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*Research Article*

# Natural Convection of Viscoelastic Fluid from a Cone Embedded in a Porous Medium with Viscous Dissipation

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We study natural convection from a downward pointing cone in a viscoelastic fluid embedded in a porous medium. The fluid properties are numerically computed for different viscoelastic, porosity, Prandtl and Eckert numbers. The governing partial differential equations are converted to a system of fourth order ordinary differential equations using the similarity transformations and then solved together by using the successive linearization method (SLM). Many studies have been carried out on natural convection from a cone but they did not consider a cone embedded in a porous medium with linear surface temperature. The results in this work are validated by the comparison with other authors.

## 1. Introduction

Natural convection of viscoelastic fluid in a porous medium with viscous dissipation is the transfer of heat due to density differences caused by temperature gradients through a permeable medium and heat generated due to the interaction of fluid molecules is considered. There are examples in practical application such as thermal insulation, extraction of petroleum resources and the so-called fracking, metal processing, performance of lubricants, application of paints, and extrusion of plastic sheets. The study of second grade fluids has been studied but there is no single constitutive equation that can fully describe non-Newtonian fluids [1]; due to this fact many authors did not consider the appropriate constitutive energy equation for second grade fluids.

Natural convection on a cone geometry has been studied by among others Alim et al. [2], Awad et al. [3], Cheng [4, 5], and Kairi and Murthy [6]. Studies have been done on other geometries such as flow over a flat plate, cylinders, vertical surfaces, stretching sheets, and inclined surfaces by, among others, Abbas et al. [7] who considered unsteady second grade fluid flow on an unsteady stretching sheet; they did not consider the energy equation mainly due to difficulties in its characterization. Anwar et al. [8] studied mixed convection boundary layer flow of a viscoelastic fluid over a horizontal circular cylinder; they solved the fourth order ordinary differential equations by considering the insufficiency of the boundary conditions by taking the zeroth, first, and second order of the viscoelastic parameter and coming up with three systems of ordinary differential equations. Cortell [9] investigated flow and heat transfer of a viscoelastic fluid over a stretching sheet. Damseh et al. [10] studied the transient mixed convection flow of a second grade viscoelastic fluid over a vertical surface. They used McCormack's method to solve their differential equations. Hayat et al. [11] studied mixed convection in a stagnation point flow adjacent to a vertical surface in a viscoelastic fluid.

The model in this work has been originally developed from the work of Ece [5] who studied heat and mass transfer from a downward pointing cone in a Newtonian fluid. In this paper the work of Ece [5] is extended to take into account the flow of a second grade fluid in a porous medium and the effect of viscous dissipation is considered. Several other studies have been done in natural convection in a viscoelastic fluid by among others Hsiao [12] who studied mixed convection for viscoelastic fluid past a porous wedge. Kasim et al. [13] investigated free convection boundary layer flow of a viscoelastic fluid in the presence of heat generation. Massoudi et al. [14]

studied natural convection flow of generalized second grade fluid between two vertical walls. Olajuwon [15] studied the convection heat and mass transfer in a hydromagnetic flow of a second grade fluid in the presence of thermal radiation and thermal diffusion; it was shown that increasing the second grade parameter causes reduction in the rate of the fluid flow and mass transfer, but heat transfer increases. Sajid et al. investigated fully developed mixed convection flow of a viscoelastic fluid between permeable parallel vertical plates [16].

Studies for viscous dissipation in a second grade fluid have been done by many authors but some assumed that fluids are more viscous than elastic resulting in the energy equation without the elastic term. Viscous dissipation has been studied by among others Subhas Abel et al. [17] who studied viscoelastic MHD flow and heat transfer over a stretching sheet with viscous and ohmic dissipations and [18] in which a Newtonian fluid was considered. The viscous dissipation term which they used in [17] assumes that the fluid is more viscous in nature than elastic. Jha [19] investigated the effects of viscous dissipation on natural convection flow between parallel plates with time periodic boundary conditions. Chen [20] studied the analytic solution of MHD flow and heat transfer for two types of viscoelastic fluids over a stretching sheet with energy dissipation, internal heat source, and thermal radiation. Cortell [21] worked on viscous dissipation and thermal radiation effects on the flow and heat transfer of a power law fluid past an infinite porous plate. Hsiao [22] investigated multimedia physical feature for unsteady MHD mixed convection viscoelastic fluid over a vertical stretching sheet with viscous dissipation. Kameswaran et al. [23] studied hydromagnetic nanofluid flow due to a stretching sheet or shrinking sheet with viscous dissipation and chemical reaction effects.

Studies have been done in porous media by among others Awad et al. [3, 4, 6, 24] and Singh and Agarwal [25] who studied heat transfer in a second grade fluid over an exponentially stretching sheet through porous medium with thermal radiation and elastic deformation under the effect of magnetic field.

An investigation of available literature shows that, to the best of our knowledge, no analysis has been done on natural convection of a viscoelastic fluid embedded in a porous medium with viscous dissipation under the given boundary conditions. The study takes into consideration a temperature that changes linearly along the surface of the cone (see Ece [5]).

## 2. Mathematical Formulation

A cone in a viscoelastic fluid embedded in a porous medium is heated and maintained at a linearly changing temperature $T$ ($> T_\infty$), and the ambient conditions are maintained at $T_\infty$; the fluid has a constant viscosity $\nu$. The vertex angle of the cone is $2\phi$. The velocity components $u$ and $v$ are in the directions of $x$ and $y$, respectively, with the $x$-axis being inclined at an angle $\phi$ to the vertical. A sketch of the system and coordinate axis is illustrated in Figure 1.

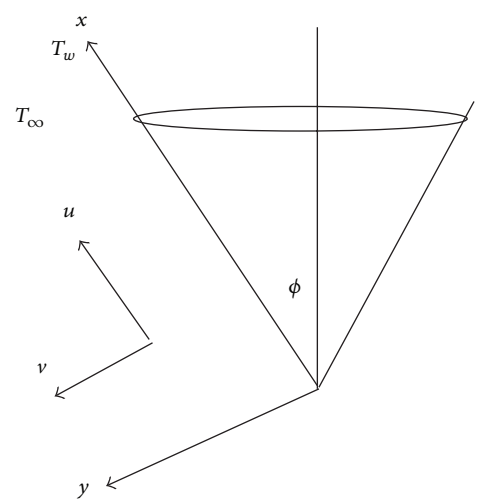


Figure 1: Physical model and coordinate system.

