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Research Article

Analysis of Sandwich Plates by Generalized Differential Quadrature Method

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We combine a layer-wise formulation and a generalized differential quadrature technique for predicting the static deformations and free vibration behaviour of sandwich plates. Through numerical experiments, the capability and efficiency of this strong-form technique for static and vibration problems are demonstrated, and the numerical accuracy and convergence are thoughtfully examined.

1. Introduction

The two big groups of theories for the analysis of laminated plates are based on displacements only or on displacements and stresses at interfaces of the laminate. This paper deals with the first group, where again two approaches can be considered: the equivalent-single layer (ESL) theories (where all the layers are referred to the same degrees of freedom) or layer-wise (LW) theories (where specific degrees of freedom are linked to specific layers).

The classical laminated plate theory and the first-order shear deformation theory [1–5] are the simplest ESL theories, at the cost of introducing artificial shear correction factors. ESL higher-order theories usually consider higher-order expansion of displacements with respect to the thickness coordinate of the plate, with the further advantage of disregarding shear correction factors [6–12]. Most of the previously mentioned ESL theories may ill-represent the transverse shear stresses for sandwich laminates, where the material properties of skins are typically much larger than those of the core. In such problems, LW theories that consider independent degrees of freedom for each layer should be considered instead. Several LW theories have been proposed in literature [13–21] usually considering translational degrees of freedom at the layer interfaces. Over the last decades, an automatic approach for generation of any kind of ESL or LW theories was developed by Carrera [18–20], which allows the user to freely develop new theories without much effort. Various examples of the use of Carrera's unified formulation with meshless methods by the first author can be found in [22–27].

Unlike LW theories that consider translations at each laminate interface, in this work we adopt a layer-wise theory that considers both translations and rotations, based on an expansion of Mindlin's first-order shear deformation theory in each layer. The displacement continuity at layer's interface is enforced. In each layer, the theory produces constant, very accurate transverse shear stress.

As is well known, the analysis of composite plates is typically performed by finite element method. On the contrary, in the present study, a layer-wise shear deformation theory is implemented within generalized differential quadrature technique. The latter meshfree method in combination with this layer-wise theory carries out an accurate calculation

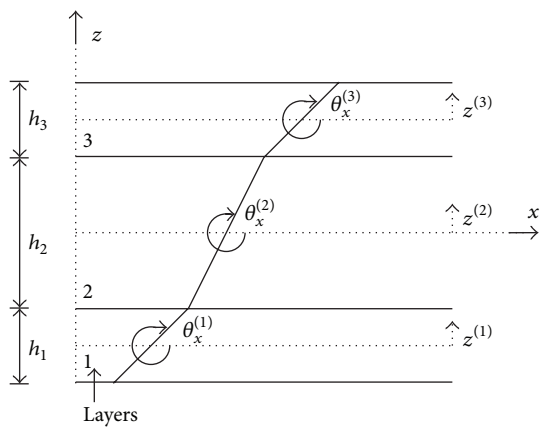

Figure 1: 1D representation of the layer-wise kinematics.

of the natural frequencies, displacements, and stresses of sandwich plates. This study appears for the first time and serves to fill the gap of knowledge in this research topic.

2. A Layer-Wise Theory

The layer-wise theory proposed in this paper is based on the assumption of a first-order shear deformation theory in each layer and the imposition of displacement continuity at layer's interfaces. Given that, a layer-wise formulation links the degrees of freedom at each layer. Nevertheless, the problem may become large if a large number of layers are considered, although the present formulation is adequate for a general laminate. In this paper, we concentrate on sandwich plates built from two external, stiff layers and one central, soft core. The three-layer laminate is shown schematically in Figure 1.

In order to simplify the imposition of displacement continuity at the core-skin interfaces, we first define the displacement field for the middle layer (the core in the sandwich plate) as

$$u^{(2)}(x,y,z) = u_0(x,y) + z^{(2)}\theta_x^{(2)}(x,y), \tag{1}$$

$$v^{(2)}(x,y,z) = v_0(x,y) + z^{(2)}\theta_y^{(2)}(x,y), \tag{2}$$

$$w^{(2)}(x,y,z) = w_0(x,y), \tag{3}$$

where u and v are the in-plane displacements at any point (x,y,z), u_0 and v_0 are the in-plane displacements of the points $(x,y,0)$ on the midplane, w is the deflection, and $\theta_x^{(2)}$ and $\theta_y^{(2)}$ are the rotations of the normals to the midplane about the y- and x-axes, respectively.

