begin{aligned} 2\dot{\mathbf{q}}(t) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \\ &\quad \times ((\mathbf{H}_{jj} + \mathbf{H}_{jj}) \mathbf{q}(t) + (\mathbf{G}_j + \mathbf{G}_j) \mathbf{w}(t)) \\ &\quad + 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\ &\quad \times ((\mathbf{H}_{jg} + \mathbf{H}_{gj}) \mathbf{q}(t) + (\mathbf{G}_j + \mathbf{G}_g) \mathbf{w}(t)), \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{\mathbf{q}}(t) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \\ &\quad \times (\mathbf{H}_{jj} \mathbf{q}(t) + \mathbf{G}_j \mathbf{w}(t)) \\ &\quad + 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\ &\quad \times \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right), \end{aligned} \quad (12)$$

respectively.

Considering this, the next lemmas can be introduced.

Lemma 1 (bounded real lemma). *The closed-loop system (7), (5) is stable with the performance $\sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \|\mathbf{C}(s\mathbf{I} - \mathbf{H}_{jg})^{-1} \mathbf{G}_j + \mathbf{D}\|_\infty^2 \leq \gamma$ if there exist a positive definite matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$ and a positive scalar $\gamma \in \mathbb{R}$ such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \quad \gamma > 0, \quad (13)$$

$$\begin{bmatrix} \mathbf{H}_{jj} \mathbf{X} + \mathbf{X} \mathbf{H}_{jj}^T & \mathbf{G}_j & \mathbf{X} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{X} + \mathbf{X} \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} & \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{X} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0, \quad (15)$$

for all $j \in \langle 1, 2, \dots, s \rangle$, and $j < g \leq s$, $j, g \in \langle 1, 2, \dots, s \rangle$, respectively, where $\mathbf{I}_r \in \mathbb{R}^{r \times r}$, $\mathbf{I}_m \in \mathbb{R}^{m \times m}$ are identity matrices and $\sqrt{\gamma}$ is an upper bound of H_∞ norm of the disturbance transfer matrix function.

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Defining Lyapunov function as

$$\begin{aligned} v(\mathbf{q}(t)) &= \mathbf{q}^T(t) \mathbf{P} \mathbf{q}(t) \\ &\quad + 3 \int_0^t (\mathbf{y}^T(\tau) \mathbf{y}(\tau) - \gamma \mathbf{w}^T(\tau) \mathbf{w}(\tau)) d\tau > 0, \end{aligned} \quad (16)$$

where $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{P} \in \mathbb{R}^{n \times n}$, $\gamma > 0$, $\gamma \in \mathbb{R}$ and evaluating the derivative of $v(\mathbf{q}(t))$ with respect to t along a system trajectory, then it yields

$$\begin{aligned} \dot{v}(\mathbf{q}(t)) &= \dot{\mathbf{q}}^T(t) \mathbf{P} \mathbf{q}(t) + \mathbf{q}^T(t) \mathbf{P} \dot{\mathbf{q}}(t) \\ &\quad + 3 (\mathbf{y}^T(t) \mathbf{y}(t) - \gamma \mathbf{w}^T(t) \mathbf{w}(t)) < 0. \end{aligned} \quad (17)$$

Substituting (5) and (12) in (17), the next inequality is obtained:

$$\begin{aligned} \dot{v}(\mathbf{q}(t)) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \\ &\quad \times \left\{ \begin{aligned} &(\mathbf{H}_{jj} \mathbf{q}(t) + \mathbf{G}_j \mathbf{w}(t))^T \mathbf{P} \mathbf{q}(t) \\ &+ \mathbf{q}^T(t) \mathbf{P} (\mathbf{H}_{jj} \mathbf{q}(t) + \mathbf{G}_j \mathbf{w}(t)) \\ &+ (\mathbf{C} \mathbf{q}(t) + \mathbf{D} \mathbf{w}(t))^T (\mathbf{C} \mathbf{q}(t) + \mathbf{D} \mathbf{w}(t)) \\ &- \gamma \mathbf{w}^T(t) \mathbf{w}(t) \end{aligned} \right\} \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\
& \times \left\{ \begin{aligned} & \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right)^T \mathbf{P} \mathbf{q}(t) \\ & + \mathbf{q}^T(t) \mathbf{P} \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right) \\ & + (\mathbf{C} \mathbf{q}(t) + \mathbf{D} \mathbf{w}(t))^T (\mathbf{C} \mathbf{q}(t) + \mathbf{D} \mathbf{w}(t)) \\ & - \gamma \mathbf{w}^T(t) \mathbf{w}(t) \end{aligned} \right\} \\
& < 0
\end{aligned} \tag{18}$$

and with the notation

$$\mathbf{q}_w^T(t) = [\mathbf{q}^T(t) \quad \mathbf{w}^T(t)], \tag{19}$$

it yields

$$\begin{aligned}
\dot{v}(\mathbf{q}(t)) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \mathbf{q}_w^T(t) \\
&\quad \times (\mathbf{R}_{jj} + \mathbf{Q}) \mathbf{q}_w(t) \\
&+ 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\
&\quad \times \mathbf{q}_w^T(t) (\mathbf{R}_{jg} + \mathbf{Q}) \mathbf{q}_w(t) < 0,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\mathbf{R}_{jj} &= \begin{bmatrix} \mathbf{P} \mathbf{H}_{jj} + \mathbf{H}_{jj}^T \mathbf{P} & \mathbf{P} \mathbf{G}_j \\ * & -\gamma \mathbf{I}_r \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{C}^T \\ \mathbf{D}^T \end{bmatrix} [\mathbf{C} \quad \mathbf{D}], \\
\mathbf{R}_{jg} &= \begin{bmatrix} \mathbf{P} \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{P} & \mathbf{P} \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \\ * & -\gamma \mathbf{I}_r \end{bmatrix}.
\end{aligned} \tag{21}$$

Since (20) implies

$$\mathbf{R}_{jj} + \mathbf{Q} < 0, \quad \mathbf{R}_{jg} + \mathbf{Q} < 0, \tag{22}$$

for all $j \in \langle 1, 2, \dots, s \rangle$ and $j < g \leq s$, $i, j \in \langle 1, 2, \dots, s \rangle$, respectively, then using the Schur complement property, (22) can be written as

$$\begin{bmatrix} \mathbf{P} \mathbf{H}_{jj} + \mathbf{H}_{jj}^T \mathbf{P} & \mathbf{P} \mathbf{G}_j & \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{P} \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{P} & \mathbf{P} \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0. \tag{23}$$

Defining the transform matrix $\mathbf{L}$ as

$$\mathbf{L} = \text{diag} [\mathbf{P}^{-1} \quad \mathbf{I}_r \quad \mathbf{I}_m], \tag{24}$$

premultiplying the left-hand side and the right-hand side of (23) by $\mathbf{L}$, gives

$$\begin{bmatrix} \mathbf{H}_{jj} \mathbf{P}^{-1} + \mathbf{P}^{-1} \mathbf{H}_{jj}^T & \mathbf{G}_j & \mathbf{P}^{-1} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{P}^{-1} + \mathbf{P}^{-1} \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} & \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{P}^{-1} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{D}^T \\ * & * & -\mathbf{I}_m \end{bmatrix} < 0. \tag{26}$$

Thus, using the substitution $\mathbf{X} = \mathbf{P}^{-1}$ then (25) and (26) imply (14) and (15), respectively. This concludes the proof. $\square$

Remark 2. Another form of bounded real lemma can be obtained using Lyapunov function of the form

$$\begin{aligned}
v(\mathbf{q}(t)) &= \mathbf{q}^T(t) \mathbf{P} \mathbf{q}(t) \\
&+ 3\gamma^{-1} \int_0^t (\mathbf{y}^T(\tau) \mathbf{y}(\tau) - \gamma^2 \mathbf{w}^T(\tau) \mathbf{w}(\tau)) d\tau > 0,
\end{aligned} \tag{27}$$

(compare, e.g., [6, 12, 13, 18]), but the BRL forms implying from (27) result, in general, the higher H_∞ norm upper bound γ then using the presented conditions (14), (15).

Lemma 3 (enhanced bounded real lemma). *The closed-loop system (7), (5) is stable with the performance $\sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \|\mathbf{C}(s\mathbf{I} - \mathbf{H}_{jg})^{-1} \mathbf{G}_j + \mathbf{D}\|_\infty^2 \leq \gamma$ if for given $\delta > 0$, $\delta \in \mathbb{R}$, there exist positive definite matrices $\mathbf{V}, \mathbf{T} \in \mathbb{R}^{n \times n}$ and a positive scalar $\gamma \in \mathbb{R}$ such that*

$$\mathbf{V} = \mathbf{V}^T > 0, \quad \gamma > 0, \quad (28)$$

$$\begin{bmatrix} \mathbf{H}_{jj}\mathbf{V} + \mathbf{V}\mathbf{H}_{jj}^T & \mathbf{G}_j & \mathbf{T} - \delta\mathbf{V} + \mathbf{V}\mathbf{H}_{jj}^T & \mathbf{V}\mathbf{C}^T \\ * & -\gamma\mathbf{I}_r & \mathbf{G}_j^T & \mathbf{D}^T \\ * & * & -2\delta\mathbf{V} & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix} < 0, \quad (29)$$

$$\begin{bmatrix} \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2}\mathbf{V} + \mathbf{V}\frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} & \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{T} - \delta\mathbf{V} + \mathbf{V}\frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} & \mathbf{V}\mathbf{C}^T \\ * & -\gamma\mathbf{I}_r & \frac{(\mathbf{G}_j + \mathbf{G}_g)^T}{2} & \mathbf{D}^T \\ * & * & -2\delta\mathbf{V} & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix} < 0, \quad (30)$$

for all $j \in \langle 1, 2, \dots, s \rangle$ and $j < g \leq s$, $j, g \in \langle 1, 2, \dots, s \rangle$, respectively.