The governing equations in this buoyant-driven flow are given by

$$\begin{aligned}
&\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0,\\
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K}u - k_o\left\{u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right.\\
&\qquad\left. + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}\right\}\\
&\qquad + g\beta(T - T_\infty)\cos\phi,\\
&u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2\\
&\qquad + \frac{k_0}{\rho C_p}\left(u\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial y}\right),
\end{aligned} \tag{1}$$

where $r = x\sin\phi$, $g$ is the acceleration due to gravity, $\nu$ is the kinematic viscosity for the fluid, $k_o$ is the non-Newtonian parameter of the viscoelastic fluid, $\beta$ is the coefficient of thermal expansion, $\alpha$ is the thermal diffusivity, $C_p$ is the specific heat capacity for the fluid, $\rho$ is the density of the fluid, and $K$ is the permeability coefficient of the porous medium. The boundary conditions are given as

$$\begin{aligned}
&u = v = 0, \qquad T = T_w(x) = T_\infty + A\left(\frac{x}{L}\right) \quad \text{at } y = 0,\\
&\frac{\partial u}{\partial y}, \quad u \longrightarrow 0, \quad T \longrightarrow T_\infty, \quad \text{as } y \longrightarrow \infty,
\end{aligned} \tag{2}$$

where $A > 0$ is a constant, $L > 0$ is the characteristic length, and the subscript $\infty$ refers to the ambient condition.

We introduce the nondimensional variables:

$$X = \frac{x}{L}, \qquad Y = \frac{\mathrm{Gr}^{1/4} y}{L}, \qquad R = \frac{r}{L},$$
$$U = \frac{u}{U_0}, \qquad V = \frac{\mathrm{Gr}^{1/4} v}{U_0}, \tag{3}$$
$$\overline{T} = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \mathrm{Gr} = \left(\frac{U_0 L}{\nu}\right)^2,$$

where $U_0 = [g\beta \cos\phi L(T_w - T_\infty)]^{1/2}$. Using (3) in (1) gives the following equations:

$$\frac{\partial}{\partial X}(RU) + \frac{\partial}{\partial Y}(RV) = 0,$$
$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} - \frac{\nu U}{K} - \Lambda\left\{U\frac{\partial^3 U}{\partial X \partial Y^2} + V\frac{\partial^3 U}{\partial Y^3} + \frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial Y^2} - \frac{\partial^2 U}{\partial X \partial Y}\frac{\partial U}{\partial Y}\right\} + \overline{T},$$
$$U\frac{\partial \overline{T}}{\partial X} + V\frac{\partial \overline{T}}{\partial Y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \overline{T}}{\partial Y^2} + \mathrm{Ec}\left(\frac{\partial U}{\partial Y}\right)^2 + \Lambda \mathrm{Ec}\left(U\frac{\partial^2 U}{\partial X \partial Y}\frac{\partial U}{\partial Y} + V\frac{\partial^2 U}{\partial Y^2}\frac{\partial U}{\partial Y}\right), \tag{4}$$

where $R = X\sin\phi$, $\Lambda = (k_0 U_0/\nu L)$ is the viscoelastic parameter known as the Deborah number, Gr is the Grashof number, $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number, and $\mathrm{Ec} = (U_0^2/C_p A)$ is the Eckert number. The corresponding boundary conditions are given as

$$U = V = 0, \quad \overline{T} = X \quad \text{at } Y = 0,$$
$$\frac{\partial U}{\partial Y}, \quad U \longrightarrow 0, \quad \overline{T} \longrightarrow 0 \quad \text{as } Y \longrightarrow \infty. \tag{5}$$

We now introduce the stream functions $\psi = XRf(Y)$ and $\overline{T} = X\theta(Y)$ defined by

$$U = \frac{1}{R}\frac{\partial \psi}{\partial Y}, \qquad V = -\frac{1}{R}\frac{\partial \psi}{\partial X}. \tag{6}$$

Substituting (6) and the similarity variables in (4) gives the following ordinary differential equations:

$$f''' + 2ff'' - (f')^2 + \theta - \gamma f' - \Lambda\left(2f'f''' - 2ff^{iv} - (f'')^2\right) = 0, \tag{7}$$
$$\theta'' + \mathrm{Pr}\left(2f\theta' - f'\theta\right) + \mathrm{Pr}\,\mathrm{Ec} f''^2 + \Lambda \mathrm{Pr}\,\mathrm{Ec}\left(f'f''^2 - ff''f'''\right) = 0. \tag{8}$$

With boundary conditions,

$$f(0) = f'(0) = 0, \quad \theta(0) = 1, \tag{9}$$
$$f'(\infty) \longrightarrow 0, \quad f''(\infty) \longrightarrow 0, \quad \theta(\infty) \longrightarrow 0. \tag{10}$$

It is of interest to discuss the skin friction and the heat transfer coefficient in this context. The shear stress at the surface of the cone is defined as (see Olajuwon [15])

$$\tau_w = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0} + k_0\left[u\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]_{y=0}, \tag{11}$$

where $\mu$ is the coefficient of viscosity. The skin friction is defined as

$$c_f = \frac{\tau_w}{(1/2)\rho U_\infty^2},$$
$$c_f = \frac{2X}{\mathrm{Gr}^{1/4}} f''(0)\left(1 + 3\Lambda f'(0)\right). \tag{12}$$

The skin friction coefficient can be expressed as

$$\frac{C_f \mathrm{Gr}^{1/4}}{2X} = f''(0). \tag{13}$$

The heat transfer rate at the surface of the cone is given by

$$q_w = -\frac{k}{X}\left[\frac{\partial T}{\partial y}\right]_{y=0}. \tag{14}$$

The Nusselt number can be expressed as

$$\mathrm{Nu} = \frac{L q_w}{k(T_w - T_\infty)}. \tag{15}$$

Using the nondimensional variables (9)-(10), the dimensionless wall heat rate is given by

$$\mathrm{Nu}\,\mathrm{Gr}^{-1/4} = -\theta'(0). \tag{16}$$

## 3. Method of Solution

In this study, (7)–(10) were solved using the successive linearization method. The inclusion of the non-Newtonian term brings about the fourth order ordinary differential equation for the momentum equation. The given boundary conditions are insufficient to obtain a unique solution. To overcome this problem the system is decomposed into the zeroth, first, and second order systems of the viscoelastic parameter. Subhas Abel et al. [17] showed that if this method is applied small values of the viscoelastic parameter can be used without difficulty in convergence. It is also noticed in this study that the direct application of the successive linearization method has difficulties in convergence for small values of the viscoelastic parameter. Anwar et al. [8] also confirmed the same observation and solved a system of differential equations simultaneously and obtained better convergence for small values of the viscoelastic parameter. In this work we solve the system using the successive linearization method. To solve the equations we seek the series solution of the form

$$\begin{aligned} f(y) &= f_0(y) + \Lambda f_1(y) + \Lambda^2 f_2(y) + \cdots, \\ \theta(y) &= \theta_0(y) + \Lambda\theta_1(y) + \Lambda^2\theta_2(y) + \cdots. \end{aligned} \tag{17}$$

The skin friction can be computed using

$$f''(0) = f_0''(0) + \Lambda f_1''(0) + \Lambda^2 f_2''(0) + \cdots. \tag{18}$$

Then substituting (17) into the system (7)–(10). We then take the zeroth, first, and second order of the viscoelastic parameter $\Lambda$. We obtain the following system.