The corresponding displacement fields for the (skins) upper layer (3) and lower layer (1) are given, respectively, as

$$\begin{aligned}
u^{(3)}(x,y,z) &= u_0(x,y) + \frac{h_2}{2}\theta_x^{(2)}(x,y) + \frac{h_3}{2}\theta_x^{(3)}(x,y) \\
&\quad + z^{(3)}\theta_x^{(3)}(x,y), \\
v^{(3)}(x,y,z) &= v_0(x,y) + \frac{h_2}{2}\theta_y^{(2)}(x,y) + \frac{h_3}{2}\theta_y^{(3)}(x,y) \\
&\quad + z^{(3)}\theta_y^{(3)}(x,y), \\
w^{(3)}(x,y,z) &= w_0(x,y), \\
u^{(1)}(x,y,z) &= u_0(x,y) - \frac{h_2}{2}\theta_x^{(2)}(x,y) - \frac{h_1}{2}\theta_x^{(1)}(x,y) \\
&\quad + z^{(1)}\theta_x^{(1)}(x,y), \\
v^{(1)}(x,y,z) &= v_0(x,y) - \frac{h_2}{2}\theta_y^{(2)}(x,y) - \frac{h_1}{2}\theta_y^{(1)}(x,y) \\
&\quad + z^{(1)}\theta_y^{(1)}(x,y), \\
w^{(1)}(x,y,z) &= w_0(x,y),
\end{aligned} \tag{4}$$

where h_k are the kth layer thickness and $z^{(k)} \in [-h_k/2, h_k/2]$ are the kth layer z coordinates.

Deformations for a generic layer k are given by

$$\begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \gamma_{xy}^{(k)} \\ \gamma_{xz}^{(k)} \\ \gamma_{yz}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u^{(k)}}{\partial x} \\ \dfrac{\partial v^{(k)}}{\partial y} \\ \dfrac{\partial u^{(k)}}{\partial y} + \dfrac{\partial v^{(k)}}{\partial x} \\ \dfrac{\partial u^{(k)}}{\partial z} + \dfrac{\partial w^{(k)}}{\partial x} \\ \dfrac{\partial v^{(k)}}{\partial z} + \dfrac{\partial w^{(k)}}{\partial y} \end{Bmatrix}. \tag{5}$$

It is common that the ESL theories decouple the in-plane deformations with the out-of-plane deformations. The in-plane deformations can be expressed as

$$\begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \gamma_{xy}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^{m(k)} \\ \epsilon_{yy}^{m(k)} \\ \gamma_{xy}^{m(k)} \end{Bmatrix} + z^{(k)} \begin{Bmatrix} \epsilon_{xx}^{f(k)} \\ \epsilon_{yy}^{f(k)} \\ \gamma_{xy}^{f(k)} \end{Bmatrix} + \begin{Bmatrix} \epsilon_{xx}^{mf(k)} \\ \epsilon_{yy}^{mf(k)} \\ \gamma_{xy}^{mf(k)} \end{Bmatrix} \tag{6}$$

and shear deformations as

$$\begin{Bmatrix} \gamma_{xz}^{(k)} \\ \gamma_{yz}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_0}{\partial x} + \theta_x^{(k)} \\ \dfrac{\partial w_0}{\partial y} + \theta_y^{(k)} \end{Bmatrix}. \tag{7}$$

The membrane components are given by

$$\begin{Bmatrix} \epsilon_{xx}^{m(k)} \\ \epsilon_{yy}^{m(k)} \\ \gamma_{xy}^{m(k)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \tag{8}$$

where superscript m denotes membrane components.

