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Research Article

A Comparison between Adomian's Polynomials and He's Polynomials for Nonlinear Functional Equations

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We will compare the standard Adomian decomposition method and the homotopy perturbation method applied to obtain the solution of nonlinear functional equations. We prove analytically that the two methods are equivalent for solving nonlinear functional equations. In Ghorbani (2009), Ghorbani presented a new definition which he called as He's polynomials. In this paper, we also show that He's polynomials are only the Adomian polynomials.

1. Introduction

The Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) are two powerful methods which consider the approximate solution of a nonlinear equation as an infinite series usually converging to the accurate solution. These methods have been used in obtaining analytic and approximate solutions to a wide class of linear and nonlinear, differential, and integral equations.

Öziş and Yıldırım compared Adomian's method and He's homotopy perturbation method [1] for solving certain nonlinear problems. Li also has shown that the ADM and HPM for solving nonlinear equations are equivalent [2]. In [3], Ghorbani has presented a definition which he called it as He's polynomials.

Consider the following nonlinear functional equation:

$$u = f + N(u), \tag{1}$$

where N is a nonlinear operator from Hilbert space H to H, u is an unknown function, and f is a known function in H. We are looking for a solution u of (1) belonging to H. We will suppose that (1) admits a unique solution. If (1) does not possess a unique solution, the ADM and HPM will give a solution among many (possible) other solutions. However, relatively few papers deal with the comparison of these methods with other existing techniques. In [4], a useful comparison between the decomposition method and the perturbation method showed the efficiency of the decomposition method compared to the tedious work required by the perturbation techniques. In [5], the advantage of the decomposition method over the Picard's method has been emphasized. Sadat has shown that the Adomian decomposition method and perturbation method are closely related and lead to the same solution in many heat conduction problems [6]. In [7, 8] the HPM has compared with Liao's homotopy analysis method and showed the HPM is special case of HAM, and the advantage of the HAM over the HPM has been emphasized.

In this paper, we want to prove that He's polynomials are only Adomian's polynomials. We will also show that the standard Adomian decomposition method and the standard HPM are equivalent when applied for solving nonlinear functional equations.

2. Adomian's Decomposition Method (ADM)

Let us consider the nonlinear equation (1) which can be written in the following canonical form:

$$u = f + N(u). \tag{2}$$

The standard ADM consists of representing the solution of (1) as a series

$$u(x) = \sum_{i=0}^{\infty} u_i(x), \tag{3}$$

and the nonlinear function as the decomposed form:

$$N(u(x)) = \sum_{i=0}^{\infty} A_i, \tag{4}$$

where A_n, $n = 0, 1, 2, \ldots$ are the Adomian polynomials of $u_0, u_1, \ldots, u_n$ given by [9, 10]

$$A_n = \frac{1}{n!} \frac{d^n}{dp^n} \left[N\left(\sum_{i=0}^{n} u_i p^i \right) \right]_{p=0}. \tag{5}$$

Substituting (3) and (4) into (1) yields

$$\sum_{i=0}^{\infty} u_i(x) = f + \sum_{i=0}^{\infty} A_i. \tag{6}$$

The convergence of the series in (6) gives the desired relation

$$\begin{gathered} u_0 = f, \\ u_{n+1} = A_n, \quad n = 0, 1, 2, \ldots. \end{gathered} \tag{7}$$

It should be pointed out that A_0 depends only on u_0, A_1 depends only on u_0 and u_1; A_2 depends only on u_0, u_1, and u_2, and so on. The Adomian technique is very simple in its principles. The difficulties consist in proving the convergence of the introduced series.

3. Homotopy Perturbation Method (HPM)

This is a basic idea of homotopy method which is to continuously deform a simple problem easy to solve into the difficult problem under study.

In this section, we apply the homotopy perturbation method [11–13] to the discussed problem. To illustrate the homotopy perturbation method (HPM), we consider (1) as

$$L(v) = v(x) - f(x) - N(v) = 0, \tag{8}$$

with solution $u(x)$. The basic idea of the HPM is to construct a homotopy $H(v; p) : R \times [0, 1] \to R$ which satisfies

$$\mathscr{H}(v; p) = (1 - p) F(v) + pL(v) = 0, \tag{9}$$

where $F(v)$ is a proper function with known solution which can be obtained easily. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(v) = 0$ is continuously transformed to the original problem $v - f - N(v) = 0$. From $\mathscr{H}(v; p) = 0$, we have $H(v; 0) = F(v) = 0$ and $H(v; 1) = v - f - N(v) = 0$.