Zeroth order:

$$f_0''' + 2f_0 f_0'' - f_0'^2 + \theta_0 - \gamma f_0' = 0, \tag{19}$$

$$\theta'' + 2\Pr f_0\theta_0' - \Pr f_0'\theta_0 + \Pr \mathrm{Ec} f_0''^2 = 0, \tag{20}$$

$$f_0(0) = 0, \qquad f_0'(0) = 0, \qquad \theta_0(0) = 1, \tag{21}$$

$$f_0'(\infty) = 0, \qquad \theta_0(\infty) = 0. \tag{22}$$

First order:

$$\begin{aligned} &f_1''' + 2f_0 f_1'' + 2f_1 f_0'' - 2f_0' f_1' + \theta_1 \\ &\quad - \gamma f_1' - 2f_0' f_0''' + 2f_0 f_0^{iv} + f_0''^2 = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &\theta_1'' + 2\Pr f_0\theta_1 + 2\Pr f_1\theta_0' - \Pr f_0'\theta_1 - \Pr f_1'\theta_0 \\ &\quad + 2\Pr \mathrm{Ec} f_0'' f_1'' + \Pr \mathrm{Ec} f_0' f_0''^2 - \Pr \mathrm{Ec} f_0 f_0'' f_0''' = 0, \end{aligned} \tag{24}$$

$$f_1(0) = 0, \qquad f_1'(0) = 0, \qquad f_1'(\infty) = 0, \tag{25}$$

$$\theta_1(0) = 0, \qquad \theta_1(\infty) = 0. \tag{26}$$

Second order:

$$\begin{aligned} &f_2''' + 2f_0 f_2'' + 2f_1 f_1'' + 2f_2' f_0'' - 2f_0' f_2' - f_1'^2 + \theta_2 - \gamma f_2' \\ &\quad - 2f_0' f_1''' - 2f_1' f_0''' + 2f_0 f_1^{iv} + 2f_1 f_0^{iv} + 2f_0'' f_1'' = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &\theta_2'' + 2\Pr f_0\theta_2' + 2\Pr f_1\theta_1' + 2\Pr f_2\theta_0' - \Pr f_0'\theta_2 - \Pr f_1'\theta_1 \\ &\quad + \Pr \mathrm{Ec}\left(2f_0'' f_2'' + f_1''^2 + 2f_0' f_0'' f_1'' + f_1' f_0''^2 + f_2' f_0''^2\right. \\ &\quad \left. + f_0 f_0'' f_1''' + f_0 f_1'' f_0''' + f_1 f_0'' f_0'''\right) = 0, \end{aligned} \tag{28}$$

$$f_2(0) = 0, \qquad f_2'(0) = 0, \qquad f_2'(\infty) = 0, \tag{29}$$

$$\theta_2(0) = 0, \qquad \theta_2(\infty) = 0. \tag{30}$$

The functions in the system (19)–(30) may be expanded in series form as

$$\begin{aligned} f_0(y) &= f_{0i}(y) + \sum_{m=0}^{i-1} f_{0m}(y), \\ \theta_0(y) &= \theta_{0i}(y) + \sum_{m=0}^{i-1} \theta_{0m}(y), \\ f_1(y) &= f_{1i}(y) + \sum_{m=0}^{i-1} f_{1m}(y), \\ \theta_1(y) &= \theta_{1i}(y) + \sum_{m=0}^{i-1} \theta_{1m}(y), \\ f_2(y) &= f_{2i}(y) + \sum_{m=0}^{i-1} f_{2m}(y), \\ \theta_2(y) &= \theta_{2i}(y) + \sum_{m=0}^{i-1} \theta_{2m}(y), \end{aligned} \tag{31}$$

where $f_{0i}$, $f_{1i}$, and $f_{2i}$ and $\theta_{0i}$, $\theta_{1i}$, and $\theta_{2i}$ ($i = 1, 2, 3, \ldots$) are unknown functions and $f_{0m}$, $f_{1m}$, and $f_{2m}$ and $\theta_{0m}$, $\theta_{1m}$, and $\theta_{2m}$ are approximations that are found by successively solving the linear part of equations that are obtained after substituting (31) into system (19)–(30). These linear equations have the form

$$f_{0i}''' + a_{01,i-1} f_{0i}'' + a_{02,i-1} f_{0i}' + a_{03,i-1} f_{0i} + a_{04,i-1}\theta_{0i} = r_{01,i-1}, \tag{32}$$

$$\begin{aligned} &\theta_{0i}'' + b_{01,i-1}\theta_{0i}' + b_{02,i-1}\theta_{0i} + b_{03,i-1} f_{0i}'' \\ &\quad + b_{04,i-1} f_{0i}' + b_{05,i-1} f_{0i} = r_{02,i-1}, \end{aligned} \tag{33}$$

$$f_{0i}(0) = 0, \qquad f_{0i}'(0) = 0, \qquad \theta_{0i}(0) = 0, \tag{34}$$

$$f_{0i}'(\infty) = 0, \qquad \theta_{0i}(\infty) = 0, \qquad f_{0i}''(\infty) = 0, \tag{35}$$

$$f_{1i}''' + a_{11,i-1} f_{1i}'' + a_{12,i-1} f_{1i}' + a_{13,i-1} f_{1i} + a_{14,i-1}\theta_{1i} = r_{11,i-1}, \tag{36}$$

$$\begin{aligned}\theta_{1i}'' + b_{11,i-1}\theta_{1i}' + b_{12,i-1}\theta_{1i} + b_{13,i-1} f_{1i}'' \\ + b_{14,i-1} f_{1i}' + b_{15,i-1} f_{1i} = r_{12,i-1},\end{aligned} \tag{37}$$

$$f_{1i}(0) = 0, \qquad f_{1i}'(0) = 0, \qquad \theta_{1i}(0) = 0, \tag{38}$$

$$f_{1i}'(\infty) = 0, \qquad \theta_{1i}(\infty) = 0, \qquad f_{1i}''(\infty) = 0, \tag{39}$$

$$f_{2i}''' + a_{21,i-1} f_{2i}'' + a_{22,i-1} f_{2i}' + a_{23,i-1} f_{2i} + a_{24,i-1}\theta_{2i} = r_{21,i-1}, \tag{40}$$

$$\begin{aligned}\theta_{2i}'' + b_{21,i-1}\theta_{2i}' + b_{22,i-1}\theta_{2i} + b_{23,i-1} f_{2i}'' \\ + b_{24,i-1} f_{2i}' + b_{25,i-1} f_{2i} = r_{22,i-1},\end{aligned} \tag{41}$$

$$f_{2i}(0) = 0, \qquad f_{2i}'(0) = 0, \qquad \theta_{2i}(0) = 0, \tag{42}$$

$$f_{2i}'(\infty) = 0, \qquad \theta_{2i}(\infty) = 0, \qquad f_{2i}''(\infty) = 0. \tag{43}$$