The bending components can be expressed as

$$\begin{Bmatrix} \epsilon_{xx}^{f(k)} \\ \epsilon_{yy}^{f(k)} \\ \gamma_{xy}^{f(k)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \theta_x^{(k)}}{\partial x} \\ \dfrac{\partial \theta_y^{(k)}}{\partial y} \\ \dfrac{\partial \theta_x^{(k)}}{\partial y} + \dfrac{\partial \theta_y^{(k)}}{\partial x} \end{Bmatrix}, \tag{9}$$

where superscript f denotes flexural or bending components and the membrane-bending coupling components for layers 2, 3, and 1 are, respectively, given as

$$\begin{Bmatrix} \epsilon_{xx}^{mf(2)} \\ \epsilon_{yy}^{mf(2)} \\ \gamma_{xy}^{mf(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\begin{Bmatrix} \epsilon_{xx}^{mf(3)} \\ \epsilon_{yy}^{mf(3)} \\ \gamma_{xy}^{mf(3)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{h_2}{2}\dfrac{\partial \theta_x^{(2)}}{\partial x} + \dfrac{h_3}{2}\dfrac{\partial \theta_x^{(3)}}{\partial x} \\ \dfrac{h_2}{2}\dfrac{\partial \theta_y^{(2)}}{\partial y} + \dfrac{h_3}{2}\dfrac{\partial \theta_y^{(3)}}{\partial y} \\ \dfrac{h_2}{2}\left(\dfrac{\partial \theta_x^{(2)}}{\partial y} + \dfrac{\partial \theta_y^{(2)}}{\partial x}\right) + \dfrac{h_3}{2}\left(\dfrac{\partial \theta_x^{(3)}}{\partial y} + \dfrac{\partial \theta_y^{(3)}}{\partial x}\right) \end{Bmatrix},$$

$$\begin{Bmatrix} \epsilon_{xx}^{mf(1)} \\ \epsilon_{yy}^{mf(1)} \\ \gamma_{xy}^{mf(1)} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{h_2}{2}\dfrac{\partial \theta_x^{(2)}}{\partial x} - \dfrac{h_1}{2}\dfrac{\partial \theta_x^{(1)}}{\partial x} \\ -\dfrac{h_2}{2}\dfrac{\partial \theta_y^{(2)}}{\partial y} - \dfrac{h_1}{2}\dfrac{\partial \theta_y^{(1)}}{\partial y} \\ -\dfrac{h_2}{2}\left(\dfrac{\partial \theta_x^{(2)}}{\partial y} + \dfrac{\partial \theta_y^{(2)}}{\partial x}\right) - \dfrac{h_1}{2}\left(\dfrac{\partial \theta_x^{(1)}}{\partial y} + \dfrac{\partial \theta_y^{(1)}}{\partial x}\right) \end{Bmatrix}, \tag{10}$$

where superscript mf denotes membrane-flexural components.

The present layer-wise formulation neglects the transverse normal deformations, by assuming a constant transverse displacement in every layer. Neglecting $\sigma_z^{(k)}$ for each orthotropic layer, the stress-strain relations in the fiber local coordinate system can be expressed as

$$\begin{Bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \tau_{12}^{(k)} \\ \tau_{23}^{(k)} \\ \tau_{31}^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & Q_{33}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & Q_{44}^{(k)} & 0 \\ 0 & 0 & 0 & 0 & Q_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^{(k)} \\ \varepsilon_2^{(k)} \\ \gamma_{12}^{(k)} \\ \gamma_{23}^{(k)} \\ \gamma_{31}^{(k)} \end{Bmatrix}, \tag{11}$$

where subscripts 1 and 2 are, respectively, the principal middle surface directions, 3 is the direction normal to the middle plane of the plate, and the reduced stiffness components, $Q_{ij}^{(k)}$, are given by

$$\begin{gathered} Q_{11}^{(k)} = \frac{E_1^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \qquad Q_{22}^{(k)} = \frac{E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \\ Q_{12}^{(k)} = \nu_{12}^{(k)} Q_{22}^{(k)}, \qquad Q_{33}^{(k)} = G_{12}^{(k)}, \\ Q_{44}^{(k)} = G_{23}^{(k)}, \qquad Q_{55}^{(k)} = G_{31}^{(k)}, \\ \nu_{21}^{(k)} = \nu_{12}^{(k)} \frac{E_2^{(k)}}{E_1^{(k)}}, \end{gathered} \tag{12}$$

in which $E_1^{(k)}$, $E_2^{(k)}$, $\nu_{12}^{(k)}$, $G_{12}^{(k)}$, $G_{23}^{(k)}$, and $G_{31}^{(k)}$ are material properties of the lamina k. Note that shear correction factors are not considered in the stress-strain laws, as they must be done in the first-order shear deformation theories [21].

