Research Article

Variable-Coefficient Exact Solutions for Nonlinear Differential Equations by a New Bernoulli Equation-Based Subequation Method

Chunxia Qi and Shunliang Huang

School of Business, Shandong University of Technology, Zibo, Shandong 255049, China

Correspondence should be addressed to Chunxia Qi; qichunxiasdut@126.com

Received 20 January 2013; Revised 10 March 2013; Accepted 11 March 2013

Academic Editor: Ebrahim Momoniat

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A new Bernoulli equation-based subequation method is proposed to establish variable-coefficient exact solutions for nonlinear differential equations. For illustrating the validity of this method, we apply it to the asymmetric (2 + 1)-dimensional NNV system and the Kaup-Kupershmidt equation. As a result, some new exact solutions with variable functions coefficients for them are successfully obtained.

1. Introduction

Nonlinear differential equations (NLDEs) can be used to describe many nonlinear phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, and chemical physics. Recently, research for seeking exact analytical solutions of NLDEs has been a hot topic, and many powerful and efficient methods to find exact solutions have been presented so far. For example, these methods include the known homogeneous balance method [1, 2], the tanh method [3–5], the inverse scattering transform [6], the Backlund transform [7, 8], the Hirota's bilinear method [9, 10], the generalized Riccati subequation method [11, 12], the Jacobi elliptic function expansion [13, 14], the F -expansion method [15], the exp-function expansion method [16, 17], and the (G'/G) -expansion method [18, 19]. However, we notice that most of the existing methods are accompanied with constant coefficients, while very few methods are concerned with variable coefficients.

In this paper, by introducing a new ansatz, we develop a new Bernoulli equation-based sub equation method for obtaining variable-coefficient exact solutions for NLDEs. First we give the description of the Bernoulli equation-based subequation method. Then we apply the method to solve the

asymmetric (2+1)-dimensional NNV system and the Kaup-Kupershmidt equation. Some conclusions are presented at the end of the paper.

2. Description of the Bernoulli Equation-Based Subequation Method

We consider the following Bernoulli equation:

$$G' + \lambda G = G^2, \quad (1)$$

where $\lambda \neq 0$ is a complex number and $G = G(\xi)$. The solutions of (1) are denoted by

$$G(\xi) = \frac{\lambda}{1 + \lambda d e^{\lambda \xi}}, \quad (2)$$

where d is an arbitrary constant. In particular, when λ is a real number and $d = 1/\lambda$, we obtain

$$G(\xi) = \frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right). \quad (3)$$

When $d = 1/\lambda$, $\lambda = i\tilde{\lambda}$, where $\tilde{\lambda}$ is a real number and i is the unit of imaginary number, we obtain

$$G(\xi) = \frac{\lambda}{2} - \frac{\lambda i}{2} \tan\left(\frac{\tilde{\lambda}\xi}{2}\right). \quad (4)$$

Suppose that a nonlinear equation, say in two or three independent variables x, y, t , is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{xy}, \dots) = 0, \quad (5)$$

where $u = u(x, y, t)$ is an unknown function and P is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Suppose that

$$u(x, y, t) = u(\xi), \quad \xi = \xi(x, y, t), \quad (6)$$

and then (5) can be turned into the following form:

$$\tilde{P}(u, u', u'', \dots) = 0. \quad (7)$$

Step 2. Suppose that the solution of (7) can be expressed by a polynomial in G as follows:

$$u(\xi) = a_m(x, y, t)G^m + a_{m-1}(x, y, t)G^{m-1} + \dots + a_0(x, y, t), \quad (8)$$

where $G = G(\xi)$ satisfies (1) and $a_m(x, y, t), a_{m-1}(x, y, t), \dots, a_0(x, y, t)$ are all unknown functions to be determined later with $a_m(x, y, t) \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7).