The coefficients $a_{jk,i-1}, b_{jk,i-1}$ $(j = 0, 1, 2,\ k = 1, \ldots, 5)$, $r_{j1,i-1}$, and $r_{j2,i-1}$ are defined as

$$a_{01,i-1} = a_{11,i-1} = a_{21,i-1} = 2\sum_{m=0}^{i-1} f_{0m},$$

$$a_{02,i-1} = a_{12,i-1} = a_{22,i-1} = -\left(2\sum_{m=0}^{i-1} f_{0m}' + \gamma\right),$$

$$a_{03,i-1} = a_{13,i-1} = a_{23,i-1} = 2\sum_{m=0}^{i-1} f_{0m}'',$$

$$a_{04,i-1} = a_{14,i-1} = a_{24,i-1} = \mathbf{I},$$

$$b_{01,i-1} = b_{11,i-1} = b_{21,i-1} = 2\Pr\sum_{m=0}^{i-1} f_{0m},$$

$$b_{02,i-1} = b_{12,i-1} = b_{22,i-1} = -\Pr\sum_{m=0}^{i-1} f_{0m}',$$

$$b_{03,i-1} = b_{13,i-1} = b_{23,i-1} = \Pr\mathrm{Ec}\sum_{m=0}^{i-1} f_{0m}'',$$

$$b_{04,i-1} = b_{14,i-1} = b_{24,i-1} = -\Pr\sum_{m=0}^{i-1}\theta_{0m},$$

$$b_{05,i-1} = b_{15,i-1} = b_{25,i-1} = 2\Pr\sum_{m=0}^{i-1}\theta_{0m}',$$

$$\begin{aligned} r_{01,i-1} = -\Bigg[&\sum_{m=0}^{i-1} f_{0m}''' + 2\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1} f_{0m}'' - \left(\sum_{m=0}^{i-1} f_{0m}'\right)^2 \\ &- \sum_{m=0}^{i-1}\theta_{0m} - \gamma\sum_{m=0}^{i-1} f_{0m}'\Bigg],\end{aligned}$$

$$\begin{aligned} r_{02,i-1} = -\Bigg[&\sum_{m=0}^{i-1}\theta_{0m}'' + 2\Pr\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1}\theta_{0m}' \\ &-\Pr\sum_{m=0}^{i-1} f_{0m}'\sum_{m=0}^{i-1}\theta_{0m} + \Pr\mathrm{Ec}\left(\sum_{m=0}^{i-1} f_{0m}''\right)^2\Bigg],\end{aligned}$$

$$\begin{aligned} r_{11,i-1} = -\Bigg[&\sum_{m=0}^{i-1} f_{1m}''' + 4\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1} f_{1m}' + 4\sum_{m=0}^{i-1} f_{1m}\sum_{m=0}^{i-1} f_{0m}'' \\ &- 6\sum_{m=0}^{i-1} f_{0m}''\sum_{m=0}^{i-1} - \gamma\sum_{m=0}^{i-1} f_{0m}' + 3\left(\sum_{m=0}^{i-1} f_{0m}''\right)^2 \\ &+6\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1} f_{0m}^{iv} - 6\sum_{m=0}^{i-1} f_{0m}'\sum_{m=0}^{i-1} f_{0m}''' + \sum_{m=0}^{i-1}\Bigg],\end{aligned}$$

$$\begin{aligned} r_{12,i-1} = -\Bigg[&\sum_{m=0}^{i-1}\theta_{1m}'' + 4\Pr\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1}\theta_{1m}' \\ &- 2\Pr\sum_{m=0}^{i-1} f_{0m}'\sum_{m=0}^{i-1}\theta_{1m} - 2\Pr\sum_{m=0}^{i-1} f_{1m}'\sum_{m=0}^{i-1}\theta_{0m} \\ &+ 4\Pr\sum_{m=0}^{i-1} f_{1m}\sum_{m=0}^{i-1}\theta_{0m}' + 4\Pr\mathrm{Ec}\sum_{m=0}^{i-1} f_{0m}''\sum_{m=0}^{i-1} f_{1m}'' \\ &+ 4\Pr\mathrm{Ec}\sum_{m=0}^{i-1} f_{0m}'\left(\sum_{m=0}^{i-1} f_{0m}\right)^2 \\ &-4\Pr\mathrm{Ec}\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1} f_{0m}''\sum_{m=0}^{i-1} f_{0m}'''\Bigg],\end{aligned}$$

$$\begin{aligned} r_{21,i-1} = -\Bigg[&\sum_{m=0}^{i-1} f_{2}''' + 4\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1} f_{2m}'' + 6\sum_{m=0}^{i-1} f_{1m}\sum_{m=0}^{i-1} f_{1m}'' \\ &- 4\sum_{m=0}^{i-1} f_{2m}\sum_{m=0}^{i-1} f_{0m}'' - 4\sum_{m=0}^{i-1} f_{2m}'\sum_{m=0}^{i-1} f_{0m}' \\ &- 3\left(\sum_{m=0}^{i-1} f_{1m}'\right)^2 - \gamma\sum_{m=0}^{i-1} f_{2m}' + \sum_{m=0}^{i-1}\theta_{2m} \\ &- 6\sum_{m=0}^{i-1} f_{1m}'''\sum_{m=0}^{i-1} f_{0m}' - 6\sum_{m=0}^{i-1} f_{0m}'''\sum_{m=0}^{i-1} f_{1m}' \\ &+ 6\sum_{m=0}^{i-1} f_{1m}^{iv}\sum_{m=0}^{i-1} f_{0m} + 6\sum_{m=0}^{i-1} f_{0m}^{iv}\sum_{m=0}^{i-1} f_{1m} \\ &+6\sum_{m=0}^{i-1} f_{1m}''\sum_{m=0}^{i-1} f_{0m}''\Bigg],\end{aligned}$$

$$r_{22,i-1} = -\Bigg[\sum_{m=0}^{i-1}\theta_{2m}'' + 4\Pr\sum_{m=0}^{i-1} f_{0m}\sum_{m=0}^{i-1}\theta_{2m}'$$

$$
\begin{aligned}
&- 6\Pr \sum_{m=0}^{i-1} f'_{1m} \sum_{m=0}^{i-1} \theta'_{1m} - 6\Pr \sum_{m=0}^{i-1} f_{1m} \sum_{m=0}^{i-1} \theta'_{1m} \\
&+ 4\Pr \sum_{m=0}^{i-1} f_{2m} \sum_{m=0}^{i-1} f'_{0m} - 2\Pr \sum_{m=0}^{i-1} \theta_{2m} \sum_{m=0}^{i-1} f'_{0m} \\
&- 3\Pr \sum_{m=0}^{i-1} \theta_{1m} \sum_{m=0}^{i-1} f'_{1m} \\
&+ 10\Pr \operatorname{Ec} \sum_{m=0}^{i-1} f'_{0m} \sum_{m=0}^{i-1} f''_{0m} \sum_{m=0}^{i-1} f'_{1m} \\
&+ 2\left( \sum_{m=0}^{i-1} f''_{0m} \right)^2 \sum_{m=0}^{i-1} f'_{1m} + 4\left( \sum_{m=0}^{i-1} f''_{0m} \right)^2 \sum_{m=0}^{i-1} f'_{2m} \\
&+ 4 \sum_{m=0}^{i-1} f_{0m} \sum_{m=0}^{i-1} f''_{0m} \sum_{m=0}^{i-1} f'''_{1m} + 4 \sum_{m=0}^{i-1} f_{0m} \sum_{m=0}^{i-1} f''_{1m} \sum_{m=0}^{i-1} f'''_{0m} \\
&+ 4 \sum_{m=0}^{i-1} f''_{0m} \sum_{m=0}^{i-1} f'''_{0m} \sum_{m=0}^{i-1} f_{1m} \Bigg] .
\end{aligned}
\tag{44}
$$