The stress-strain relations in the global x-y-z coordinate system can be obtained by coordinate transformation as

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \tau_{xy}^{(k)} \\ \tau_{yz}^{(k)} \\ \tau_{zx}^{(k)} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11}^{(k)} & \overline{Q}_{12}^{(k)} & \overline{Q}_{16}^{(k)} & 0 & 0 \\ \overline{Q}_{12}^{(k)} & \overline{Q}_{22}^{(k)} & \overline{Q}_{26}^{(k)} & 0 & 0 \\ \overline{Q}_{16}^{(k)} & \overline{Q}_{26}^{(k)} & \overline{Q}_{66}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44}^{(k)} & \overline{Q}_{45}^{(k)} \\ 0 & 0 & 0 & \overline{Q}_{45}^{(k)} & \overline{Q}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ \varepsilon_{yy}^{(k)} \\ \gamma_{xy}^{(k)} \\ \gamma_{yz}^{(k)} \\ \gamma_{zx}^{(k)} \end{Bmatrix}. \tag{13}$$

By considering θ as the angle between x-axis and 1-axis, with 1-axis being the first principal material axis, connected usually with fiber direction, the components $\overline{Q}_{ij}^{(k)}$ can be computed by coordinate transformation (see [13] for details).

The equations of motion of this layer-wise theory are derived from the dynamic version of the principle of virtual displacements. In the present work, only symmetric laminates are considered; therefore, u_0, v_0, and related stress resultants can be discarded. The virtual strain energy (δU), the virtual kinetic energy (δK), and the virtual work done

by applied forces (δV), assuming a three-layer laminate, are given by

$$\begin{aligned}\delta U = \int_{\Omega_0} \sum_{k=1}^{3} \Bigg\{ \int_{-h_k/2}^{h_k/2} \Big[& \sigma_{xx}\left(z\delta\epsilon_{xx}^{f(k)} + \delta\epsilon_{xx}^{mf(k)}\right) \\ & + \sigma_{yy}\left(z\delta\epsilon_{yy}^{f(k)} + \delta\epsilon_{yy}^{mf(k)}\right) \\ & + \tau_{xy}\left(z\delta\gamma_{xy}^{f(k)} + \delta\gamma_{xy}^{mf(k)}\right) \\ & + \tau_{xz}\delta\gamma_{xz}^{(k)} + \tau_{yz}\delta\gamma_{yz}^{(k)}\Big] dz \Bigg\} dx\, dy \\ = \int_{\Omega_0} \sum_{k=1}^{3} \Big(& N_{xx}^{(k)}\delta\epsilon_{xx}^{mf(k)} + M_{xx}^{(k)}\delta\epsilon_{xx}^{f(k)} + N_{yy}^{(k)}\delta\epsilon_{yy}^{mf(k)} \\ & + M_{yy}^{(k)}\delta\epsilon_{yy}^{f(k)} + N_{xy}^{(k)}\delta\gamma_{xy}^{mf(k)} + M_{xy}^{(k)}\delta\gamma_{xy}^{f(k)} \\ & + Q_x^{(k)}\delta\gamma_{xz}^{(k)} + Q_y^{(k)}\delta\gamma_{yz}^{(k)}\Big) dx\, dy.\end{aligned} \tag{14}$$