It is better to take $F(v)$ as a deformation of $L(v)$. For example, in (9), $F(v) = v - f(x)$. By selecting $F(v) = v - f(x)$ we can define another convex homotopy $\mathscr{H}(v; p)$ by

$$\mathscr{H}(v; p) = v(x) - f(x) - pN(v) = 0. \tag{10}$$

The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [14, 15]. The HPM uses the embedding parameter p as a "small parameter," and writes the solution of (10) as a power series of p, that is,

$$v = v_0 + v_1 p + v_2 p^2 + \cdots. \tag{11}$$

Setting $p = 1$ results in the approximate solution of (10):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{12}$$

Substituting (11) into (10) and equating the terms with identical powers of p, we can obtain a series of equations of the following form:

$$\begin{gathered} p^0 \colon v_0 - f(x) = 0, \\ p^1 \colon v_1 - H(v_0) = 0, \\ p^2 \colon v_2 - H(v_0, v_1) = 0, \\ p^3 \colon v_3 - H(v_0, v_1, v_2) = 0, \\ \vdots, \end{gathered} \tag{13}$$

where $H(v_0, v_1, \ldots, v_j)$ depend upon $v_0, v_1, \ldots, v_j$. In view of (10) to determine $H(v_0, v_1, \ldots, v_j)$, we use [16]

$$H(v_0, v_1, \ldots, v_j) = \frac{1}{j!} \frac{\partial^j}{\partial p^j} N\left(\sum_{i=0}^{j} v_i p^i \right) \Bigg|_{p=0}. \tag{14}$$

It is obvious that the system of nonlinear equations in (13) is easy to solve, and the components v_i, $i \geq 0$ of the homotopy perturbation method can be completely determined, and the series solutions are thus entirely determined. For the convergence of the previous method we refer the reader to the work of He [12, 17, 18].

4. Equivalence between ADM and HPM

In this section, we prove that the HPM and the ADM give same solution for solving nonlinear functional equations. We also show that the He polynomials are like the Adomian polynomials. In [3], Ghorbani has presented the following definition.

Definition 1 (see [3]). The He polynomials are defined as follows:

$$H_n(v_0, \ldots, v_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} v_i p^i \right) \Bigg|_{p=0}, \quad n = 0, 1, 2, \ldots. \tag{15}$$

Note 1. Comparison between (5) and (15) has shown that the He polynomials are only Adomian's polynomials, and it is calculated like Adomian's polynomials.

Theorem 2. *Suppose that nonlinear function $N(u)$ and the parameterized representation of v are $v(p) = \sum_{i=0}^{\infty} v_i p^i$, where p is a parameter, then we have*

$$\left.\frac{\partial^n N(v(p))}{\partial p^n}\right|_{p=0} = \left.\frac{\partial^n N\left(\sum_{i=0}^{\infty} v_i p^i\right)}{\partial p^n}\right|_{p=0} = \left.\frac{\partial^n N\left(\sum_{i=0}^{n} v_i p^i\right)}{\partial p^n}\right|_{p=0}. \tag{16}$$

Proof (see [3, 19]). In Theorem 3, we prove that the He polynomials are the Adomian polynomials. □

Theorem 3. *The He polynomials which are given by* (15) *are the Adomian polynomials.*

Proof. From Taylor's expansion of $N(v)$, we have

$$N(v) = N(v_0) + N'(v_0)(v - v_0) + \frac{1}{2!}N''(v_0)(v - v_0)^2 + \cdots \tag{17}$$

substituting (11) in (17) and expanding it in terms of p leads to

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} v_i p^i\right) &= N(v_0) + N'(v_0)\left(v_1 p + v_2 p^2 + \cdots\right) \\ &\quad + \frac{1}{2!}N''(v_0)\left(v_1 p + v_2 p^2 + \cdots\right)^2 + \cdots \\ &= N(v_0) + N'(v_0) v_1 p \\ &\quad + \left(N'(v_0) v_2 + \frac{1}{2!}N''(v_0) v_1^2\right) p^2 + \cdots \\ &= H_0 + H_1 p + H_2 p^2 + \cdots, \end{aligned} \tag{18}$$

where H_i, $i = 0, 1, 2, \ldots$, depends only on $v_0, v_1, \ldots, v_i$.

In order to obtain H_n, we give n-order derivative of both sides of (18) with respect to p and let $p = 0$, that is,

$$\left.\frac{\partial^n N(v(p))}{\partial p^n}\right|_{p=0} = \left.\frac{\partial^n \sum_{i=0}^{\infty} H_i p^i}{\partial p^n}\right|_{p=0}. \tag{19}$$

According to Theorem 2

$$\begin{aligned} &\left.\frac{\partial^n N\left(\sum_{i=0}^{\infty} v_i p^i\right)}{\partial p^n}\right|_{p=0} = \left.\frac{\partial^n N\left(\sum_{i=0}^{n} v_i p^i\right)}{\partial p^n}\right|_{p=0}, \\ &\left.\frac{\partial^n \sum_{i=0}^{\infty} H_i p^i}{\partial p^n}\right|_{p=0} = \left.\frac{\partial^n \sum_{i=0}^{n} H_i p^i}{\partial p^n}\right|_{p=0} = n! H_n. \end{aligned} \tag{20}$$

We know that H_i just depends on $v_0, v_1, \ldots, v_i$ so $(\partial^n \sum_{i=0}^{n} H_i p^i)/\partial p^n|_{p=0} = n! H_n$. Substituting (20) in (19) leads us to find H_i in the following form:

$$H_n = \frac{1}{n!}\left.\frac{\partial^n N\left(\sum_{i=0}^{n} v_i p^i\right)}{\partial p^n}\right|_{p=0} \tag{21}$$

□

which is called for the first time by Ghorbani as the He polynomials [3]!