Equations (32)–(43) must be solved simultaneously subject to certain initial approximations $f_0$ and $\theta_0$. We choose these initial approximations so that they satisfy the given boundary conditions. In this case suitable initial approximations are

$$
f_0(Y) = 1 - e^{-Y} - Ye^{-Y}, \qquad \theta_0(Y) = e^{-Y}. \tag{45}
$$

We note that when $f_i$ and $\theta_i$ $(i > 1)$ have been found, the approximate solutions $f(Y)$ and $\theta(Y)$ are obtained as

$$
f(Y) \approx \sum_{n=0}^{M} f_n(Y), \qquad \theta(Y) \approx \sum_{n=0}^{M} \theta_n(Y), \tag{46}
$$

where $M$ is the order of the SLM approximation. Equations (32) and (43) can be solved by any numerical method. In this work the equations have been solved by the Chebyshev spectral collocation method. The method of solution is fully described in Awad et al. [3]. The system of differential equations is solved simultaneously using the MATLAB SLM code.

*3.1. Results and Discussion.* The problem that is investigated in this study is the steady laminar flow and natural convection from a cone in a viscoelastic fluid in the presence of viscous dissipation in a porous medium. The coupled nonlinear differential equations (7)–(10) were solved numerically using the successive linearisation method (SLM). In this section we discuss the effects of the viscoelastic parameter ($\Lambda$), porosity parameter ($\gamma$), Prandtl number (Pr), and Eckert numbers (Ec) on both the velocity and temperature profiles.

In Table 1 the comparison between our results for the local skin friction and Nusselt numbers and those of Ece [5] who used the Thomas algorithm shows that our method gives satisfactory results, thus confirming that the method is accurate.

Table 1: Comparison of the values of $f''(0)$ obtained by SLM against the Thomas algorithm of Ece [5] when $\Lambda = 0$.

| Pr | Ece [5] | | Present | |
|---|---|---|---|---|
| | $f''(0)$ | $-\theta'(0)$ | $f''(0)$ | $-\theta'(0)$ |
| 1 | 0.681483 | 0.638855 | 0.68148334 | 0.63885473 |
| 10 | 0.433268 | 1.275499 | 0.43327820 | 1.27552877 |

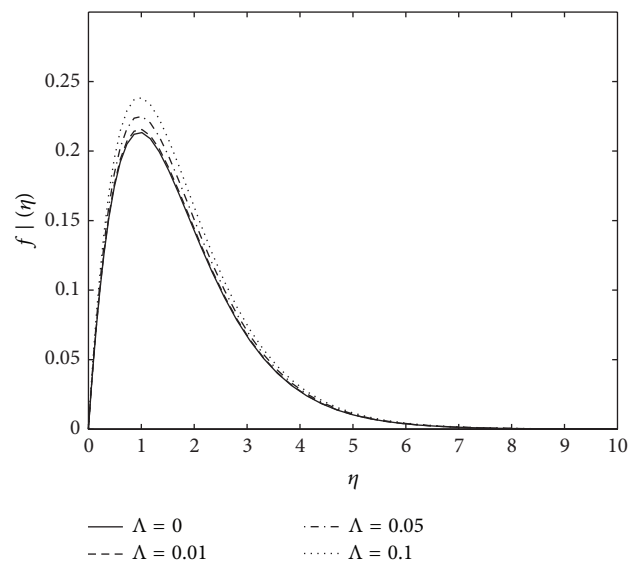


Figure 2: Velocity profiles for different values of the viscoelastic parameter $\Lambda$ at Pr = 1, Ec = 0.1, and $\gamma = 1$.

To get a clear understanding of natural convection effects on the physics of the problem of a flow from a cone in a viscoelastic fluid with viscous dissipation, the investigation has been carried out for different viscoelastic numbers $\Lambda$, porosity parameter $\gamma$, the Eckert number Ec, and the Prandtl number Pr. The results for the skin friction and heat transfer coefficients are depicted in Tables 1 and 2.

In Table 2 the effect of increasing the viscoelastic parameter increases the skin friction coefficient and the opposite effect is noted on the Nusselt number in the presence of the porous medium and viscous dissipation. Cortell [9] noted the same result. A faster increase is noted in the absence of the porous medium and the Eckert number. Increasing the porosity parameter reduces local skin friction and the same trend is noted on the Nusselt number. Skin friction increases with increasing Eckert number and the opposite trend is noted on the Nusselt number. The skin friction decreases with increasing Prandtl number, and the opposite trend is noted on the Nusselt number.

Figures 2–9 show the effects of various fluid properties on the velocity and temperature profiles.

Figure 2 shows that increasing the viscoelastic parameter increases the velocity across the boundary layer (see Butt et al. [24]).

Increasing the Prandtl number decreases the velocity profile in the boundary layer as shown in Figure 3; This is

Table 2: Effect of the viscoelastic and porosity parameters and Eckert number $\Lambda$, $\gamma$, and Ec on the local skin friction and heat transfer for Pr = 1.

| $\Lambda$ | $\gamma$ | Ec | Pr | $f''(0)$ | $-\theta'(0)$ |
|---|---|---|---|---|---|
| −0.1 | 1 | 0.1 | 1 | 0.51437649 | 0.64214087 |
| −0.05 | 1 | 0.1 | 1 | 0.53489736 | 0.59964040 |
| −0.01 | 1 | 0.1 | 1 | 0.55491411 | 0.56204002 |
| 0 | 1 | 0.1 | 1 | 0.56041829 | 0.55213993 |
| 0.01 | 1 | 0.1 | 1 | 0.56612247 | 0.54203983 |
| 0.05 | 1 | 0.1 | 1 | 0.59093920 | 0.49963495 |
| 0.1 | 1 | 0.1 | 1 | 0.62646010 | 0.44213898 |
| 0.01 | 0 | 0.1 | 1 | 0.68990728 | 0.61800108 |
| 0.01 | 1 | 0.1 | 1 | 0.56612247 | 0.54203983 |
| 0.01 | 2 | 0.1 | 1 | 0.49213165 | 0.48868800 |
| 0.01 | 3 | 0.1 | 1 | 0.44155007 | 0.44852699 |
| 0.01 | 1 | 0.1 | 1 | 0.56612247 | 0.54203983 |
| 0.01 | 1 | 0.2 | 1 | 0.56678919 | 0.53544933 |
| 0.01 | 1 | 0.3 | 1 | 0.56745956 | 0.52882111 |
| 0.01 | 1 | 0.4 | 1 | 0.56813364 | 0.52215479 |
| 0.01 | 1 | 0.1 | 0.7 | 0.59466242 | 0.47714847 |
| 0.01 | 1 | 0.1 | 1 | 0.56612247 | 0.54203983 |
| 0.01 | 1 | 0.1 | 2 | 0.50981746 | 0.68640396 |
| 0.01 | 1 | 0.1 | 10 | 0.38403617 | 1.13367723 |