It should be noted that the integrals in the thickness direction are now in the layer k domain, from $-h_k/2$ to $h_k/2$. Consider

$$\begin{aligned}\delta K = \int_{\Omega_0} \sum_{k=1}^{3} \int_{-h_k/2}^{h_k/2} & \rho^{(k)} \\ & \times \left(\dot{\mathbf{u}}^{(k)}\delta\dot{\mathbf{u}}^{(k)} + \dot{\mathbf{v}}^{(k)}\delta\dot{\mathbf{v}}^{(k)} + \dot{\mathbf{w}}^{(k)}\delta\dot{\mathbf{w}}^{(k)}\right) dz\, dx\, dy,\end{aligned} \tag{15}$$

where boldface quantities are vectors and $(\cdot)$ denotes time differentiation. Finally, the virtual work is given as

$$\delta V = -\int_{\Omega_0} q\delta w_0 dx\, dy, \tag{16}$$

where δ quantities are virtual, Ω_0 denotes the midplane of the laminate, q is the external distributed load, and

$$\begin{aligned}\left(N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)}\right) &= \int_{-h_k/2}^{h_k/2} \sigma_{\alpha\beta}^{(k)}(1, z)\, dz_k, \\ Q_\alpha^{(k)} &= \int_{-h_k/2}^{h_k/2} \tau_{\alpha z}^{(k)} dz_k,\end{aligned} \tag{17}$$

where α, β take the symbols x, y.

Substituting δU, δK, and δV into the virtual work statement, using the fundamental lemma of the calculus of variations, and integrating by parts to relieve from any derivatives of the generalized displacements (see [13] for details), the equations of motion with respect to 7 degrees of freedom (w_0, $\theta_x^{(1)}$, $\theta_y^{(1)}$, $\theta_x^{(2)}$, $\theta_y^{(2)}$, $\theta_x^{(3)}$, and $\theta_y^{(3)}$) are obtained. Consider

$$\begin{aligned}
\delta w_0 &: \sum_{k=1}^{3}\left(\frac{\partial Q_x^{(k)}}{\partial x} + \frac{\partial Q_y^{(k)}}{\partial y}\right) - q = \sum_{k=1}^{3} I_0^{(k)}\ddot{w}_0, \\
\delta\theta_x^{(1)} &: \frac{h_1}{2}\frac{\partial N_{xx}^{(1)}}{\partial x} - \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{h_1}{2}\frac{\partial N_{xy}^{(1)}}{\partial y} - \frac{\partial M_{xy}^{(1)}}{\partial y} + Q_x^{(1)} \\
&\quad = I_0^{(1)}\left(\frac{h_1h_2}{4}\ddot{\theta}_{x2} + \frac{h_1^2}{4}\ddot{\theta}_{x1}\right) + I_2^{(1)}\ddot{\theta}_{x1}, \\
\delta\theta_y^{(1)} &: \frac{h_1}{2}\frac{\partial N_{yy}^{(1)}}{\partial y} - \frac{\partial M_{yy}^{(1)}}{\partial y} + \frac{h_1}{2}\frac{\partial N_{xy}^{(1)}}{\partial x} - \frac{\partial M_{xy}^{(1)}}{\partial x} + Q_y^{(1)} \\
&\quad = I_0^{(1)}\left(\frac{h_1h_2}{4}\ddot{\theta}_{y2} + \frac{h_1^2}{4}\ddot{\theta}_{y1}\right) + I_2^{(1)}\ddot{\theta}_{y1}, \\
\delta\theta_x^{(2)} &: \frac{h_2}{2}\frac{\partial N_{xx}^{(1)}}{\partial x} - \frac{h_2}{2}\frac{\partial N_{xx}^{(3)}}{\partial x} - \frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{h_2}{2}\frac{\partial N_{xy}^{(1)}}{\partial y} \\
&\quad - \frac{h_2}{2}\frac{\partial N_{xy}^{(3)}}{\partial y} - \frac{\partial M_{xy}^{(2)}}{\partial y} + Q_x^{(2)} \\
&\quad = I_0^{(1)}\left(\frac{h_2^2}{4}\ddot{\theta}_{x2} + \frac{h_1h_2}{4}\ddot{\theta}_{x1}\right) + I_0^{(3)}\left(\frac{h_2^2}{4}\ddot{\theta}_{x2} + \frac{h_2h_3}{4}\ddot{\theta}_{x3}\right) \\
&\quad + I_2^{(2)}\ddot{\theta}_{x2}, \\
\delta\theta_y^{(2)} &: \frac{h_2}{2}\frac{\partial N_{yy}^{(1)}}{\partial y} - \frac{h_2}{2}\frac{\partial N_{yy}^{(3)}}{\partial y} - \frac{\partial M_{yy}^{(2)}}{\partial y} + \frac{h_2}{2}\frac{\partial N_{xy}^{(1)}}{\partial x} \\
&\quad - \frac{h_2}{2}\frac{\partial N_{xy}^{(3)}}{\partial x} - \frac{\partial M_{xy}^{(2)}}{\partial x} + Q_y^{(2)} \\
&\quad = I_0^{(1)}\left(\frac{h_2^2}{4}\ddot{\theta}_{y2} + \frac{h_1h_2}{4}\ddot{\theta}_{y1}\right) + I_0^{(3)}\left(\frac{h_2^2}{4}\ddot{\theta}_{y2} + \frac{h_2h_3}{4}\ddot{\theta}_{y3}\right) \\
&\quad + I_2^{(2)}\ddot{\theta}_{y2}, \\
\delta\theta_x^{(3)} &: -\frac{h_3}{2}\frac{\partial N_{xx}^{(3)}}{\partial x} - \frac{\partial M_{xx}^{(3)}}{\partial x} - \frac{h_3}{2}\frac{\partial N_{xy}^{(3)}}{\partial y} - \frac{\partial M_{xy}^{(3)}}{\partial y} + Q_x^{(3)} \\
&\quad = I_0^{(3)}\left(\frac{h_2h_3}{4}\ddot{\theta}_{x2} + \frac{h_3^2}{4}\ddot{\theta}_{x3}\right) + I_2^{(3)}\ddot{\theta}_{x3}, \\
\delta\theta_y^{(3)} &: \frac{h_3}{2}\frac{\partial N_{yy}^{(3)}}{\partial y} - \frac{\partial M_{yy}^{(3)}}{\partial y} - \frac{h_3}{2}\frac{\partial N_{xy}^{(3)}}{\partial x} - \frac{\partial M_{xy}^{(3)}}{\partial x} + Q_y^{(3)} \\
&\quad = I_0^{(3)}\left(\frac{h_2h_3}{4}\ddot{\theta}_{y2} + \frac{h_3^2}{4}\ddot{\theta}_{y3}\right) + I_2^{(3)}\ddot{\theta}_{y3},
\end{aligned} \tag{18}$$