Proof. Since (12) implies

$$\begin{aligned} & \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) (\mathbf{H}_{jj}\mathbf{q}(t) + \mathbf{G}_j\mathbf{w}(t)) \\ & + 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \end{aligned}$$

$$\times \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right) - \dot{\mathbf{q}}(t)$$

$$= \mathbf{0},$$

(31)

then, with arbitrary regular symmetric square matrices $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{R}^{n \times n}$, it yields

$$\begin{pmatrix} \mathbf{q}^T(t) \mathbf{S}_1 + \dot{\mathbf{q}}^T(t) \mathbf{S}_2 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) (\mathbf{H}_{jj}\mathbf{q}(t) + \mathbf{G}_j\mathbf{w}(t)) \\ + 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right) - \dot{\mathbf{q}}(t) \end{pmatrix} = \mathbf{0}. \quad (32)$$

Thus, adding (32), as well as its transposition to (17), and substituting (5) into (17) give

$$0 > \dot{v}(\mathbf{q}(t))$$

$$= \dot{\mathbf{q}}^T(t) \mathbf{P}\mathbf{q}(t) + \mathbf{q}^T(t) \mathbf{P}\dot{\mathbf{q}}(t)$$

$$+ 3(\mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{w}(t))^T (\mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{w}(t))$$

$$- 3\gamma \mathbf{w}^T(t) \mathbf{w}(t)$$

$$+ 2(\mathbf{q}^T(t) \mathbf{S}_1 + \dot{\mathbf{q}}^T(t) \mathbf{S}_2)$$

$$\begin{aligned} & \times \left(\sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \right. \\ & \quad \left. \times (\mathbf{H}_{jj}\mathbf{q}(t) + \mathbf{G}_j\mathbf{w}(t)) - \dot{\mathbf{q}}(t) \right) \end{aligned}$$

$$+ 4(\mathbf{q}^T(t) \mathbf{S}_1 + \dot{\mathbf{q}}^T(t) \mathbf{S}_2)$$

$$\times \sum_{j=1}^{s-1} \sum_{g=j+1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t))$$

$$\times \left(\frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{q}(t) + \frac{\mathbf{G}_j + \mathbf{G}_g}{2} \mathbf{w}(t) \right),$$

(33)

and using the notation

$$\mathbf{q}_w^{\diamond T}(t) = [\mathbf{q}^T(t) \quad \mathbf{w}^T(t) \quad \mathbf{q}^T(t)] \quad (34)$$

can be obtained the following:

$$\dot{\mathbf{v}}(\mathbf{q}(t)) = \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_j(\boldsymbol{\theta}(t)) \times \mathbf{q}_w^{\diamond T}(t) (\mathbf{R}_{jj}^{\diamond} + \mathbf{Q}^{\diamond}) \mathbf{q}_w^{\diamond}(t) < 0, \quad (35)$$

where

$$\mathbf{R}_{jj}^{\diamond} = \begin{bmatrix} \mathbf{S}_1 \mathbf{H}_{jj} + \mathbf{H}_{jj}^T \mathbf{S}_1 & \mathbf{S}_1 \mathbf{G}_j & \mathbf{P} - \mathbf{S}_1 + \mathbf{H}_{jj}^T \mathbf{S}_2 \\ * & -\gamma \mathbf{I}_r & \mathbf{G}_j^T \mathbf{S}_2 \\ * & * & -2\mathbf{S}_2 \end{bmatrix},$$

$$\mathbf{R}_{jg}^{\diamond} = \begin{bmatrix} \mathbf{S}_1 \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{S}_1 & \mathbf{S}_1 \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{P} - \mathbf{S}_1 + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{S}_2 \\ * & -\gamma \mathbf{I}_r & \frac{(\mathbf{G}_j + \mathbf{G}_g)^T}{2} \mathbf{S}_2 \\ * & * & -2\mathbf{S}_2 \end{bmatrix}, \quad (36)$$

$$\mathbf{Q}^{\diamond} = \begin{bmatrix} \mathbf{C}^T \\ \mathbf{D}^T \\ \mathbf{0} \end{bmatrix} [\mathbf{C} \quad \mathbf{D} \quad \mathbf{0}].$$

Since (35) implies

$$\mathbf{R}_{jj}^{\diamond} + \mathbf{Q}^{\diamond} < 0, \quad \mathbf{R}_{jg}^{\diamond} + \mathbf{Q}^{\diamond} < 0, \quad (37)$$

for all $j \in \langle 1, 2, \dots, s \rangle$ and $j < g \leq s$, $i, j \in \langle 1, 2, \dots, s \rangle$, respectively, using the Schur complement property, (37) can be written as

$$\begin{bmatrix} \mathbf{S}_1 \mathbf{H}_{jj} + \mathbf{H}_{jj}^T \mathbf{S}_1 & \mathbf{S}_1 \mathbf{G}_j & \mathbf{P} - \mathbf{S}_1 + \mathbf{H}_{jj}^T \mathbf{S}_2 & \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{G}_j^T \mathbf{S}_2 & \mathbf{D}^T \\ * & * & -2\mathbf{S}_2 & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{S}_1 \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{S}_1 & \mathbf{S}_1 \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{P} - \mathbf{S}_1 + \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} \mathbf{S}_2 & \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \frac{(\mathbf{G}_j + \mathbf{G}_g)^T}{2} \mathbf{S}_2 & \mathbf{D}^T \\ * & * & -2\mathbf{S}_2 & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix}. \quad (38)$$

Since $\mathbf{S}_1, \mathbf{S}_2$ are supposed to be regular and symmetric, the transform matrix $\mathbf{L}^{\diamond}$ can be defined as

$$\mathbf{L}^{\diamond} = \text{diag} [\mathbf{S}_1^{-1} \quad \mathbf{I}_r \quad \mathbf{S}_2^{-1} \quad \mathbf{I}_m], \quad (39)$$

and pre-multiplying the left-hand and the right-hand side of (38) by $\mathbf{L}^{\diamond}$ gives

$$\begin{bmatrix} \mathbf{H}_{jj} \mathbf{S}_1^{-1} + \mathbf{S}_1^{-1} \mathbf{H}_{jj}^T & \mathbf{G}_j & \mathbf{T} - \mathbf{S}_2^{-1} + \mathbf{S}_1^{-1} \mathbf{H}_{jj}^T & \mathbf{S}_1^{-1} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \mathbf{G}_j^T & \mathbf{D}^T \\ * & * & -2\mathbf{S}_2^{-1} & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix}, \quad (40)$$

$$\begin{bmatrix} \Pi_{jg} & \frac{\mathbf{G}_j + \mathbf{G}_g}{2} & \mathbf{T} - \mathbf{S}_2^{-1} + \mathbf{S}_1^{-1} \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2} & \mathbf{S}_1^{-1} \mathbf{C}^T \\ * & -\gamma \mathbf{I}_r & \frac{(\mathbf{G}_j + \mathbf{G}_g)^T}{2} & \mathbf{D}^T \\ * & * & -2\mathbf{S}_2^{-1} & \mathbf{0} \\ * & * & * & -\mathbf{I}_m \end{bmatrix}, \quad (41)$$

$$\mathbf{T} = \mathbf{S}_1^{-1} \mathbf{P} \mathbf{S}_2^{-1}, \quad \Pi_{jg} = \frac{\mathbf{H}_{jg} + \mathbf{H}_{gj}}{2} \mathbf{S}_1^{-1} + \mathbf{S}_1^{-1} \frac{(\mathbf{H}_{jg} + \mathbf{H}_{gj})^T}{2}. \quad (42)$$

Thus, using the notation

$$\mathbf{S}_2^{-1} = \delta \mathbf{V}, \quad \mathbf{S}_1^{-1} = \mathbf{V}, \quad (43)$$

(40), (41) imply (29), (30). This concludes the proof. $\square$

This enhanced form of the bounded real lemma eliminates products of a Lyapunov matrix $\mathbf{T}$ and the system matrix parameters $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}$, and $\mathbf{D}$ in the LMI stability conditions. When used in the synthesis of controllers or observers for TS systems (that are, evidently, the systems with polytopic uncertainties), the enhanced form of BRL gives solutions that are less conservative than ones given by the standard form of BRL. In that sense, the enhanced form of BRL can be preferred in TS systems analysis and design [13] giving less conservative equivalency to the synthesis based on the parameter dependent Lyapunov functions principle [4].

3. Pairwise Distributed Principle in Control Design

The main property of the pairwise distributable structure of TS large-scale systems is given by the next two lemmas.

Lemma 4. *Let the system (1), (2) is structured in p subsystems in such way that, for $j = 1, 2, \dots, s$, $r \geq 3$, $i, l = 1, 2, \dots, p$,*

$$\mathbf{A}_j = \text{blk} [\mathbf{A}_{jil}], \quad \mathbf{C} = \text{blk} [\mathbf{C}_{il}], \quad (44)$$

$$\mathbf{B}_j = \text{diag} [\mathbf{B}_{ji}], \quad \mathbf{D} = \mathbf{0},$$

$$\begin{aligned} \dot{\mathbf{q}}_h(t) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \\ &\times \left(\mathbf{A}_{jhh} \mathbf{q}_h(t) + \sum_{l=1, l \neq h}^p (\mathbf{A}_{jhl} \mathbf{q}_l(t) + \mathbf{B}_{jh} \mathbf{u}_h(t)) \right), \end{aligned} \quad (45)$$

$$\mathbf{y}_h(t) = \mathbf{C}_{hh} \mathbf{q}_h(t) + \sum_{l=1, l \neq h}^p \mathbf{C}_{hl} \mathbf{q}_l(t), \quad (46)$$

where $\mathbf{q}_h(t) \in \mathbb{R}^{n_h}$, $\mathbf{u}_h(t) \in \mathbb{R}^{r_h}$, $\mathbf{y}_h(t) \in \mathbb{R}^{m_h}$, $\mathbf{A}_{jhl} \in \mathbb{R}^{n_h \times n_l}$, $\mathbf{B}_{jh} \in \mathbb{R}^{r_h \times n_h}$, and $\mathbf{C}_{hl} \in \mathbb{R}^{m_h \times n_h}$, respectively, and $n = \sum_{h=1}^p n_h$, $r = \sum_{h=1}^p r_h$, and $m = \sum_{h=1}^p m_h$.