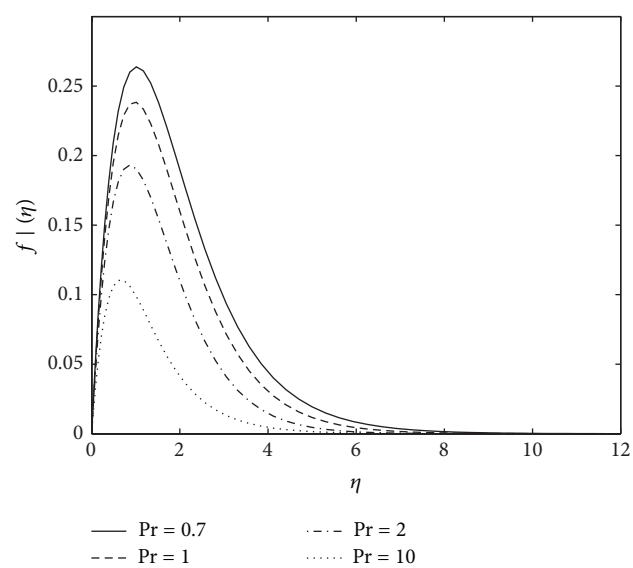


Figure 3: Velocity profiles for different values of the Prandtl number Pr at Ec = 0.1, $\gamma$ = 1, and $\Lambda$ = 0.1.

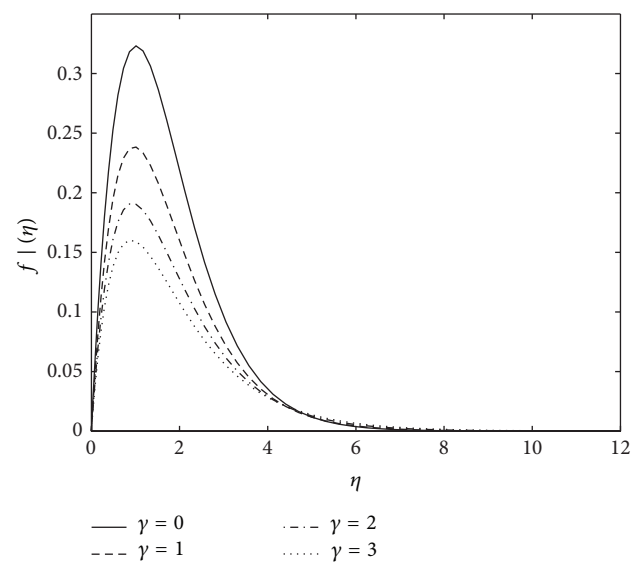


Figure 4: Velocity profiles for different values of the porosity parameter $\gamma$ at Pr = 1, Ec = 0.1, and $\Lambda$ = 0.1.

because when the Prandtl number is increased the conduction process is more enhanced than convection suggesting lower molecular motion causing fluid velocity to decrease.

Figure 4 shows the variation of the porosity parameter with velocity profile for the linear surface temperature. Increasing porosity parameter reduces the velocity profile across the boundary layer. The fluid particles move slower as the medium becomes less porous (see Singh and Agarwal [25]).

Figure 5 shows the variation of the Eckert number with velocity profile across the boundary layer. Increasing the Eckert number increases the velocity profile; this is caused by the increase in the kinetic energy caused by viscous dissipation in the boundary layer which leads to a small temperature gradient.

Figure 6 shows the effect of increasing the viscoelastic parameter on the temperature profiles. Increasing the viscoelastic parameter increases the temperature profile.

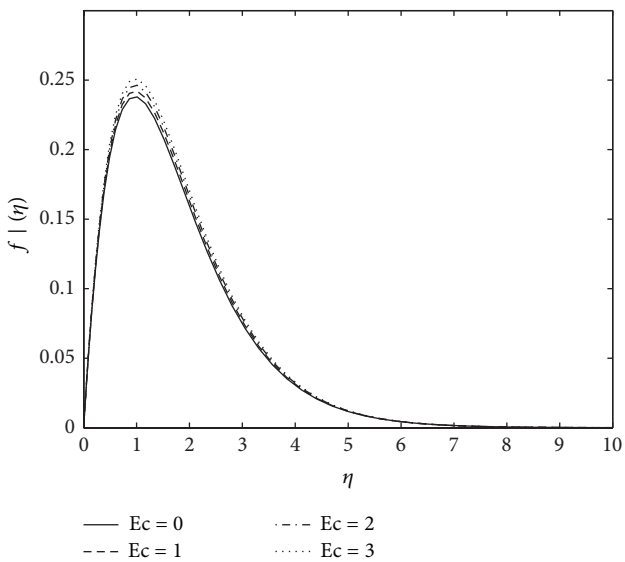


Figure 5: Velocity profiles for different values of the Eckert number Ec at Pr = 1, $\gamma$ = 1, and $\Lambda$ = 0.1.

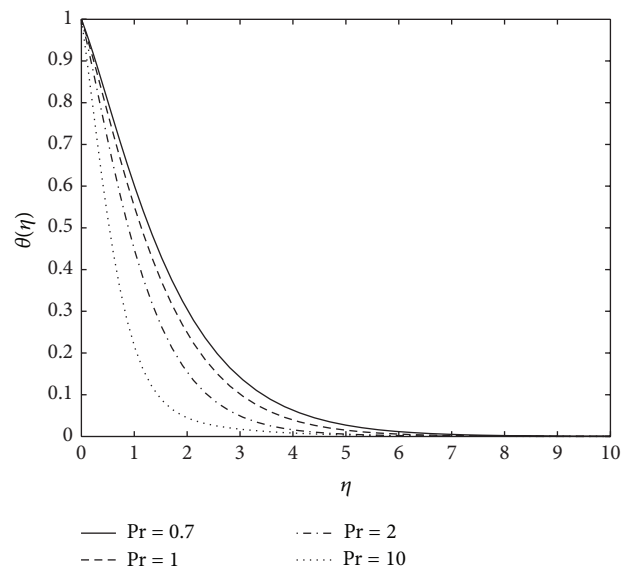


Figure 7: Temperature profiles for different values of the Prandtl number Pr at Ec = 0.1, $\gamma$ = 0.1, and $\Lambda$ = 0.1.

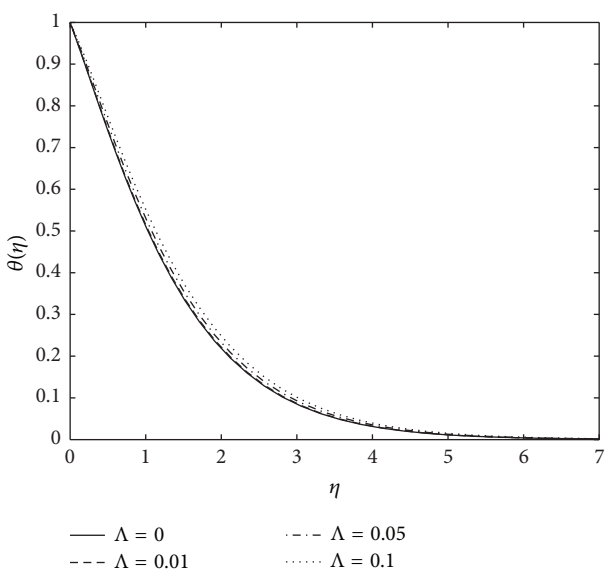


Figure 6: Temperature profiles for different values of the viscoelastic parameter $\Lambda$ at Pr = 1, $\gamma$ = 1, and Ec = 0.1.