Step 3. Substituting (8) into (7) and using (1), collecting all terms with the same order of G together, the left-hand side of (7) is converted to another polynomial in G . Equating each coefficient of this polynomial to zero yields a set of partial differential equations for $a_m(x, y, t), a_{m-1}(x, y, t), \dots, \xi(x, y, t), \lambda$.

Step 4. Solving the equations system in Step 3, and using the solutions of (1), we can construct exact coefficient function solutions of (7).

Remark 1. As the partial differential equations in Step 3 are usually overdetermined, we may choose some special forms of $a_m, a_{m-1}, \dots, a_0$ as done in the following.

3. Application of the Bernoulli Equation-Based Subequation Method to Some NLDEs

3.1. Asymmetric (2+1)-Dimensional NNV System. We consider the (2+1)-dimensional asymmetric NNV system [20–22]:

$$\begin{aligned} u_t - u_{xxx} + \alpha(uv)_x &= 0, \\ u_x + \beta v_y &= 0, \end{aligned} \quad (9)$$

where α and β are arbitrary nonzero constants.

Assume that $u(x, y, t) = U(\xi)$, where $\xi = \xi(x, y, t)$, and then (9) can be turned into

$$\xi'_t U' - (\xi_x^3 U''' + 3\xi_x \xi_{xx} U'' + \xi_{xxx} U') + \alpha \xi_x (UV)' = 0, \quad (10)$$

$$\xi'_x U' + \beta \xi_y V' = 0. \quad (11)$$

Suppose that the solutions of (10)-(11) can be expressed by a polynomial in G as follows:

$$U(\xi) = \sum_{i=0}^m a_i(y, t) G^i, \quad V(\xi) = \sum_{i=0}^n b_i(y, t) G^i, \quad (12)$$

where $a_i(y, t), b_i(y, t)$ are underdetermined functions and $G = G(\xi)$ satisfies (1). Balancing the order of U''' and $(UV)'$ in (10) and the order of U' and V' in (11), we can obtain $m + 3 = m + n + 1, m + 1 = n + 1 \Rightarrow m = 2, n = 2$. So we have

$$\begin{aligned} U(\xi) &= a_2(y, t)G^2 + a_1(y, t)G + a_0(y, t), \\ V(\xi) &= b_2(y, t)G^2 + b_1(y, t)G + b_0(y, t). \end{aligned} \quad (13)$$

Substituting (13) into (10)-(11) and collecting all the terms with the same power of G together, equating each coefficient to zero yields a set of underdetermined partial differential equations for $a_i(x, y, t), b_i(x, y, t), i = 0, 1, 2$, and $\xi(x, y, t)$. Solving these equations yields the following.

Case 1. Consider

$$\begin{aligned} \xi(x, y, t) &= C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y), \\ a_2(y, t) &= -\frac{6\beta C_1 F_2'(y)}{\alpha}, \quad a_1(y, t) = \frac{6\beta \lambda C_1 F_2'(y)}{\alpha}, \\ a_0(y, t) &= 0, \quad b_2(y, t) = \frac{6C_1^2}{\alpha}, \\ b_1(y, t) &= -\frac{6C_1^2 \lambda}{\alpha}, \quad b_0(y, t) = F_1(t), \end{aligned} \quad (14)$$

where C_1 is an arbitrary constant and $F_1(t)$ and $F_2(y)$ are two arbitrary functions with respect to the variables t and y , respectively.

Case 2. Consider

$$\begin{aligned} \xi(x, y, t) &= C_1 x + \int \frac{-\alpha \beta C_1 C_2 F_1(t) + C_1^2 (\alpha + C_1 C_2 \beta \lambda^2)}{\beta C_2} dt \end{aligned}$$

$$\begin{aligned}
& + C_2 \int F_2(y) dy + C_3, \\
a_2(y, t) &= -\frac{6\beta C_1 C_2 F_2(y)}{\alpha}, \quad a_1(y, t) = \frac{6\beta \lambda C_1 C_2 F_2(y)}{\alpha}, \\
a_0(y, t) &= F_2(y), \quad b_2(y, t) = \frac{6C_1^2}{\alpha}, \\
b_1(y, t) &= -\frac{6C_1^2 \lambda}{\alpha}, \quad b_0(y, t) = F_1(t),
\end{aligned} \tag{15}$$

where C_1 , C_2 , and C_3 are arbitrary constants and $F_1(t)$ and $F_2(y)$ are two arbitrary functions with respect to the variables t and y , respectively.