Then the h th unforced subsystem pair in unforced system (1), (2), and (44) is stable if there exists a set of symmetric matrices

$$\mathbf{P}_{hk}^\circ = \begin{bmatrix} \mathbf{P}_h^k & \mathbf{P}_{hk}^k \\ \mathbf{P}_{kh} & \mathbf{P}_k^k \end{bmatrix}, \quad h = 1, 2, \dots, p-1, \quad (47)$$

$$k = h+1, \quad h+2, \dots, p,$$

such that

$$\begin{aligned} &\sum_{h=1}^{p-1} \sum_{k=h+1}^p \left(\dot{\mathbf{q}}_{hk}^T(t) \begin{bmatrix} \mathbf{P}_h^k & \mathbf{P}_{hk}^k \\ \mathbf{P}_{kh} & \mathbf{P}_k^k \end{bmatrix} \mathbf{q}_{hk}(t) \right. \\ &\quad \left. + \mathbf{q}_{hk}^T(t) \begin{bmatrix} \mathbf{P}_h^k & \mathbf{P}_{hk}^k \\ \mathbf{P}_{kh} & \mathbf{P}_k^k \end{bmatrix} \dot{\mathbf{q}}_{hk}(t) \right) < 0, \end{aligned} \quad (48)$$

where

$$\dot{\mathbf{q}}_{hk}^T(t) = [\dot{\mathbf{q}}_h^T(t) \quad \dot{\mathbf{q}}_k^T(t)], \quad (49)$$

$$\begin{aligned} \dot{\mathbf{q}}_{hk}(t) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \\ &\times \left(\begin{bmatrix} \mathbf{A}_{jhh} & \mathbf{A}_{jhk} \\ \mathbf{A}_{jkh} & \mathbf{A}_{jkk} \end{bmatrix} \mathbf{q}_{hk}(t) + \sum_{l=1, l \neq h,k}^p \begin{bmatrix} \mathbf{A}_{jhl} \\ \mathbf{A}_{jkl} \end{bmatrix} \mathbf{q}_l(t) \right). \end{aligned} \quad (50)$$

Proof. Defining Lyapunov function as

$$v(\mathbf{q}(t)) = \mathbf{q}^T(t) \mathbf{P} \mathbf{q}(t) > 0, \quad (51)$$

where $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{P} \in \mathbb{R}^{n \times n}$, then the time derivative of $v(\mathbf{q}(t))$ along a solution of the system (1), (2) is

$$\dot{v}(\mathbf{q}(t)) = \dot{\mathbf{q}}^T(t) \mathbf{P} \mathbf{q}(t) + \mathbf{q}^T(t) \dot{\mathbf{P}} \mathbf{q}(t) < 0. \quad (52)$$

Considering $\mathbf{P}$ of the next form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \dots & \mathbf{P}_{1p} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \dots & \mathbf{P}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{p1} & \mathbf{P}_{p2} & \dots & \mathbf{P}_{pp} \end{bmatrix}, \quad (53)$$

$$\mathbf{P}_{hh} = \sum_{l=1, l \neq h}^p \mathbf{P}_{hl}^l, \quad h = 1, 2, \dots, p,$$

then the next separation is possible

$$\begin{aligned} \mathbf{P} &= \left\{ \left(\begin{bmatrix} \mathbf{P}_1^2 & \mathbf{P}_{12} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_2^1 & \mathbf{0} & \dots & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \right. \right. \\ &\quad \left. \left. + \dots + \begin{bmatrix} \mathbf{P}_1^p & \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_{1p} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ & & \ddots & & \\ \mathbf{P}_{p1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_p^1 \end{bmatrix} \right) \right. \\ &\quad \left. + \dots + \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & & & \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_{p-1}^p & \mathbf{P}_{p-1,p}^{p-1,p} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_{p,p-1} & \mathbf{P}_p^{p-1,p} \end{bmatrix} \right\}. \end{aligned} \quad (54)$$

Using (49) and writing (45) as

$$\begin{aligned} \dot{\mathbf{q}}_{hk}(t) &= \sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \\ &\times \left(\begin{bmatrix} \mathbf{A}_{jhh} & \mathbf{A}_{jhk} \\ \mathbf{A}_{jkh} & \mathbf{A}_{jkk} \end{bmatrix} \mathbf{q}_{hk}(t) + \sum_{l=1, l \neq h,k}^p \begin{bmatrix} \mathbf{A}_{jhl} \\ \mathbf{A}_{jkl} \end{bmatrix} \mathbf{q}_l(t) \right. \\ &\quad \left. + \begin{bmatrix} \mathbf{B}_{jh} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{jk} \end{bmatrix} \begin{bmatrix} \mathbf{u}_h(t) \\ \mathbf{u}_k(t) \end{bmatrix} \right) \end{aligned} \quad (55)$$

and considering that for an unforced regime $\mathbf{u}_l(t) = \mathbf{0}$, $l = 1, 2, \dots, p$, then (55) implies (50). Subsequently, (1), (52), and (54) give

$$\sum_{j=1}^s \mu_j(\boldsymbol{\theta}(t)) \sum_{h=1}^{p-1} \sum_{k=h+1}^p \left(\begin{aligned} &\dot{\mathbf{q}}_{hk}^T(t) \begin{bmatrix} \mathbf{P}_h^k & \mathbf{P}_{hk}^h \\ \mathbf{P}_{kh} & \mathbf{P}_k^h \end{bmatrix} \mathbf{q}_{hk}(t) \\ &+ \mathbf{q}_{hk}^T(t) \begin{bmatrix} \mathbf{P}_h^k & \mathbf{P}_{hk}^h \\ \mathbf{P}_{kh} & \mathbf{P}_k^h \end{bmatrix} \dot{\mathbf{q}}_{hk}(t) \end{aligned} \right) < 0, \quad (56)$$

and the inequality (56) implies (48). This concludes the proof. $\square$

Lemma 5. System (1), (2), (44) with the pairwise distributed control takes the form

$$\begin{aligned} \dot{\mathbf{q}}_{hk}(t) &= \sum_{j=1}^s \sum_{g=1}^s \mu_j(\boldsymbol{\theta}(t)) \mu_g(\boldsymbol{\theta}(t)) \\ &\quad \times \left((\mathbf{A}_{jkh}^\circ - \mathbf{B}_{jkh}^\circ \mathbf{K}_{ghk}^\circ) \mathbf{q}_{hk}(t) + \boldsymbol{\omega}_{jghk}^\circ(t) \right), \end{aligned} \quad (57)$$

$$\mathbf{y}_{hk}(t) = \mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \sum_{\substack{l=1 \\ l \neq h,k}}^p \mathbf{C}_{hk}^{l\circ} \mathbf{q}_l(t) + \mathbf{0} \boldsymbol{\omega}_{hk}(t), \quad (58)$$

where

$$\mathbf{A}_{jkh}^\circ = \begin{bmatrix} \mathbf{A}_{jhh} & \mathbf{A}_{jkh} \\ \mathbf{A}_{jkh} & \mathbf{A}_{jkk} \end{bmatrix}, \quad \mathbf{B}_{jkh}^\circ = \begin{bmatrix} \mathbf{B}_{jhh} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{jkk} \end{bmatrix}, \quad (59)$$

$$\begin{aligned} &\mathbf{K}_{ghk}^\circ \begin{bmatrix} \mathbf{K}_{gh}^k & \mathbf{K}_{ghk}^h \\ \mathbf{K}_{gkh} & \mathbf{K}_{gk}^h \end{bmatrix}, \\ \boldsymbol{\omega}_{jghk}^\circ(t) &= \sum_{\substack{l=1 \\ l \neq h,k}}^p \mathbf{A}_{jkh}^{l\circ} \mathbf{q}_l(t) + \mathbf{B}_{jkh}^\circ \boldsymbol{\omega}_{ghk}(t), \end{aligned} \quad (60)$$

$$\mathbf{A}_{jkh}^{l\circ} = \begin{bmatrix} \mathbf{A}_{jhl} \\ \mathbf{A}_{jkl} \end{bmatrix}, \quad \boldsymbol{\omega}_{ghk}(t) = \sum_{\substack{l=1 \\ l \neq h,k}}^p \begin{bmatrix} \mathbf{u}_{gh}^l(t) \\ \mathbf{u}_{gk}^l(t) \end{bmatrix}, \quad (61)$$

$$\mathbf{u}_{gh}^l(t) = \begin{bmatrix} \mathbf{K}_{gh}^l & \mathbf{K}_{ghl} \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_l(t) \end{bmatrix},$$

$$\begin{aligned} \mathbf{C}_{hk}^\circ &= \begin{bmatrix} \mathbf{C}_h^k & \mathbf{C}_{hk}^h \\ \mathbf{C}_{kh} & \mathbf{C}_k^h \end{bmatrix}, \quad \mathbf{C}_{hk}^{l\circ} = \begin{bmatrix} \mathbf{C}_{hl} \\ \mathbf{C}_{kl} \end{bmatrix}, \\ \mathbf{C}_{hh} &= \sum_{\substack{l=1 \\ l \neq h}}^p \mathbf{C}_h^l. \end{aligned} \quad (62)$$

Proof. Considering in (6) the same structure of $\mathbf{K}_g$, $g = 1, 2, \dots, s$, as is defined for $\mathbf{P}$ in (53), that is,

$$\mathbf{K}_g = \begin{bmatrix} \mathbf{K}_{g11} & \mathbf{K}_{g12} & \cdots & \mathbf{K}_{g1p} \\ \mathbf{K}_{g21} & \mathbf{K}_{g22} & \cdots & \mathbf{K}_{g2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{gp1} & \mathbf{K}_{gp2} & \cdots & \mathbf{K}_{gpp} \end{bmatrix}, \quad \mathbf{K}_{ghh} = \sum_{\substack{l=1 \\ l \neq h}}^p \mathbf{K}_{gh}^l, \quad (63)$$

then the control law $\mathbf{u}_h(t)$ takes the form

$$\mathbf{u}_h(t) = - \sum_{g=1}^s \mu_g(\boldsymbol{\theta}(t)) \left(\mathbf{K}_{ghh} \mathbf{q}_h(t) + \sum_{\substack{l=1 \\ l \neq h}}^p \mathbf{K}_{ghl} \mathbf{q}_l(t) \right), \quad (64)$$

where $\mathbf{K}_{ghl}(t)$, $h, l = 1, 2, \dots, p$, are non-zero gain matrices.