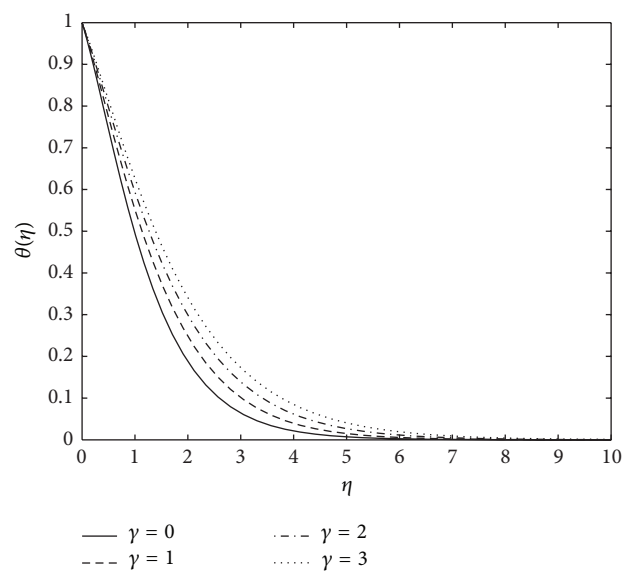


Figure 8: Temperature profiles for different values of the porosity parameter $\gamma$ at Pr = 1, $\Lambda$ = 0.1, and Ec = 0.1.

Figure 7 depicts the variation of the Prandtl number with temperature profiles. Increasing the Prandtl number decreases the temperature profile; The thermal diffusivity becomes smaller than the viscous diffusion rate causing smaller temperature profiles.

Figure 8 shows the variation of the porosity parameter with the temperature profile. Increasing the porosity parameter increases the temperature profile; when the fluid moves much slower due to the reduction in porosity heat transfer becomes more rapid.

In Figure 9 increasing the Eckert number increases the temperature profile; the heat produced due to viscous dissipation increases the temperature across the boundary layer.

Figure 10 shows the variation of the skin friction with the viscoelastic parameter at different values of the porosity parameter. Skin friction increases with increasing viscoelastic parameter and increasing the porosity parameter reduces skin friction.

Figure 11 shows the variation of the Nusselt number with the viscoelastic parameter; increasing the viscoelastic parameter reduces Nusselt number and increasing the porosity parameter reduces the Nusselt number.

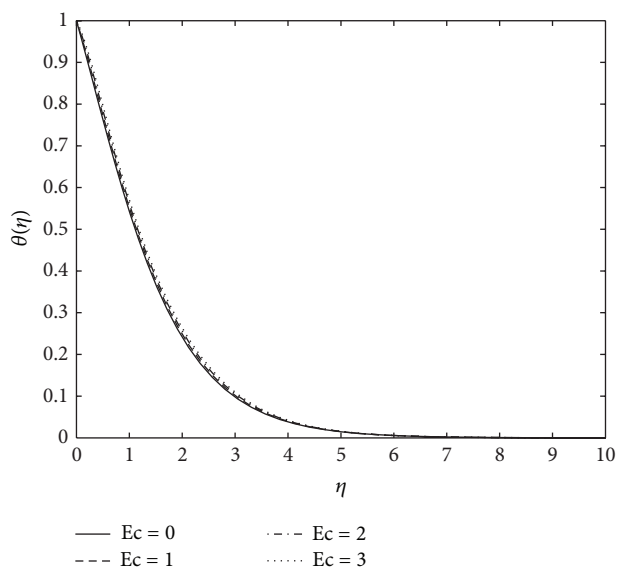


Figure 9: Temperature profiles for different values of the Eckert number Ec at Pr = 1, $\gamma = 1$, and $\Lambda = 0.1$.

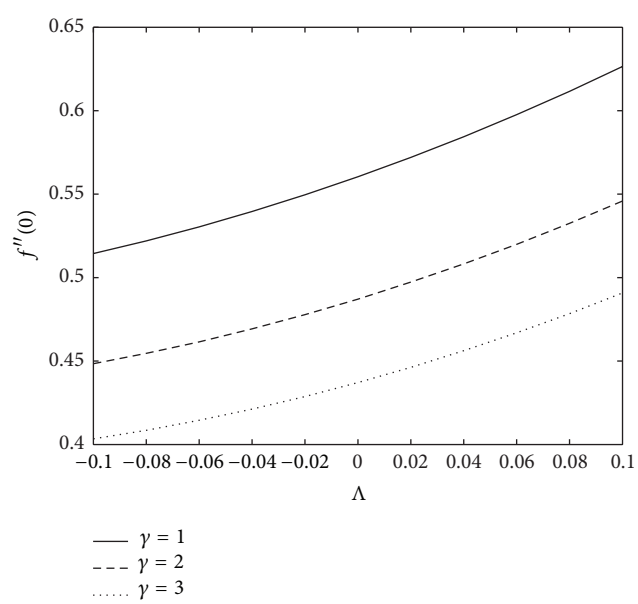


Figure 10: Skin friction $f''(0)$ versus viscoelastic parameter $\Lambda$ for different values of porosity parameter.

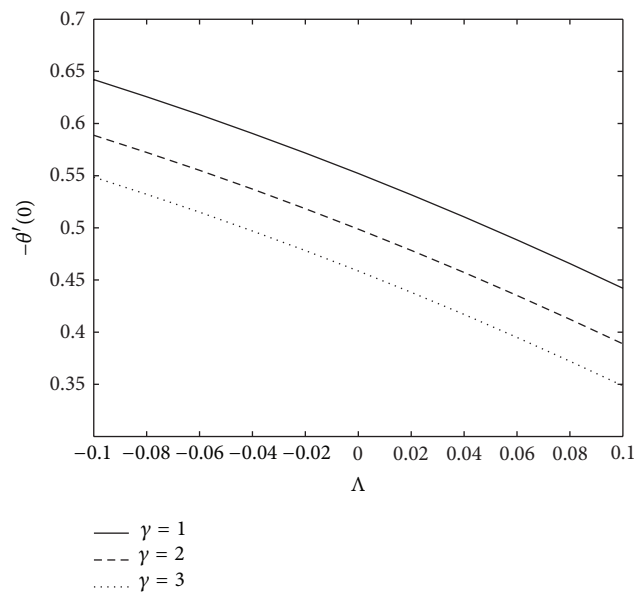


Figure 11: Nusselt number $-\theta'(0)$ versus viscoelastic parameter $\Lambda$ for different values of porosity parameter.