Case 3. Consider

$$\begin{aligned}
\xi(x, y, t) &= -\frac{\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}}{2\beta C_2 \lambda^2 F_1(y)} x \\
&+ C_2 \int F_1(y) dy + C_3, \\
a_0(y, t) &= F_1(y), \quad b_0(y, t) = C_1, \\
a_2(y, t) &= \frac{3\left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}\right)}{\lambda^2 \alpha}, \\
a_1(y, t) &= -\frac{3\left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}\right)}{\lambda \alpha}, \\
b_2(y, t) &= \frac{3\left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}\right]^2}{2\alpha\beta^2\lambda^4 C_2^2 F_1^2(y)}, \\
b_1(y, t) &= -\frac{3\left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}\right]^2}{2\alpha\beta^2\lambda^3 C_2^2 F_1^2(y)},
\end{aligned} \tag{16}$$

where C_1 , C_2 , and C_3 are arbitrary constants and $F_1(y)$ is an arbitrary function.

Case 4. Consider

$$\begin{aligned}
\xi(x, y, t) &= -\frac{\sqrt{C_1} \alpha x}{\lambda} + F_1(y), \\
a_2(y, t) &= \frac{6\beta C_1 F_1'(y)}{\lambda \sqrt{C_1} \alpha}, \\
a_1(y, t) &= -\frac{6\beta C_1 F_1'(y)}{\sqrt{C_1} \alpha}, \quad a_0(y, t) = 0, \\
b_2(y, t) &= \frac{6C_1}{\lambda^2}, \quad b_1(y, t) = -\frac{6C_1}{\lambda}, \quad b_0(y, t) = C_1,
\end{aligned} \tag{17}$$

where C_1 is an arbitrary constant with $C_1 \alpha > 0$ and $F_1(y)$ is an arbitrary function.

Substituting the results in the four cases mentioned previously into (13), and combining with the solutions of (1) as denoted in (2), we can obtain a rich variety of exact solutions to the asymmetric (2+1)-dimensional NNV system as follows.

Family 1. One has

$$\begin{aligned}
u_1(x, y, t) &= -\frac{6\beta C_1 F_2'(y)}{\alpha} \\
&\times \left(\frac{\lambda}{1 + \lambda d e^{\lambda[C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)]}} \right)^2 \\
&+ \frac{6\beta \lambda C_1 F_2'(y)}{\alpha} \\
&\times \left(\frac{\lambda}{1 + \lambda d e^{\lambda[C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)]}} \right), \\
v_1(x, y, t) &= \frac{6C_1^2}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda[C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)]}} \right)^2 \\
&- \frac{6C_1^2 \lambda}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda[C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)]}} \right) \\
&+ F_1(t).
\end{aligned} \tag{18}$$

Family 2. One has

$$\begin{aligned}
u_2(x, y, t) &= -\frac{6\beta C_1 C_2 F_2(y)}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2 \\
&+ \frac{6\beta \lambda C_1 C_2 F_2(y)}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right) + F_2(y), \\
v_2(x, y, t) &= \frac{6C_1^2}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2 \\
&- \frac{6C_1^2 \lambda}{\alpha} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right) + F_1(t),
\end{aligned} \tag{19}$$

where $\xi = C_1 x + \int ((-\alpha\beta C_1 C_2 F_1(t) + C_1^2(\alpha + C_1 C_2 \beta \lambda^2))/\beta C_2) dt + C_2 \int F_2(y) dy + C_3$.