Exploiting the main diagonal block property (63), then

$$\begin{aligned} \mathbf{u}_h(t) &= - \sum_{g=1}^s \mu_g(\boldsymbol{\theta}(t)) \\ &\quad \times \sum_{\substack{l=1 \\ l \neq h}}^p \begin{bmatrix} \mathbf{K}_{gh}^l & \mathbf{K}_{ghl} \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_l(t) \end{bmatrix} \\ &= - \sum_{g=1}^s \mu_g(\boldsymbol{\theta}(t)) \left(\begin{aligned} &\begin{bmatrix} \mathbf{K}_{gh}^k & \mathbf{K}_{ghk} \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_k(t) \end{bmatrix} \\ &+ \sum_{\substack{l=1 \\ l \neq h,k}}^p \begin{bmatrix} \mathbf{K}_{gh}^l & \mathbf{K}_{ghl} \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_l(t) \end{bmatrix} \end{aligned} \right) \\ &= - \sum_{g=1}^s \mu_g(\boldsymbol{\theta}(t)) \left(\mathbf{u}_{gh}^k(t) + \sum_{\substack{l=1 \\ l \neq h,k}}^p \mathbf{u}_{gh}^l(t) \right), \end{aligned} \quad (65)$$

where for $l = 1, 2, \dots, p$, $l \neq h, k$,

$$\mathbf{u}_{gh}^l(t) = \begin{bmatrix} \mathbf{K}_{gh}^l & \mathbf{K}_{ghl} \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_l(t) \end{bmatrix}. \quad (66)$$

Defining for $h = 1, 2, \dots, p-1$, $k = h+1, h+2, \dots, p$ and with respect to the notations (59) that

$$\begin{aligned} \begin{bmatrix} \mathbf{u}_{gh}^k(t) \\ \mathbf{u}_{gk}^h(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{K}_{gh}^k & \mathbf{K}_{ghk} \\ \mathbf{K}_{gkh} & \mathbf{K}_{gk}^h \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_k(t) \end{bmatrix} \\ &= \mathbf{K}_{ghk}^\circ \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_k(t) \end{bmatrix} \end{aligned} \quad (67)$$

and combining (64) for h and k , is obtained the following:

$$\begin{aligned} \begin{bmatrix} \mathbf{u}_h(t) \\ \mathbf{u}_k(t) \end{bmatrix} &= -\sum_{g=1}^s \mu_g(\theta(t)) \\ &\times \left(\begin{bmatrix} \mathbf{K}_{gh}^k & \mathbf{K}_{ghk}^h \\ \mathbf{K}_{ghk}^h & \mathbf{K}_{gk}^h \end{bmatrix} \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_k(t) \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} \sum_{l=1, l \neq h,k}^p [\mathbf{K}_{gh}^l & \mathbf{K}_{ghl}] \begin{bmatrix} \mathbf{q}_h(t) \\ \mathbf{q}_l(t) \end{bmatrix} \\ \sum_{l=1, l \neq h,k}^p [\mathbf{K}_{gk}^l & \mathbf{K}_{gkl}] \begin{bmatrix} \mathbf{q}_k(t) \\ \mathbf{q}_l(t) \end{bmatrix} \end{bmatrix} \right), \end{aligned} \quad (68)$$

$$\begin{aligned} \begin{bmatrix} \mathbf{u}_h(t) \\ \mathbf{u}_k(t) \end{bmatrix} &= -\sum_{g=1}^s \mu_g(\theta(t)) \\ &\times \left(\begin{bmatrix} \mathbf{u}_{gh}^k(t) \\ \mathbf{u}_{gk}^h(t) \end{bmatrix} + \sum_{l=1, l \neq h,k}^p \begin{bmatrix} \mathbf{u}_{gh}^l(t) \\ \mathbf{u}_{gk}^l(t) \end{bmatrix} \right), \end{aligned} \quad (69)$$

respectively. Then, substituting (69) in (55) gives

$$\dot{\mathbf{q}}_{hk}(t) = \sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) (\mathbf{A}_{jghkc}^\circ \mathbf{q}_{hk}(t) + \omega_{jghk}^\circ(t)), \quad (70)$$

where

$$\mathbf{A}_{jghkc}^\circ = \begin{bmatrix} \mathbf{A}_{jhh} & \mathbf{A}_{jkh} \\ \mathbf{A}_{jkh} & \mathbf{A}_{jkk} \end{bmatrix} - \begin{bmatrix} \mathbf{B}_{jh} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{jk} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{gh}^k & \mathbf{K}_{ghk}^h \\ \mathbf{K}_{ghk}^h & \mathbf{K}_{gk}^h \end{bmatrix}, \quad (71)$$

$$\omega_{jghk}^\circ(t) = \sum_{l=1, l \neq h,k}^p \begin{bmatrix} \mathbf{B}_{jh} \mathbf{u}_{gh}^l(t) + \mathbf{A}_{jhl} \mathbf{q}_l(t) \\ \mathbf{B}_{jk} \mathbf{u}_{gk}^l(t) + \mathbf{A}_{jkl} \mathbf{q}_l(t) \end{bmatrix}. \quad (72)$$

Using (59) and denoting

$$\mathbf{A}_{jghkj}^\circ = \mathbf{A}_{jghk}^\circ - \mathbf{B}_{jghk}^\circ \mathbf{K}_{ghk}^\circ, \quad (73)$$

then (71) implies (57).

Rewriting (72) in the form

$$\omega_{jghk}^\circ(t) = \sum_{l=1, l \neq h,k}^p \left(\mathbf{B}_{jghk}^\circ \begin{bmatrix} \mathbf{u}_{gh}^l(t) \\ \mathbf{u}_{gk}^l(t) \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{jhl} \\ \mathbf{A}_{jkl} \end{bmatrix} \mathbf{q}_l(t) \right) \quad (74)$$

and using (61), then (74) implies (60).

Finally, using (62), it can be written as

$$\mathbf{y}(t) = \sum_{h=1}^{p-1} \sum_{k=h+1}^p \left(\mathbf{C}_{hkh}^\circ \mathbf{q}_{hk}(t) + \sum_{l=1, l \neq h}^p \mathbf{C}_{hl}^l \mathbf{q}_l(t) \right), \quad (75)$$

$$\mathbf{y}_{hk}(t) = \mathbf{C}_{hkh}^\circ \mathbf{q}_{hk}(t) + \sum_{l=1, l \neq h}^p \mathbf{C}_{hl}^l \mathbf{q}_l(t), \quad (76)$$

and (76) implies (58). This concludes the proof. $\square$

4. Pairwise Control Law Parameter Design

The design conditions, formulated as the set of LMIs, imply from the next theorems.

Theorem 6. *Controlled subsystem pair (57), (58) in the system (1), (3) is stable with performances $\sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) \|\mathbf{C}_{hkh}^\circ (s\mathbf{I} - \mathbf{H}_{jghk}^\circ)^{-1} \mathbf{B}_{jghk}^\circ + \mathbf{D}_{jghk}^\circ\|_\infty^2 \leq \gamma_{hk}$, if there exist a symmetric positive definite matrix $\mathbf{X}_{hk}^\circ \in \mathbb{R}^{(n_h+n_k) \times (n_h+n_k)}$, matrices $\mathbf{Y}_{jghk}^\circ \in \mathbb{R}^{(r_h+r_k) \times (n_h+n_k)}$, and positive scalars $\gamma_{hk}, \varepsilon_{hkl} \in \mathbb{R}$ such that*

$$\mathbf{X}_{hk}^\circ = \mathbf{X}_{hk}^{\circ T} > 0, \quad \varepsilon_{hkl} > 0, \quad \gamma_{hk} > 0, \quad (77)$$

for $h, l = 1, \dots, p, l \neq h, k, h < k \leq p$,

$$\begin{bmatrix} \Phi_{jjhk}^\circ & \mathbf{A}_{jghk}^{1^\circ} & \cdots & \mathbf{A}_{jghk}^{p^\circ} & \mathbf{B}_{jghk}^\circ & \mathbf{X}_{hk}^\circ \mathbf{C}_{hkh}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{hkh}^{1^\circ T} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{C}_{hkh}^{p^\circ T} \\ * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{0} \\ * & * & \cdots & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0, \quad (78)$$

for $j = 1, 2, \dots, s, h = 1, 2, \dots, p-1, k = h+1, h+2, \dots, p, l = 1, 2, \dots, p, l \neq h, k$,

$$\begin{bmatrix} \Phi_{jghk}^\circ & \frac{\mathbf{A}_{jghk}^{1^\circ} + \mathbf{A}_{ghk}^{1^\circ}}{2} & \cdots & \frac{\mathbf{A}_{jghk}^{p^\circ} + \mathbf{A}_{ghk}^{p^\circ}}{2} & \frac{\mathbf{B}_{jghk}^\circ + \mathbf{B}_{ghk}^\circ}{2} & \mathbf{X}_{hk}^\circ \mathbf{C}_{hkh}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{hkh}^{1^\circ T} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{C}_{hkh}^{p^\circ T} \\ * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{0} \\ * & * & \cdots & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0, \quad (79)$$

for $j = 1, 2, \dots, s-1, j = 2, 3, \dots, s, h = 1, 2, \dots, p-1, k = h+1, h+2, \dots, p, l = 1, 2, \dots, p, l \neq h, k, j < g \leq s$, with $\mathbf{A}_{jkh}^\circ, \mathbf{B}_{jkh}^\circ, \mathbf{A}_{jkh}^{l^\circ}, \mathbf{C}_{hk}^\circ$, and $\mathbf{C}_{hk}^{l^\circ}$ defined in (59), (61), and (62), respectively,

$$\Phi_{jjhk}^\circ = \mathbf{A}_{jkh}^\circ \mathbf{X}_{hk}^\circ + \mathbf{X}_{hk}^\circ \mathbf{A}_{jkh}^{\circ T} - \mathbf{B}_{jkh}^\circ \mathbf{Y}_{jkh}^{\circ T} - \mathbf{Y}_{jkh}^{\circ T} \mathbf{B}_{jkh}^{\circ T},$$

$$\begin{aligned} \Phi_{jghk}^\circ &= \frac{\mathbf{A}_{jkh}^\circ + \mathbf{A}_{ghk}^\circ}{2} \mathbf{X}_{hk}^\circ + \mathbf{X}_{hk}^\circ \frac{(\mathbf{A}_{jkh}^\circ + \mathbf{A}_{ghk}^\circ)^T}{2} \\ &\quad - \frac{\mathbf{B}_{jkh}^\circ \mathbf{Y}_{ghk}^\circ + \mathbf{B}_{ghk}^\circ \mathbf{Y}_{jkh}^\circ}{2} \\ &\quad - \frac{(\mathbf{B}_{jkh}^\circ \mathbf{Y}_{ghk}^\circ + \mathbf{B}_{ghk}^\circ \mathbf{Y}_{jkh}^\circ)^T}{2}, \end{aligned} \quad (80)$$

and where, for given $h, k, \mathbf{A}_{jkh}^{h^\circ}, \mathbf{A}_{jkh}^{k^\circ}$, and $\mathbf{C}_{hk}^{h^\circ}, \mathbf{C}_{hk}^{k^\circ}$ are not included into the structure of (78).