Theorem 4. *The homotopy perturbation method for solving nonlinear functional equations is the Adomian decomposition method with the homotopy $\mathscr{H}(v; p)$ given by*

$$\mathscr{H}(v; p) = v - f(x) - pN(v). \tag{22}$$

Proof. Substituting (11) and (18) into (10) and equating the terms with the identical powers of p, we have

$$\begin{aligned} \mathscr{H}(v; p) &= \sum_{i=0}^{\infty} v_i p^i - f(x) - p\sum_{i=0}^{\infty} H_i p^i = 0, \\ \mathscr{H}(v; p) &= v_0 - f(x) + \sum_{i=0}^{\infty}(v_{i+1} - H_i) p^{i+1} = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} p^0&: v_0 - f(x) = 0, \\ p^{n+1}&: v_{n+1} - H_n = 0, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{24}$$

From (24) we have

$$\begin{aligned} v_0 &= f(x), \\ v_{n+1} &= H_n, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{25}$$

According to Theorem 3 we have $H_n = A_n$. Substituting (25) in (11) leads us to

$$\begin{aligned} v &= v_0 + v_1 p + v_2 p^2 + \cdots \\ &= f(x) + A_0 p + A_1 p^2 + \cdots, \end{aligned} \tag{26}$$

so

$$\begin{aligned} \lim_{p \to 1} v &= f(x) + A_1 + A_2 + \cdots \\ &= f(x) + \sum_{i=0}^{\infty} A_i = \sum_{i=0}^{\infty} u_i = u. \end{aligned} \tag{27}$$

Therefore, by letting

$$\mathscr{H}(v; p) = v - f(x) - pN(v), \tag{28}$$

we observe that the power series $v_0 + v_1 p + v_2 p^2 + \cdots$ corresponds to the solution of the equation $\mathscr{H}(v; p) = v - f(x) - pN(v) = 0$ and becomes the approximate solution of (1) if $p \to 1$. This shows that the homotopy perturbation method is the Adomian decomposition method with the homotopy $\mathscr{H}(v; p)$ given by (28). The proof of Theorem 4 is completed. □

These two approaches give the same equations for high-order approximations. This is mainly because the Taylor series of a given function is unique, which is a basic theory in calculus. Thus, nothing is new in Ghorbani's definition, except the new name "He's polynomial." He just employed the early ideas of ADM.

Example 5 (see [20]). Consider the following nonlinear Volterra integral equation:

$$y(x) = x + \int_0^x y^2(t)\,dt, \tag{29}$$

with the exact solution $y(x) = \tan x$.

We apply standard ADM and HPM. For applying standard HPM, we construct following homotopy:

$$\mathscr{H}(u;p) = u(x) - x - p\int_0^x [u(t)]^2 dt = 0. \tag{30}$$

In view of (13), we have

$$\begin{aligned} &p^0\colon v_0(x) - x = 0, \\ &p^n\colon v_{n+1}(x) - \int_0^x H(v_0, v_1, \ldots, v_n)\,dt = 0, \quad n \geq 0. \end{aligned} \tag{31}$$

Now if we apply ADM for solving (29), substituting (3) and (4) in (29) leads to

$$\sum_{i=0}^{\infty} u_i(x) = x + \int_0^x \sum_{i=0}^{\infty} A_i dt. \tag{32}$$

In view of (7), we have following recursive formula:

$$\begin{aligned} &u_0(x) = x, \\ &u_{n+1}(x) = \int_0^x A_n dt \quad n \geq 0. \end{aligned} \tag{33}$$

According to Theorem 3, we have $A_n = H(v_0, v_1, \ldots, v_n)$. By solving (31) and (33), we have

$$\begin{aligned} u(x) &= \sum_{i=0}^{\infty} u_i(x) = \lim_{p\to 1} v_0 + v_1 p + v_2 p^2 + \cdots \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots = \tan x. \end{aligned} \tag{34}$$

5. Conclusion

It has been shown that the standard HPM provides exactly the same solutions as the standard Adomian decomposition method for solving functional equations. It has been proved that He's polynomials are only Adomian's polynomials with different name.

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