Family 3. One has

$$\begin{aligned}
u_3(x, y, t) &= \frac{3\left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}\right)}{\lambda^2 \alpha} \\
&\times \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2
\end{aligned}$$

$$- \frac{3 \left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right)}{\lambda \alpha} \times \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right) + F_1(y), \quad (20)$$

$$\begin{aligned} v_3(x, y, t) &= \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha\beta^2\lambda^4 C_2^2 F_1^2(y)} \\ &\times \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2 \\ &- \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha\beta^2\lambda^3 C_2^2 F_1^2(y)} \\ &\times \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right) + C_1, \end{aligned} \quad (21)$$

where $\xi = -((\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}) / 2\beta C_2 \lambda^2 F_1(y))x + C_2 \int F_1(y) dy + C_3$.

Family 4. Consider

$$\begin{aligned} u_4(x, y, t) &= \frac{6\beta C_1 F_1'(y)}{\lambda \sqrt{C_1 \alpha}} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2 \\ &- \frac{6\beta C_1 F_1'(y)}{\sqrt{C_1 \alpha}} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right), \\ v_4(x, y, t) &= \frac{6C_1}{\lambda^2} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right)^2 - \frac{6C_1}{\lambda} \left(\frac{\lambda}{1 + \lambda d e^{\lambda \xi}} \right) + C_1, \end{aligned} \quad (22)$$

where $\xi = -(\sqrt{C_1 \alpha} x / \lambda) + F_1(y)$.

Particularly, by a combination between Cases 1–4 and (3) we can obtain some special hyperbolic function solutions as follows:

$$\begin{aligned} u_5(x, y, t) &= -\frac{6\beta C_1 F_2'(y)}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &+ \frac{6\beta \lambda C_1 F_2'(y)}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right], \\ v_5(x, y, t) &= \frac{6C_1^2}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &- \frac{6C_1^2 \lambda}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] + F_1(t), \end{aligned} \quad (23)$$

where $\xi = C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)$.

Consider

$$\begin{aligned} u_6(x, y, t) &= -\frac{6\beta C_1 C_2 F_2(y)}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &+ \frac{6\beta \lambda C_1 C_2 F_2(y)}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] \\ &+ F_2(y), \\ v_6(x, y, t) &= \frac{6C_1^2}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &- \frac{6C_1^2 \lambda}{\alpha} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] + F_1(t), \end{aligned} \quad (24)$$

where $\xi = C_1 x + \int ((-\alpha\beta C_1 C_2 F_1(t) + C_1^2(\alpha + C_1 C_2 \beta \lambda^2)) / \beta C_2) dt + C_2 \int F_2(y) dy + C_3$.

Consider

$$\begin{aligned} u_7(x, y, t) &= \frac{3 \left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right)}{\lambda^2 \alpha} \\ &\times \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &- \frac{3 \left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right)}{\lambda \alpha} \\ &\times \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] + F_1(y), \\ v_7(x, y, t) &= \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha\beta^2\lambda^4 C_2^2 F_1^2(y)} \\ &\times \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &- \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha\beta^2\lambda^3 C_2^2 F_1^2(y)} \\ &\times \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] + C_1, \end{aligned} \quad (25)$$

where $\xi = -((\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha\beta^2\lambda^2 C_1 C_2^2) F_1^2(y)}) / 2\beta C_2 \lambda^2 F_1(y))x + C_2 \int F_1(y) dy + C_3$.

Consider

$$\begin{aligned} u_8(x, y, t) &= \frac{6\beta C_1 F_1'(y)}{\lambda \sqrt{C_1 \alpha}} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &\quad - \frac{6\beta C_1 F_1'(y)}{\sqrt{C_1 \alpha}} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right], \\ v_8(x, y, t) &= \frac{6C_1}{\lambda^2} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right]^2 \\ &\quad - \frac{6C_1}{\lambda} \left[\frac{\lambda}{2} \left(1 - \tanh \left(\frac{\lambda \xi}{2} \right) \right) \right] + C_1, \end{aligned} \quad (26)$$

where $\xi = (-\sqrt{C_1 \alpha} x / \lambda) + F_1(y)$.