When the above conditions hold, the gain matrices $\mathbf{K}_{jkh}^\circ$ are given by

$$\mathbf{K}_{jkh}^\circ = \mathbf{Y}_{jkh}^\circ \mathbf{X}_{hk}^{\circ-1}. \quad (81)$$

Proof. Considering $\sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) \omega_{jghk}^\circ(t)$, with $\omega_{jghk}^\circ(t)$ given in (60), as a generalized disturbance acting on the subsystem pair (57), (58), and introducing the notations

$$\begin{aligned} \mathbf{G}_{jkh}^\circ &= \left[\left\{ \mathbf{A}_{jkh}^{l^\circ} \right\}_{l=1, l \neq h, k}^P \quad \mathbf{B}_{jkh}^\circ \right], \\ \omega_{hk}^{\circ T} &= \left[\left\{ \mathbf{q}_l^T \right\}_{l=1, l \neq h, k}^P \quad \omega_{hk}^T(t) \right], \end{aligned} \quad (82)$$

(57) takes the form

$$\begin{aligned} \dot{\mathbf{q}}_{hk}(t) &= \sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) \\ &\quad \times \left((\mathbf{A}_{jkh}^\circ - \mathbf{B}_{jkh}^\circ \mathbf{K}_{ghk}^\circ) \mathbf{q}_{hk}(t) + \mathbf{G}_{jkh}^\circ \omega_{hk}^\circ(t) \right). \end{aligned} \quad (83)$$

Analogously, (58) can be rewritten as

$$\mathbf{y}_{hk}(t) = \mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^\circ \omega_{hk}^{l^\circ}, \quad (84)$$

where

$$\mathbf{D}_{hk}^\circ = \left[\left\{ \mathbf{C}_{hk}^{l^\circ} \right\}_{l=1, l \neq h, k}^P \quad \mathbf{0} \right]. \quad (85)$$

Since (56) also gives

$$\begin{aligned} &\sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) \\ &\quad \times \sum_{h=1}^{p-1} \sum_{k=h+1}^p \left(\mathbf{q}_{hk}^T(t) \mathbf{P}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{q}_{hk}^T(t) \mathbf{P}_{hk}^\circ \dot{\mathbf{q}}_{hk}(t) \right) < 0, \end{aligned} \quad (86)$$

defining the matrices

$$\mathbf{\Gamma}_{hk}^\circ = \text{diag} \left[\left\{ \varepsilon_{hkl} \mathbf{I}_{n_l} \right\}_{l=1, l \neq h, k}^P \quad \gamma_{hk} \mathbf{I}_{(r_h+r_k)} \right], \quad (87)$$

(18) gives

$$\begin{aligned} 0 > \dot{v}(\mathbf{q}_{hk}(t)) &= \sum_{j=1}^s \sum_{h=1}^{p-1} \sum_{k=h+1}^p \mu_j \mu_k \times \left\{ \begin{aligned} &(\mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^\circ \omega_{hk}^\circ(t))^T (\mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^\circ \omega_{hk}^\circ(t)) \\ &- \omega_{hk}^{\circ T}(t) \mathbf{\Gamma}_{hk}^\circ \omega_{hk}^\circ(t) \\ &+ \mathbf{q}_{hk}^T(t) \mathbf{P}_{hk}^\circ (\mathbf{H}_{jjhk}^\circ \mathbf{q}_{hk}(t) + \mathbf{G}_{jkh}^\circ \omega_{hk}^\circ(t)) \\ &+ (\mathbf{H}_{jjhk}^\circ \mathbf{q}_{hk}(t) + \mathbf{G}_{jkh}^\circ \omega_{hk}^\circ(t))^T \mathbf{P}_{hk}^\circ \mathbf{q}_{hk}(t) \end{aligned} \right\} \\ &+ 2 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \sum_{h=1}^{p-1} \sum_{k=h+1}^p \mu_j \mu_g \left\{ \begin{aligned} &(\mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^\circ \omega_{hk}^\circ(t))^T (\mathbf{C}_{hk}^\circ \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^\circ \omega_{hk}^\circ(t)) \\ &- \omega_{hk}^{\circ T}(t) \mathbf{\Gamma}_{hk}^\circ \omega_{hk}^\circ(t) \\ &+ \mathbf{q}_{hk}^T(t) \mathbf{P}_{hk}^\circ \left(\frac{\mathbf{H}_{jghk}^\circ + \mathbf{H}_{gjhk}^\circ}{2} \mathbf{q}_{hk}(t) + \frac{\mathbf{G}_{jkh}^\circ + \mathbf{G}_{ghk}^\circ}{2} \omega_{hk}^\circ(t) \right) \\ &+ \left(\frac{\mathbf{H}_{jghk}^\circ + \mathbf{H}_{gjhk}^\circ}{2} \mathbf{q}_{hk}(t) + \frac{\mathbf{G}_{jkh}^\circ + \mathbf{G}_{ghk}^\circ}{2} \omega_{hk}^\circ(t) \right)^T \mathbf{P}_{hk}^\circ \mathbf{q}_{hk}(t) \end{aligned} \right\}, \end{aligned} \quad (88)$$

where, for simplicity, the argument of membership functions was omitted.

Therefore, inserting appropriate into (14), (15), it is obtained the following:

$$\begin{bmatrix} \Psi_{jjhk}^\circ & \mathbf{A}_{jkh}^{1^\circ} & \cdots & \mathbf{A}_{jkh}^{p^\circ} & \mathbf{B}_{jkh}^\circ & \mathbf{X}_{hk}^\circ \mathbf{C}_{hk}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_l} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{hk}^{1^\circ T} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{C}_{hk}^{p^\circ T} \\ * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{0} \\ * & * & \cdots & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0, \quad (89)$$

$$\begin{bmatrix} \Psi_{jghk}^\circ & \frac{\mathbf{A}_{jkh}^{1^\circ} + \mathbf{A}_{ghk}^{1^\circ}}{2} & \cdots & \frac{\mathbf{A}_{jkh}^{p^\circ} + \mathbf{A}_{ghk}^{p^\circ}}{2} & \frac{\mathbf{B}_{jkh}^\circ + \mathbf{B}_{ghk}^\circ}{2} & \mathbf{X}_{hk}^\circ \mathbf{C}_{hk}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_l} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{hk}^{1^\circ T} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{C}_{hk}^{p^\circ T} \\ * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{0} \\ * & * & \cdots & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Psi_{jjhk}^\circ &= \mathbf{A}_{jkh}^\circ \mathbf{X}_{hk}^\circ + \mathbf{X}_{hk}^\circ \mathbf{A}_{jkh}^{\circ T} - \mathbf{B}_{jkh}^\circ \mathbf{K}_{jkh}^{\circ T} \mathbf{X}_{hk}^\circ - \mathbf{X}_{hk}^\circ \mathbf{K}_{jkh}^{\circ T} \mathbf{B}_{jkh}^{\circ T}, \\ \Psi_{jghk}^\circ &= \frac{\mathbf{A}_{jkh}^\circ + \mathbf{A}_{ghk}^\circ}{2} \mathbf{X}_{hk}^\circ + \mathbf{X}_{hk}^\circ \frac{(\mathbf{A}_{jkh}^\circ + \mathbf{A}_{ghk}^\circ)^T}{2} \\ &\quad - \frac{\mathbf{B}_{jkh}^\circ \mathbf{K}_{ghk}^\circ + \mathbf{B}_{ghk}^\circ \mathbf{K}_{jkh}^\circ}{2} \mathbf{X}_{hk}^\circ \\ &\quad - \mathbf{X}_{hk}^\circ \frac{(\mathbf{B}_{jkh}^\circ \mathbf{K}_{ghk}^\circ + \mathbf{B}_{ghk}^\circ \mathbf{K}_{jkh}^\circ)^T}{2}. \end{aligned} \quad (90)$$

Thus, with the substitutions

$$\mathbf{Y}_{jkh}^\circ = \mathbf{K}_{jkh}^\circ \mathbf{X}_{hk}^\circ, \quad \mathbf{Y}_{ghk}^\circ = \mathbf{K}_{ghk}^\circ \mathbf{X}_{hk}^\circ, \quad (91)$$

(89)-(90) imply (78)-(79). This concludes the proof. $\square$

Theorem 7. Controlled subsystem pair (57), (58) in the system (1), (3) is stable with performances $\sum_{j=1}^s \sum_{g=1}^s \mu_j(\theta(t)) \mu_g(\theta(t)) \|\mathbf{C}_{hk}^\circ (s\mathbf{I} - \mathbf{H}_{jghk}^\circ)^{-1} \mathbf{B}_{jkh}^\circ + \mathbf{D}_{jghk}^\circ\|_\infty^2 \leq \gamma_{hk}$, if for given $\delta > 0$, $\delta \in \mathbb{R}$ there exist symmetric positive definite matrices $\mathbf{V}_{hk}^\circ, \mathbf{T}_{hk}^\circ \in \mathbb{R}^{(n_h+n_k) \times (n_h+n_k)}$, matrices $\mathbf{W}_{jkh}^\circ \in \mathbb{R}^{(r_h+r_k) \times (n_h+n_k)}$, and positive scalars $\gamma_{hk}, \varepsilon_{hkl} \in \mathbb{R}$ such that

$$\mathbf{V}_{hk}^\circ = \mathbf{V}_{hk}^{\circ T} > 0, \quad \mathbf{T}_{hk}^\circ = \mathbf{T}_{hk}^{\circ T} > 0, \quad (92)$$

$$\varepsilon_{hkl} > 0, \quad \gamma_{hk} > 0,$$

for $h, l = 1, \dots, p, l \neq h, k, h < k \leq p$,

$$\begin{bmatrix} \Sigma_{jjhk}^\circ & \mathbf{A}_{jkh}^{1^\circ} & \cdots & \mathbf{A}_{jkh}^{p^\circ} & \mathbf{B}_{jkh}^\circ & \Gamma_{jjhk}^\circ & \mathbf{V}_{hk}^\circ \mathbf{C}_{hk}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_l} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jkh}^{1^\circ T} & \mathbf{C}_{hk}^{1^\circ T} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{A}_{jkh}^{p^\circ T} & \mathbf{C}_{hk}^{p^\circ T} \\ * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{B}_{jkh}^{\circ T} & \mathbf{0} \\ * & * & \cdots & * & * & -2\delta \mathbf{V}_{hk}^\circ & \mathbf{0} \\ * & * & \cdots & * & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0, \quad (93)$$

for $j = 1, 2, \dots, s, h = 1, 2, \dots, p-1, k = h+1, h+2, \dots, p,$
 $l = 1, 2, \dots, p, l \neq h, k,$