where

$$\left(I_0^{(k)}, I_2^{(k)}\right) = \int_{-h_k/2}^{h_k/2} \rho^{(k)}\left(1, z^2\right) dz, \tag{19}$$

where ρ^k, the specific mass of the material of the k is layer and h_k is the thickness of the kth layer.

3. The Generalized Differential Quadrature Method

The differential quadrature (DQ) method was presented by Bellman and his associates in the early 1970s [28]. It is a numerical discretization technique for the approximation of derivatives. The DQ method was initiated from the idea of conventional integral quadrature. In fact, one problem which arises frequently in structural mechanics and in many other engineering problems is the evaluation of the integral $\int_a^b f(x)dx$. If F is a function such that $dF/dx = f$, then the value of the given integral is $F(b) - F(a)$. Unfortunately, in practical problems, it is extremely difficult, and most of the times impossible, to obtain an explicit expression for F. The values of f, perhaps, can be known at a discrete set of points, and, in this situation, a numerical approach is essential.

Following the idea of integral quadrature, Bellman et al. [28] suggested that the first-order derivative of the function $f(x)$ with respect to x at a grid point x_i is approximated by a linear sum of all the functional values in the whole domain as

$$\left.\frac{df(x)}{dx}\right|_{x=x_i} = \sum_{j=1}^{N} a_{ij}^{x,(1)} f\left(x_j\right) \quad \text{for } i = 1, 2, \ldots, N, \tag{20}$$

where $a_{ij}^{x,(1)}$ represent the weighting coefficients, N is the total number of grid points $x_1, x_2, \ldots, x_N$ in the whole domain, and $f(x_j)$ is the calculated value of $f(x)$ at the point $x = x_j$. However, the weighting coefficients which were used depend, in the Bellman approach, on the number of grid points and the type of discretization that was used. Some years later the DQ approach has been extended by Shu [29], who introduced the generalized differential quadrature (GDQ) method. In [29], the weighting coefficients calculation is performed through recursive formulae, and they are independent of the number of grid points N and the discretization type. The interested reader can find a brief review on GDQ applications in [30–40].