By a combination with (4) we can obtain some special trigonometric function solutions as follows:

$$\begin{aligned} u_9(x, y, t) &= -\frac{6\beta C_1 F_2'(y)}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad + \frac{6\beta \lambda C_1 F_2'(y)}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right], \\ v_9(x, y, t) &= \frac{6C_1^2}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{6C_1^2 \lambda}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] + F_1(t), \end{aligned} \quad (27)$$

where $\xi = C_1 x - \int C_1 \alpha F_1(t) dt + C_1^3 \lambda^2 t + F_2(y)$, $\tilde{\lambda}$ is a real number and $\lambda = i\tilde{\lambda}$.

One has

$$\begin{aligned} u_{10}(x, y, t) &= -\frac{6\beta C_1 C_2 F_2(y)}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad + \frac{6\beta \lambda C_1 C_2 F_2(y)}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] \\ &\quad + F_2(y), \\ v_{10}(x, y, t) &= \frac{6C_1^2}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{6C_1^2 \lambda}{\alpha} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] + F_1(t), \end{aligned} \quad (28)$$

where $\xi = C_1 x + \int ((-\alpha \beta C_1 C_2 F_1(t) + C_1^2(\alpha + C_1 C_2 \beta \lambda^2)) / \beta C_2) dt + C_2 \int F_2(y) dy + C_3$.

One has

$$\begin{aligned} u_{11}(x, y, t) &= \frac{3 \left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha \beta^2 \lambda^2 C_1 C_2^2) F_1^2(y)} \right)}{\lambda^2 \alpha} \\ &\quad \times \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{3 \left(\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha \beta^2 \lambda^2 C_1 C_2^2) F_1^2(y)} \right)}{\lambda \alpha} \\ &\quad \times \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] + F_1(y), \\ v_{11}(x, y, t) &= \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha \beta^2 \lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha \beta^2 \lambda^4 C_2^2 F_1^2(y)} \\ &\quad \times \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{3 \left[\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha \beta^2 \lambda^2 C_1 C_2^2) F_1^2(y)} \right]^2}{2\alpha \beta^2 \lambda^3 C_2^2 F_1^2(y)} \\ &\quad \times \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] + C_1, \end{aligned} \quad (29)$$

where $\xi = -((\alpha F_1(y) + \sqrt{(\alpha^2 + 4\alpha \beta^2 \lambda^2 C_1 C_2^2) F_1^2(y)}) / 2\beta C_2 \lambda^2 F_1(y)) x + C_2 \int F_1(y) dy + C_3$.

$$\begin{aligned} u_{12}(x, y, t) &= \frac{6\beta C_1 F_1'(y)}{\lambda \sqrt{C_1 \alpha}} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{6\beta C_1 F_1'(y)}{\sqrt{C_1 \alpha}} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right], \\ v_{12}(x, y, t) &= \frac{6C_1}{\lambda^2} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right]^2 \\ &\quad - \frac{6C_1}{\lambda} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan \left(\frac{\tilde{\lambda} \xi}{2} \right) \right] + C_1, \end{aligned} \quad (30)$$

where $\xi = -(\sqrt{C_1 \alpha} x / \lambda) + F_1(y)$.

Remark 2. In [20–22], some exact solutions for the asymmetric (2+1)-dimensional NNV system are established using different methods. We note that the established solutions mentioned previously are different from them essentially as they are new exact solutions with variable functions coefficients and have been reported by other authors in the literature.