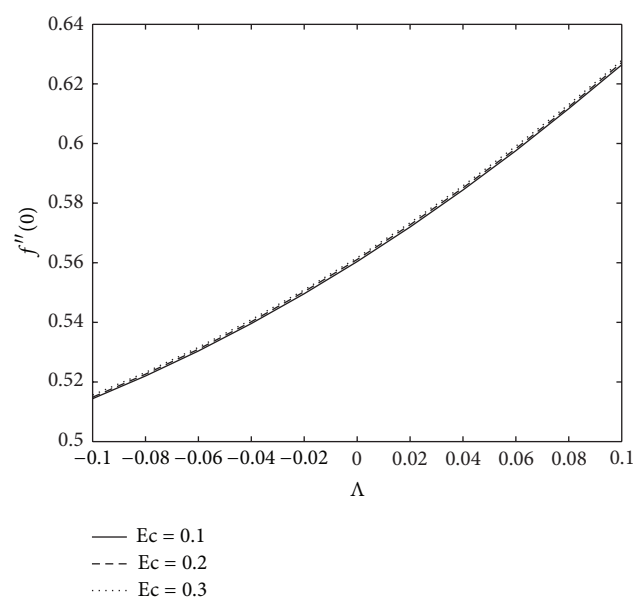


Figure 12: Skin friction $f''(0)$ versus viscoelastic parameter $\Lambda$ for different values of Eckert numbers.

Figure 12 shows the effect of increasing the Eckert number on the skin friction and viscoelastic parameter. Increasing viscoelastic parameter increases skin friction and increasing the Eckert number increases the skin friction.

In Figure 13 the increase of viscoelastic parameter reduces the Nusselt number and increasing the Eckert number reduces the Nusselt number.

Figure 14 shows that generally increasing the viscoelastic parameter increases the skin friction and increasing the Prandtl number reduces skin friction.

In Figure 15 increasing the viscoelastic parameter reduces the Nusselt number and increasing the Prandtl number increases the Nusselt number.

## 4. Conclusion

This study presented an analysis of flow and heat transfer in natural convection of viscoelastic fluid from a cone embedded in a porous medium with viscous dissipation. The nonlinear coupled governing equations were solved using the successive linearization method (SLM). The equations

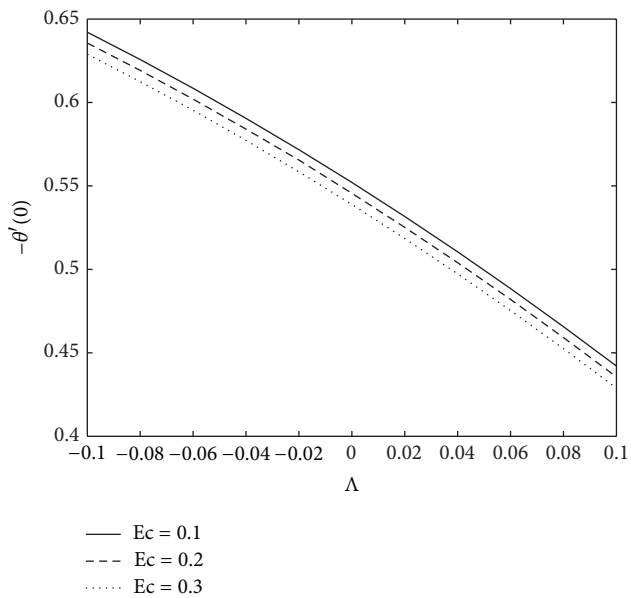


Figure 13: Nusselt number $-\theta'(0)$ versus viscoelastic parameter $\Lambda$ for different values of Eckert numbers.

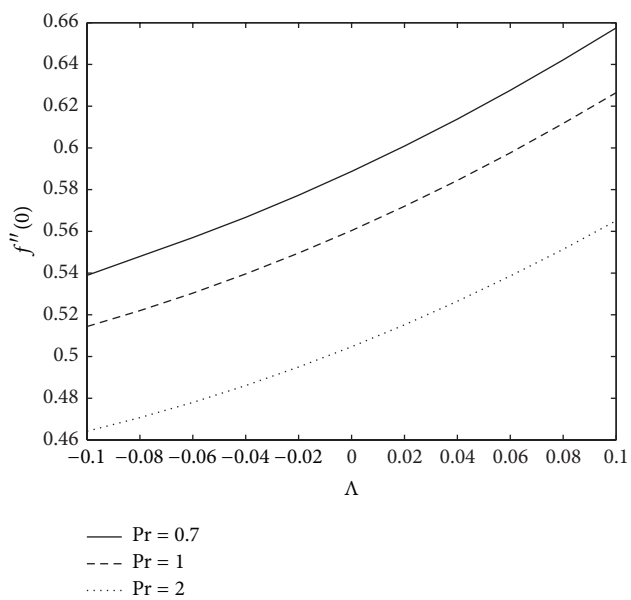


Figure 14: Skin friction $f''(0)$ versus viscoelastic parameter $\Lambda$ for different values of the Prandtl numbers.

were first split into the zeroth, first, and second order of the viscoelastic parameter and solved together under the linear surface boundary conditions. The velocity and temperature profiles together with local skin friction and local Nusselt numbers were presented and investigated. It was found that increasing the viscoelastic parameter increased the skin friction, reduced the Nusselt number, and increased the velocity and temperature profiles. Increasing the porosity parameter decreased the skin friction and Nusselt number and decreased the velocity profile and the opposite effect was noted in the temperature profile. Increasing the Eckert

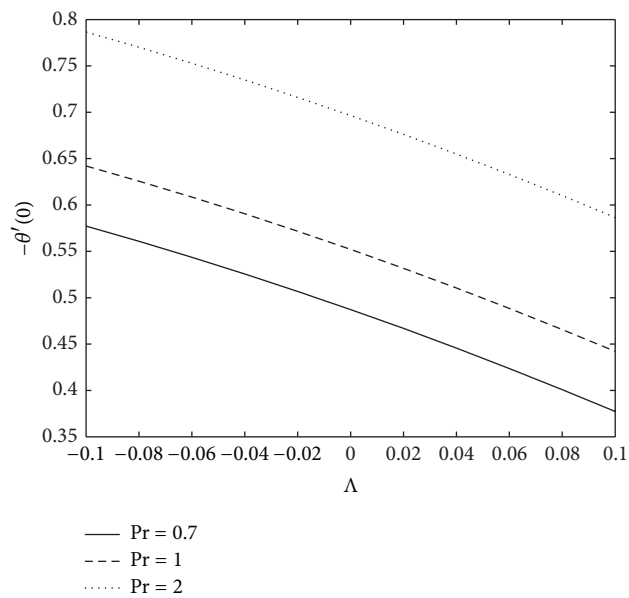


Figure 15: Nusselt number $-\theta'(0)$ versus viscoelastic parameter $\Lambda$ for different values of Prandtl numbers.

number increased both velocity and temperature profiles and decreased the Nusselt number and the opposite was noted on the skin friction. The results compared well with those of Ece [5] in case when $\gamma = \Lambda = \mathrm{Ec} = 0$.

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