$$\begin{bmatrix} \Sigma_{jghk}^{\circ} & \frac{\mathbf{A}_{jghk}^{1^{\circ}} + \mathbf{A}_{ghk}^{1^{\circ}}}{2} & \dots & \frac{\mathbf{A}_{jghk}^{p^{\circ}} + \mathbf{A}_{ghk}^{p^{\circ}}}{2} & \frac{\mathbf{B}_{jghk}^{\circ} + \mathbf{B}_{ghk}^{\circ}}{2} & \Gamma_{jghk}^{\circ} & \mathbf{V}_{hk}^{\circ} \mathbf{C}_{hk}^{\circ T} \\ * & -\varepsilon_{hkl} \mathbf{I}_{n_l} & \dots & \mathbf{0} & \mathbf{0} & \frac{(\mathbf{A}_{jghk}^{1^{\circ}} + \mathbf{A}_{ghk}^{1^{\circ}})^T}{2} & \mathbf{C}_{hk}^{1^{\circ T}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \frac{(\mathbf{A}_{jghk}^{p^{\circ}} + \mathbf{A}_{ghk}^{p^{\circ}})^T}{2} & \mathbf{C}_{hk}^{p^{\circ T}} \\ * & * & \dots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \frac{(\mathbf{B}_{jghk}^{\circ} + \mathbf{B}_{ghk}^{\circ})^T}{2} & \mathbf{0} \\ * & * & \dots & * & * & -2\delta \mathbf{V}_{hk}^{\circ} & \mathbf{0} \\ * & * & \dots & * & * & * & -\mathbf{I}_{(m_h+m_k)} \end{bmatrix} < 0, \quad (94)$$

for $j = 1, 2, \dots, s-1, j = 2, 3, \dots, s, h = 1, 2, \dots, p-1, k = h+1, h+2, \dots, p, l = 1, 2, \dots, p, l \neq h, k, j < g \leq s,$ with $\mathbf{A}_{jghk}^{\circ}, \mathbf{B}_{jghk}^{\circ}, \mathbf{A}_{jghk}^{l^{\circ}}, \mathbf{C}_{hk}^{\circ},$ and $\mathbf{C}_{hk}^{l^{\circ}}$ defined in (59), (61), and (62), respectively, and with

$$\Sigma_{jjhk}^{\circ} = \mathbf{A}_{jghk}^{\circ} \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \mathbf{A}_{jghk}^{\circ T} - \mathbf{B}_{jghk}^{\circ} \mathbf{W}_{jghk}^{\circ} - \mathbf{W}_{jghk}^{\circ T} \mathbf{B}_{jghk}^{\circ T}, \quad (95)$$

$$\begin{aligned} \Sigma_{jghk}^{\circ} = & \frac{\mathbf{A}_{jghk}^{\circ} + \mathbf{A}_{ghk}^{\circ}}{2} \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \frac{(\mathbf{A}_{jghk}^{\circ} + \mathbf{A}_{ghk}^{\circ})^T}{2} \\ & - \frac{\mathbf{B}_{jghk}^{\circ} \mathbf{W}_{ghk}^{\circ} + \mathbf{B}_{ghk}^{\circ} \mathbf{W}_{jghk}^{\circ}}{2} \\ & - \frac{(\mathbf{B}_{jghk}^{\circ} \mathbf{W}_{ghk}^{\circ} + \mathbf{B}_{ghk}^{\circ} \mathbf{W}_{jghk}^{\circ})^T}{2}, \end{aligned} \quad (96)$$

$$\Gamma_{jjhk}^{\circ} = \mathbf{T}_{hk}^{\circ} - \delta \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \mathbf{A}_{jghk}^{\circ T} - \mathbf{W}_{jghk}^{\circ T} \mathbf{B}_{jghk}^{\circ T}, \quad (97)$$

$$\begin{aligned} \Gamma_{jghk}^{\circ} = & \mathbf{T}_{hk}^{\circ} - \delta \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \frac{\mathbf{A}_{jghk}^{\circ T} + \mathbf{A}_{ghk}^{\circ T}}{2} \\ & - \frac{\mathbf{W}_{ghk}^{\circ T} \mathbf{B}_{jghk}^{\circ T} + \mathbf{W}_{jghk}^{\circ T} \mathbf{B}_{ghk}^{\circ T}}{2}, \end{aligned} \quad (98)$$

where, for given $h, k, \mathbf{A}_{jghk}^{h^{\circ}}, \mathbf{A}_{jghk}^{k^{\circ}},$ and $\mathbf{C}_{hk}^{h^{\circ}}, \mathbf{C}_{hk}^{k^{\circ}}$ are not included into the structure of (78).

When the above conditions hold, the gain matrices $\mathbf{K}_{jghk}^{\circ}$ are given by

$$\mathbf{K}_{jghk}^{\circ} = \mathbf{W}_{jghk}^{\circ} \mathbf{V}_{hk}^{\circ -1}. \quad (99)$$

Proof. Using the above given notations, (33) gives

$$\begin{aligned} 0 > \dot{v}(\mathbf{q}_{hk}(t)) = & \sum_{j=1}^s \sum_{h=1}^{p-1} \sum_{k=h+1}^p \mu_j \mu_j \left\{ \begin{aligned} & \dot{\mathbf{q}}_{hk}^T(t) \mathbf{P}_{hk} \mathbf{q}_{hk}(t) + \mathbf{q}_{hk}^T(t) \mathbf{P}_{hk} \dot{\mathbf{q}}_{hk}(t) \\ & + 3(\mathbf{C}_{hk}^{\circ} \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^{\circ} \omega_{hk}^{\circ}(t))^T (\mathbf{C}_{hk}^{\circ} \mathbf{q}_{hk}(t) + \mathbf{D}_{hk}^{\circ} \omega_{hk}^{\circ}(t)) \\ & - 3\gamma \omega_{hk}^{\circ T}(t) \omega_{hk}^{\circ}(t) \\ & + 2(\mathbf{q}_{hk}^T(t) \mathbf{S}_1^{\circ} + \dot{\mathbf{q}}_{hk}^T(t) \mathbf{S}_2^{\circ}) \left\{ \begin{aligned} & \mathbf{H}_{jjhk}^{\circ} \mathbf{q}_{hk}(t) + \mathbf{G}_{jghk}^{\circ} \omega_{hk}^{\circ}(t) \\ & - \dot{\mathbf{q}}_{hk}(t) \end{aligned} \right\} \end{aligned} \right\} \\ & + 4 \sum_{j=1}^{s-1} \sum_{g=j+1}^s \sum_{h=1}^{p-1} \sum_{k=h+1}^p \mu_j \mu_g \left(\mathbf{q}_{hk}^T(t) \mathbf{S}_1^{\circ} + \dot{\mathbf{q}}_{hk}^T(t) \mathbf{S}_2^{\circ} \right) \left\{ \begin{aligned} & \frac{\mathbf{H}_{jghk} + \mathbf{H}_{gjhk}}{2} \mathbf{q}_{hk}(t) \\ & + \frac{\mathbf{G}_{jghk} + \mathbf{G}_{gjhk}}{2} \omega_{hk}^{\circ}(t) \end{aligned} \right\}, \end{aligned} \quad (100)$$

and so (29), (30) take the forms

$$\begin{bmatrix}
 \Xi_{jjhk}^{\circ} & \mathbf{A}_{jkh}^{1\circ} & \cdots & \mathbf{A}_{jkh}^{p\circ} & \mathbf{B}_{jkh}^{\circ} & \mathbf{Y}_{jjhk}^{\circ} & \mathbf{X}_{hk}^{\circ} \mathbf{C}_{hk}^{\circ T} \\
 * & -\varepsilon_{hk1} \mathbf{I}_{n_1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jkh}^{1\circ T} & \mathbf{C}_{hk}^{1\circ T} \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \mathbf{A}_{jkh}^{p\circ T} & \mathbf{C}_{hk}^{p\circ T} \\
 * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \mathbf{B}_{jkh}^{\circ T} & \mathbf{0} \\
 * & * & \cdots & * & * & -2\delta \mathbf{V}_{hk}^{\circ} & \mathbf{0} \\
 * & * & \cdots & * & * & * & -\mathbf{I}_{(m_h+m_k)}
 \end{bmatrix} < 0,$$

$$\begin{bmatrix}
 \Xi_{jghk}^{\circ} & \frac{\mathbf{A}_{jkh}^{1\circ} + \mathbf{A}_{ghk}^{1\circ}}{2} & \cdots & \frac{\mathbf{A}_{jkh}^{p\circ} + \mathbf{A}_{ghk}^{p\circ}}{2} & \frac{\mathbf{B}_{jkh}^{\circ} + \mathbf{B}_{ghk}^{\circ}}{2} & \mathbf{Y}_{jghk}^{\circ} & \mathbf{V}_{hk}^{\circ} \mathbf{C}_{hk}^{\circ T} \\
 * & -\varepsilon_{hk1} \mathbf{I}_{n_1} & \cdots & \mathbf{0} & \mathbf{0} & \frac{(\mathbf{A}_{jkh}^{1\circ} + \mathbf{A}_{ghk}^{1\circ})^T}{2} & \mathbf{C}_{hk}^{1\circ T} \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 * & * & \cdots & -\varepsilon_{hkp} \mathbf{I}_{n_p} & \mathbf{0} & \frac{(\mathbf{A}_{jkh}^{p\circ} + \mathbf{A}_{ghk}^{p\circ})^T}{2} & \mathbf{C}_{hk}^{p\circ T} \\
 * & * & \cdots & * & -\gamma_{hk} \mathbf{I}_{(r_h+r_k)} & \frac{(\mathbf{B}_{jkh}^{\circ} + \mathbf{B}_{ghk}^{\circ})^T}{2} & \mathbf{0} \\
 * & * & \cdots & * & * & -2\delta \mathbf{V}_{hk}^{\circ} & \mathbf{0} \\
 * & * & \cdots & * & * & * & -\mathbf{I}_{(m_h+m_k)}
 \end{bmatrix} < 0, \quad (101)$$

respectively, where

$$\Xi_{jjhk}^{\circ} = \mathbf{A}_{jkh}^{\circ} \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \mathbf{A}_{jkh}^{\circ T} - \mathbf{B}_{jkh}^{\circ} \mathbf{K}_{jkh}^{\circ} \mathbf{V}_{hk}^{\circ} - \mathbf{V}_{hk}^{\circ} \mathbf{K}_{jkh}^{\circ T} \mathbf{B}_{jkh}^{\circ T}, \quad (102)$$

$$\begin{aligned}
 \Xi_{jghk}^{\circ} &= \frac{\mathbf{A}_{jkh}^{\circ} + \mathbf{A}_{ghk}^{\circ}}{2} \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \frac{(\mathbf{A}_{jkh}^{\circ} + \mathbf{A}_{ghk}^{\circ})^T}{2} \\
 &\quad - \frac{\mathbf{B}_{jkh}^{\circ} \mathbf{K}_{ghk}^{\circ} + \mathbf{B}_{ghk}^{\circ} \mathbf{K}_{jkh}^{\circ}}{2} \mathbf{V}_{hk}^{\circ} \\
 &\quad - \mathbf{V}_{hk}^{\circ} \frac{(\mathbf{B}_{jkh}^{\circ} \mathbf{K}_{ghk}^{\circ} + \mathbf{B}_{ghk}^{\circ} \mathbf{K}_{jkh}^{\circ})^T}{2}, \quad (103)
 \end{aligned}$$

$$\mathbf{Y}_{jjhk}^{\circ} = \mathbf{T}_{hk}^{\circ} - \delta \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \mathbf{A}_{jkh}^{\circ T} - \mathbf{V}_{hk}^{\circ} \mathbf{K}_{jkh}^{\circ T} \mathbf{B}_{jkh}^{\circ T}, \quad (104)$$

$$\begin{aligned}
 \mathbf{Y}_{jghk}^{\circ} &= \mathbf{T}_{hk}^{\circ} - \delta \mathbf{V}_{hk}^{\circ} + \mathbf{V}_{hk}^{\circ} \frac{\mathbf{A}_{jkh}^{\circ} + \mathbf{A}_{ghk}^{\circ}}{2} \\
 &\quad - \mathbf{V}_{hk}^{\circ} \frac{\mathbf{K}_{ghk}^{\circ T} \mathbf{B}_{jkh}^{\circ} + \mathbf{K}_{jkh}^{\circ T} \mathbf{B}_{ghk}^{\circ}}{2}. \quad (105)
 \end{aligned}$$