3.2. *Kaup-Kupershmidt Equation.* We consider the following Kaup-Kupershmidt equation [23–25]:

$$u_{xxxxx} + u_t + 45u_x u^2 - \frac{75}{2}u_x u_{xx} - 15uu_{xxx} = 0. \quad (31)$$

Suppose that $u(x, t) = U(\xi)$, $\xi = \xi(x, t)$, and, furthermore, by balancing the order in (31), we can suppose that

$$U(\xi) = a_2(x, t)G^2 + a_1(x, t)G + a_0(x, t), \quad (32)$$

where $G = G(\xi)$ satisfies (1). Similar to the process mentioned before, we obtain

$$\begin{aligned} \xi(x, t) &= \frac{2}{\lambda i} \arctan\left(\frac{C_1(x + C_2)}{2\lambda i}\right), \\ a_2(x, t) &= \frac{128C_1^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2}, \\ a_1(x, t) &= -\frac{64C_1^3(x + C_2)}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2}, \\ a_0(x, t) &= \frac{8C_1^4(x + C_2)^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} + \frac{\lambda}{2}. \end{aligned} \quad (33)$$

Then we can obtain a kind of exact solutions of (31) which is unrelated to time variable as follows:

$$\begin{aligned} u_1(x, t) &= \frac{128C_1^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \\ &\times \left(\frac{\lambda}{1 + \lambda de^{(2/i) \arctan(C_1(x+C_2)/2\lambda i)}} \right)^2 \\ &- \frac{64C_1^3(x + C_2)}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \\ &\times \left(\frac{\lambda}{1 + \lambda de^{(2/i) \arctan(C_1(x+C_2)/2\lambda i)}} \right) \\ &+ \frac{8C_1^4(x + C_2)^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} + \frac{\lambda}{2}. \end{aligned} \quad (34)$$

In particular, by the combination with (3)-(4) we can obtain hyperbolic function and trigonometric function solutions as follows:

$$\begin{aligned} u_2(x, t) &= \frac{128C_1^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \left[\frac{\lambda}{2} \left(1 - \tanh\left(\frac{\lambda\xi}{2}\right) \right) \right]^2 \\ &- \frac{64C_1^3(x + C_2)}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \left[\frac{\lambda}{2} \left(1 - \tanh\left(\frac{\lambda\xi}{2}\right) \right) \right] \\ &+ \frac{8C_1^4(x + C_2)^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} + \frac{\lambda}{2}, \end{aligned} \quad (35)$$

$$\begin{aligned} u_3(x, t) &= \frac{128C_1^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan\left(\frac{\tilde{\lambda}\xi}{2}\right) \right]^2 \\ &- \frac{64C_1^3(x + C_2)}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} \left[\frac{\lambda}{2} - \frac{\lambda i}{2} \tan\left(\frac{\tilde{\lambda}\xi}{2}\right) \right] \\ &+ \frac{8C_1^4(x + C_2)^2}{(C_1^2(x + C_2)^2 - 4\lambda^2)^2} + \frac{\lambda}{2}, \end{aligned} \quad (36)$$

where $\xi = (2/\lambda i) \arctan(C_1(x + C_2)/2\lambda i)$, $\tilde{\lambda}$ is a real number and $\lambda = i\tilde{\lambda}$.

Remark 3. The previous established solutions for the Kaup-Kupershmidt equation cannot be obtained by the methods in [23–25] and are new exact solutions to our best knowledge.

4. Conclusions

We have proposed a Bernoulli equation-based subequation method for solving nonlinear differential equations and applied it to find exact solutions with variable functions coefficients of the asymmetric (2+1)-dimensional asymmetric NNV system and the Kaup-Kupershmidt equation. As a result, some new exact solutions for them have been successfully found. Finally, we note that the proposed method can be applied to solve other nonlinear evolution equations.

Acknowledgments

This work is partially supported by Humanity and Social Science Youth Foundation of Ministry of Education of China (11YJCZH070). The authors would like to thank the reviewers very much for their valuable suggestions on this paper.

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