Therefore, with the substitutions

$$\mathbf{W}_{jkh}^{\circ} = \mathbf{K}_{jkh}^{\circ} \mathbf{V}_{hk}^{\circ}, \quad \mathbf{W}_{ghk}^{\circ} = \mathbf{K}_{ghk}^{\circ} \mathbf{V}_{hk}^{\circ}, \quad (106)$$

(101)–(105) imply (93)–(98). This concludes the proof. $\square$

To bet both of these theorems in the context of the control design for systems specified by TS models, it is necessary to make some remarks.

Although both theorems solve the same problem, the role of Theorem 6 is primarily methodological and in particular shows that the design problem of pairwise distributable control of large-scale TS systems can be formulated in the terms of H_{∞} approach. Because it is based on the default structure of BRL, the Lyapunov matrices $\mathbf{X}_{hk}^{\circ}$ and the subsystem dynamic matrices $\mathbf{A}_{jkh}^{\circ}$ form bound pairs with regard to the operation of multiplication. Consequently, when using the design conditions proposed in Theorem 6 and the LMI task is feasible, the obtained solution is very conservative.

Under the conditions defined by Theorem 7, the set of subsystem matrices $\mathbf{A}_{jkh}^{\circ}$ are decoupled from the set of Lyapunov matrices $\mathbf{T}_{hk}^{\circ}$ by using the set of slack matrix variables $\mathbf{V}_{hk}^{\circ}$. This enables the design conditions in respect of natural affine properties of TS models and, compared with the result by Theorem 6, a less conservative solution. In addition, by tuning parameter δ , the stability conditions setting of the whole system can be modified. Less conservatism in this case also means that the linear control of certain subsystem pair can be used instead of the nonlinear TS fuzzy control algorithm (see Section 5).

These theorems, which could be potentially considered as equivalent, give generally different solutions, except for the special case when for a pair h, k by chance is $\delta = 1$ and $\mathbf{V}_{hk}^{\circ} = \mathbf{T}_{hk}^{\circ} = \mathbf{X}_{hk}^{\circ}$.

In terms of computational complexity, if there is, for example, only one nonlinear sector in each block of the matrix $\mathbf{A}$ and every sector is described only by a pair of sector functions, the number of fuzzy rules is $s = 2^{p^2}$ to formulate the global control design and $s = 2^{2p}$ if pairwise distributed control principle is preferred. This can play a major role due to boundaries of LMI solvers. Moreover, as mentioned above, the proposed method can reduce this number in some subsystem pair structures.

On the other hand, although the pairwise subsystem control principle brings more complex control gain, the use of the state control laws does not substantially modify the computational complexity in dependency on the dimensionality of global state vector parts in the control algorithms.

5. Illustrative Example

To demonstrate properties of this approach, a system with four inputs and four outputs is used in the example. The parameters of (1)–(3) are [22]

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 2 & -1 & 4 & 0 \\ -1 & -2 & 1 & 1 & 5 & 2 \\ 1 & -1 & 1 & -a_j & -2 & 4 \\ 1 & -2 & -2 & 2 & 1 & 2 \\ 3 & -1 & -2 & 0 & 1 & 5 \\ 0 & 0 & -1 & 3 & -2 & -4 \end{bmatrix},$$

$$\mathbf{C}^T = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 6 & -1 & 0 \\ 1 & 6 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 2 & \end{bmatrix},$$

$$a_1 = 0.8, \quad a_2 = 1.2,$$

$$\mu_1(\vartheta(t)) = \frac{a_1 - q_1(t)}{a_1 - a_2}, \quad \mu_2(\vartheta(t)) = 1 - \mu_1(\vartheta(t)),$$

$$\vartheta(t) = q_1(t).$$
(107)

To solve this problem, the next pairs grouping were done

$$\mathbf{A}_{\bullet 12}^{\circ} = \begin{bmatrix} -3 & 1 & 2 \\ -1 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{A}_{112}^{3\circ} = \begin{bmatrix} -1.0 \\ 1.0 \\ -0.8 \end{bmatrix},$$

$$\mathbf{A}_{212}^{3\circ} = \begin{bmatrix} -1.0 \\ 1.0 \\ -1.2 \end{bmatrix}, \quad \mathbf{A}_{\bullet 12}^{4\circ} = \begin{bmatrix} 4 & 0 \\ 5 & 2 \\ -2 & 4 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 13}^{\circ} = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{A}_{\bullet 13}^{2\circ} = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 13}^{4\circ} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 14}^{\circ} = \begin{bmatrix} -3 & 4 & 0 \\ 3 & 2 & 5 \\ 0 & 5 & -4 \end{bmatrix}, \quad \mathbf{A}_{\bullet 14}^{2\circ} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 14}^{3\circ} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{A}_{\bullet 23}^{1\circ} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{A}_{123}^{\circ} = \begin{bmatrix} -2 & 1 & 1.0 \\ -1 & 1 & -0.8 \\ -2 & -2 & 2.0 \end{bmatrix}, \quad \mathbf{A}_{223}^{\circ} = \begin{bmatrix} -2 & 1 & 1.0 \\ -1 & 1 & -1.2 \\ -2 & -2 & 2.0 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 23}^{4\circ} = \begin{bmatrix} 5 & 2 \\ -2 & 4 \\ 1 & 2 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 24}^{\circ} = \begin{bmatrix} -2 & 1 & 5 & 2 \\ -1 & 1 & -2 & 4 \\ -1 & -2 & 1 & 5 \\ 0 & -1 & -2 & -4 \end{bmatrix}, \quad \mathbf{A}_{\bullet 24}^{1\circ} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 0 \end{bmatrix},$$

$$\mathbf{A}_{124}^{3\circ} = \begin{bmatrix} 1 \\ -0.8 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{A}_{224}^{3\circ} = \begin{bmatrix} 1 \\ -1.2 \\ 0 \\ 3 \end{bmatrix},$$

$$\mathbf{A}_{\bullet 34}^{\circ} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 5 \\ 3 & -2 & -4 \end{bmatrix}, \quad \mathbf{A}_{\bullet 34}^{1\circ} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{A}_{\bullet 34}^{2\circ} = \begin{bmatrix} -2 & -2 \\ -1 & -2 \\ 0 & -1 \end{bmatrix},$$

$$\mathbf{B}_{\bullet 12} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_{\bullet 13} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B}_{\bullet 14} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix},$$

$$\mathbf{B}_{\bullet 23} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B}_{\bullet 24} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B}_{\bullet 34} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 12}^{\circ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}, \quad \mathbf{C}_{\bullet 12}^{3\circ} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{C}_{\bullet 12}^{4\circ} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 13}^{\circ} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 13}^{2\circ} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 13}^{4\circ} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 14}^{\circ} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 14}^{2\circ} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}_{\bullet 14}^{3\circ} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 23}^{\circ} = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 23}^{1\circ} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C}_{\bullet 23}^{4\circ} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 24}^{\circ} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 24}^{1\circ} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_{\bullet 24}^{3\circ} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\mathbf{C}_{\bullet 34}^{\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{C}_{\bullet 34}^{1\circ} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{C}_{\bullet 34}^{2\circ} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix},$$
(108)

where $\bullet hk$ denotes that the matrix parameters with this subscript are independent of the fuzzy rules. That means that in the pairwise partially decentralized control structure of this example, the control loop pairs 13, 14, and 34 will be surely fuzzy independent.

Solving for given tuning parameter δ , for example, with respect to $\mathbf{V}_{23}^\circ$, $\mathbf{T}_{23}^\circ$, $\mathbf{W}_{123}^\circ$, $\mathbf{W}_{223}^\circ$, ε_{231} , ε_{234} , and γ_{23} , it means writing (92)–(98) as

$$\begin{aligned} \mathbf{V}_{23}^\circ &= \mathbf{V}_{23}^{\circ T} > 0, & \mathbf{T}_{23}^\circ &= \mathbf{T}_{23}^{\circ T} > 0, \\ \varepsilon_{231} &> 0, & \varepsilon_{234} &> 0, & \gamma_{23} &> 0, \\ \left[\begin{array}{cccccc} \Sigma_{jg23}^\circ & \mathbf{A}_{23}^{1^\circ} & \mathbf{A}_{23}^{4^\circ} & \mathbf{B}_{23}^\circ & \Gamma_{jg23}^\circ & \mathbf{V}_{23}^\circ \mathbf{C}_{23}^{\circ T} \\ * & -\varepsilon_{231} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{23}^{1^\circ T} & \mathbf{C}_{23}^{1^\circ T} \\ * & * & -\varepsilon_{234} & \mathbf{0} & \mathbf{A}_{23}^{4^\circ T} & \mathbf{C}_{23}^{4^\circ T} \\ * & * & * & -\gamma_{23} \mathbf{I}_2 & \mathbf{B}_{23}^{\circ T} & \mathbf{0} \\ * & * & * & * & -2\delta \mathbf{Z}_{23}^\circ & \mathbf{0} \\ * & * & * & * & * & -\mathbf{I}_2 \end{array} \right] < 0, \\ \Sigma_{jj23}^\circ &= \mathbf{V}_{23}^\circ \mathbf{A}_{j23}^{\circ T} + \mathbf{A}_{j23}^\circ \mathbf{V}_{23}^\circ \\ &\quad - \mathbf{B}_{23}^\circ \mathbf{W}_{j23}^\circ - \mathbf{W}_{j23}^{\circ T} \mathbf{B}_{23}^{\circ T}, \quad j = 1, 2, \\ \Sigma_{jg23}^\circ &= \frac{\mathbf{A}_{j23}^\circ + \mathbf{A}_{g23}^\circ}{2} \mathbf{V}_{23}^\circ + \mathbf{V}_{23}^\circ \frac{(\mathbf{A}_{j23}^\circ + \mathbf{A}_{g23}^\circ)^T}{2} \\ &\quad - \frac{\mathbf{B}_{23}^\circ \mathbf{W}_{g23}^\circ + \mathbf{B}_{23}^\circ \mathbf{W}_{j23}^\circ}{2} \\ &\quad - \frac{(\mathbf{B}_{23}^\circ \mathbf{W}_{g23}^\circ + \mathbf{B}_{23}^\circ \mathbf{W}_{j23}^\circ)^T}{2}, \\ &\quad j = 1, \quad g = 2, \\ \Gamma_{jj23}^\circ &= \mathbf{T}_{23}^\circ - \delta \mathbf{V}_{23}^\circ + \mathbf{V}_{23}^\circ \mathbf{A}_{j23}^{\circ T} - \mathbf{W}_{23}^{\circ T} \mathbf{B}_{23}^{\circ T}, \\ &\quad j = 1, 2, \\ \Gamma_{jg23}^\circ &= \mathbf{T}_{23}^\circ - \delta \mathbf{V}_{23}^\circ + \mathbf{V}_{23}^\circ \frac{\mathbf{A}_{j23}^{\circ T} + \mathbf{A}_{g23}^{\circ T}}{2} \\ &\quad - \frac{\mathbf{W}_{g23}^{\circ T} \mathbf{B}_{23}^{\circ T} + \mathbf{W}_{j23}^{\circ T} \mathbf{B}_{23}^{\circ T}}{2}, \\ &\quad j = 1, \quad g = 2, \end{aligned} \quad (109)$$

and using SeDuMi package for Matlab [23], the feasible task for subsystem 23 and $\delta = 10$ yields the parameters

$$\varepsilon_{231} = 58.0063, \quad \varepsilon_{234} = 70.1966, \quad \gamma_{23} = 32.4549,$$

$$\mathbf{T}_{23}^\circ = \begin{bmatrix} 27.9470 & 10.1742 & -4.8315 \\ 10.1742 & 8.0186 & 3.1112 \\ -4.8315 & 3.1112 & 38.0154 \end{bmatrix},$$

$$\mathbf{W}_{j23}^\circ = \begin{bmatrix} 7.5666 & 4.4717 & 0.6808 \\ -1.9191 & -0.3955 & 13.6818 \end{bmatrix},$$

$$\begin{aligned} \mathbf{V}_{23}^\circ &= \begin{bmatrix} 1.2004 & 0.2170 & -0.5773 \\ 0.2170 & 0.3793 & 0.2991 \\ -0.5773 & 0.2991 & 2.2580 \end{bmatrix}, \\ \mathbf{K}_{\diamond 23}^\circ &= \begin{bmatrix} 4.9482 & 8.6232 & 0.4245 \\ 4.3630 & -10.2695 & 8.5351 \end{bmatrix}, \end{aligned} \quad (110)$$

where $\mathbf{K}_{\diamond 23}^\circ$ means that $\mathbf{K}_{j23}^\circ$ were computed equally to $g = 1, 2$.

By the same way, solving the rest LMIs, the gain matrix set is computed as

$$\begin{aligned} \mathbf{K}_{112}^\circ &= \begin{bmatrix} 7.9346 & -2.8080 & 12.4851 \\ 0.7258 & 5.3963 & 6.0160 \end{bmatrix}, \\ \mathbf{K}_{212}^\circ &= \begin{bmatrix} 7.8740 & -2.8497 & 12.3828 \\ 0.7250 & 5.4268 & 6.0364 \end{bmatrix}, \\ \mathbf{K}_{\cdot 14}^\circ &= \begin{bmatrix} 4.0106 & 5.9223 & 1.4848 \\ 2.0249 & 4.1941 & 3.8514 \end{bmatrix}, \\ \mathbf{K}_{\diamond 34}^\circ &= \begin{bmatrix} 5.4294 & 1.1356 & 1.5393 \\ 2.0320 & 4.0784 & 3.0034 \end{bmatrix}, \\ \mathbf{K}_{\cdot 13}^\circ &= \begin{bmatrix} 6.2603 & 2.7947 \\ 2.0298 & 5.8502 \end{bmatrix}, \\ \mathbf{K}_{\diamond 24}^\circ &= \begin{bmatrix} 0.8349 & 2.2731 & 0.5904 & 1.1688 \\ -0.0995 & -0.1542 & 2.2100 & 1.8754 \end{bmatrix}, \\ \gamma_{12} &= 22.9891, & \gamma_{13} &= 9.5395, \\ \gamma_{14} &= 9.4173, & \gamma_{24} &= 9.5641, \\ \gamma_{34} &= 11.0660. \end{aligned} \quad (111)$$

The results imply that in this example only the control of the subsystem pair 1, 2 has to be realized in a TS fuzzy structure and all other pairs controls can be realized in constant linear pairwise control structure.

Generalizing, with respect the results presented in [13], the design methods based on the enhanced principle tend to produce for TS models the same control gain matrices which radically reduce the control structure, since such results mean very often stabilizing linear control laws for the nonlinear system or, as here, for its pair parts.

It is possible to verify—routinely—that the resulting closed-loop large-scale system will be stable if the tuning parameter is chosen as $\delta = 10$. Moreover, this example implies that with $\delta \leq 5$ such pairwise distributed TS fuzzy control structure ensures all stable closed-loop pairs but does not ensure the stability of the whole closed-loop system. That means that using this design principle, the stability of all closed-loop subsystem pairs does not mean automatically the stability of the whole TS large-scale closed-loop system. Note that the control laws are given in the partly autonomous structure (68), (69), which ensures now that every closed-loop subsystem pair is stable and designed for $\delta = 10$, and the large-scale closed-loop system will be stable too.

FIGURE 1: The general closed-loop structure for control of the subsystems pair (23) in the autonomous regime.

FIGURE 2: State response of the global closed-loop TS system in the autonomous regime.

The block diagram on Figure 1 describes how the control of the subsystems pair (23) is, in general, implemented. This specific subsystem pair in the example is the only one that is controlled by the fuzzy PDC controller. As stated earlier, $\mathbf{B}_{123}^o = \mathbf{B}_{223}^o = \mathbf{B}_{23}^o$, while, for example, $\mathbf{q}_2(t)$ of the global TS system can be obtained as the sum of the second component of vector $\mathbf{q}_{12}(t)$ and the first components of vectors $\mathbf{q}_{23}(t)$ and $\mathbf{q}_{24}(t)$. It is assumed, of course, that bonds to create $\mathbf{A}_{123}^o$, $\mathbf{A}_{223}^o$, and $\mathbf{B}_{23}^o$ structures can be realized.

Using autonomous control regime, defined by (68), (69), the illustrative simulations were realized under the initial condition

$$\mathbf{q}^T(0) = [0.02 \quad -0.10 \quad 0.00 \quad 0.10 \quad -0.05 \quad 0.05]. \quad (112)$$

Figure 2 gives the global closed-loop TS system state response in the used simulation conditions.

For comparison, an equivalent set of gain matrices for the centralized fuzzy control can be constructed using above obtained results, where, for the control law (6),

$$\mathbf{K}_1 = \begin{bmatrix} 18.2055 & -2.8080 & 12.4851 & 2.7947 & 5.9223 & 1.4848 \\ 0.7258 & 13.4370 & 20.2510 & 0.4245 & -1.1914 & 1.7646 \\ 2.0298 & 4.3630 & -10.2695 & 19.8147 & 1.1356 & 1.5393 \\ 2.0249 & 1.5018 & -5.8333 & 2.0320 & 19.8317 & 10.8627 \end{bmatrix}, \quad (113)$$

$$\mathbf{K}_2 = \begin{bmatrix} 18.1449 & -2.8497 & 12.3828 & 2.7947 & 5.9223 & 1.4848 \\ 0.7250 & 13.4675 & 20.2714 & 0.4245 & -1.1914 & 1.7646 \\ 2.0298 & 4.3630 & -10.2695 & 19.8147 & 1.1356 & 1.5393 \\ 2.0249 & 1.5018 & -5.8333 & 2.0320 & 19.8317 & 10.8627 \end{bmatrix},$$

and the resulting closed-loop eigenvalue spectrums are stable, where

$$\rho(\mathbf{A}_1 - \mathbf{BK}_1) = \{-0.3278, -8.1143, -18.2169, -35.0038, -41.2029, -48.6512\}, \quad (114)$$

$$\rho(\mathbf{A}_2 - \mathbf{BK}_2) = \{-0.3267, -8.1208, -18.1581, -35.0064, -41.1984, -48.7273\}.$$

Note that the structure of the matrices $\mathbf{K}_g$, $g = 1, 2$ evidently implies that the control laws are block diagonal dominant.

6. Concluding Remarks

The main difficulty of solving the decentralized control problem comes from the fact that the feedback gain is subject to structural constraints. At the beginning the study of large scale system theory, there was prevailing idea that a large of scale system is decentrally stabilizable under controllability condition by strengthening the stability degree of subsystems. But, because of the existence of decentralized fixed modes, some large scale systems can not be decentrally stabilized at all. In this paper, the idea to stabilize all subsystems and the whole system simultaneously by using decentralized controllers is replaced by another one, to stabilize all subsystems pairs and the whole system simultaneously by using partly pairwise decentralized control. In this sense the final scope of the paper is the design conditions for pairwise control of one class of pairwise-distributable continuous time TS large-scale systems. It is shown, based on the equivalent BRL formulations, how to expand the Lyapunov condition for pairwise TS control by using slack matrix variables in LMIs. As mentioned above, such matrix inequalities are linear with respect to the subsystem pairs variables and do not involve any product of the matrices, obtained by the Lyapunov matrix block separation and the subsystem pairs matrices. This enables us to derive a sufficient design condition for control with quadratic performances, optimizing subsystems interaction H_∞ norm bounds.

The method generally requires to solve, but separate, a larger number of linear matrix inequalities, and so more computational efforts are needed to provide design control parameters. However, used design conditions are less restrictive and are more close to necessity conditions in the sense of [24]. It is a very useful extension to TS system control performance synthesis problem. Numerical example demonstrates the principle effectiveness, although some computational complexity is increased.

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