To compute the first-order weighting coefficients $a_{ij}^{x,(1)}$, the following algebraic formulae are derived:

$$\begin{gathered} a_{ij}^{x,(1)} = \frac{L^{(1)}\left(x_i\right)}{\left(x_i - x_j\right) L^{(1)}\left(x_j\right)} \quad \text{for } i, j = 1, 2, \ldots, N,\ i \neq j \\ a_{ii}^{x,(1)} = -\sum_{k=1, k\neq i}^{N} a_{ik}^{(1)} \quad \text{for } i = j, \end{gathered} \tag{21}$$

where $L^{(1)}(x_k) = \prod_{j=1, j\neq k}^{N}(x_k - x_j)$, which are the derivatives of the Lagrange polynomials $L(x) = (x - x_1)(x - x_2)\cdots(x - x_N) = \prod_{j=1}^{N}(x - x_j)$. As reported above, Shu [29] found out a recurrence formulation for the weighting coefficients calculation of the second- and higher-order derivatives. Consider

$$\begin{gathered} a_{ij}^{x,(n)} = n\left(a_{ii}^{x,(n-1)} a_{ij}^{x,(1)} - \frac{a_{ij}^{x,(n-1)}}{x_i - x_j}\right) \\ \text{for } i \neq j,\ n = 2, 3, \ldots, N-1, \\ a_{ii}^{x,(n)} = -\sum_{k=1, k\neq i}^{N} a_{ik}^{x,(n)} \quad \text{for } i = j. \end{gathered} \tag{22}$$

So any derivative of an order greater than the first can be written as

$$\begin{gathered} \left.\frac{d^{(n)} f(x)}{dx^n}\right|_{x=x_i} = \sum_{j=1}^{N} a_{ij}^{x,(n)} f\left(x_j\right) \\ \text{for } i = 1, 2, \ldots, N,\ n = 2, 3, \ldots, N-1. \end{gathered} \tag{23}$$

The previous derivative discretization was presented for the one-dimension problems only. Nevertheless, the simple one-dimensional case can be directly extended to the multidimensional one for any regular shape, such as a rectangular or circular. Given a two-dimensional physical system, in Cartesian coordinates, described by a function $f(x, y)$, the problem values depend on the nodal coordinates x and y. Since GDQ works with regular grid along the main axes, N and M will indicate the points along x and y, respectively. Using the discretization of the derivatives in the one-dimensional case (23), a generic higher-order derivative along x and y for the two-dimensional case can be reported. Consider

$$\begin{gathered} \left.\frac{\partial^{(n)} f(x, y)}{\partial x^n}\right|_{x=x_i,\, y=y_j} = \sum_{k=1}^{N} a_{ik}^{x,(n)} f\left(x_k, y_j\right) \\ \text{for } i = 1, 2, \ldots, N,\ n = 1, 2, \ldots, N-1, \\ \left.\frac{\partial^{(m)} f(x, y)}{\partial y^m}\right|_{x=x_i,\, y=y_j} = \sum_{l=1}^{M} a_{jl}^{y,(m)} f\left(x_i, y_l\right) \\ \text{for } j = 1, 2, \ldots, M,\ m = 1, 2, \ldots, M-1, \end{gathered} \tag{24}$$

where $a_{ik}^{x,(n)}$ and $a_{jl}^{y,(m)}$ are the weighting coefficients of order n and m along x and y, respectively. Moreover, the mixed derivative can be written in the same way as (24). Consider

$$\begin{gathered} \left.\frac{\partial^{(n+m)} f(x, y)}{\partial x^n \partial y^m}\right|_{x=x_i,\, y=y_j} = \sum_{k=1}^{N} a_{ik}^{x,(n)} \left(\sum_{l=1}^{M} a_{jl}^{y,(m)} f\left(x_k, y_l\right)\right) \\ \text{for } i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M,\ n = 1, 2, \ldots, N-1, \\ m = 1, 2, \ldots, M-1, \end{gathered} \tag{25}$$

where $a_{ik}^{x,(n)}$ and $a_{jl}^{y,(m)}$ have the same meaning in (24).

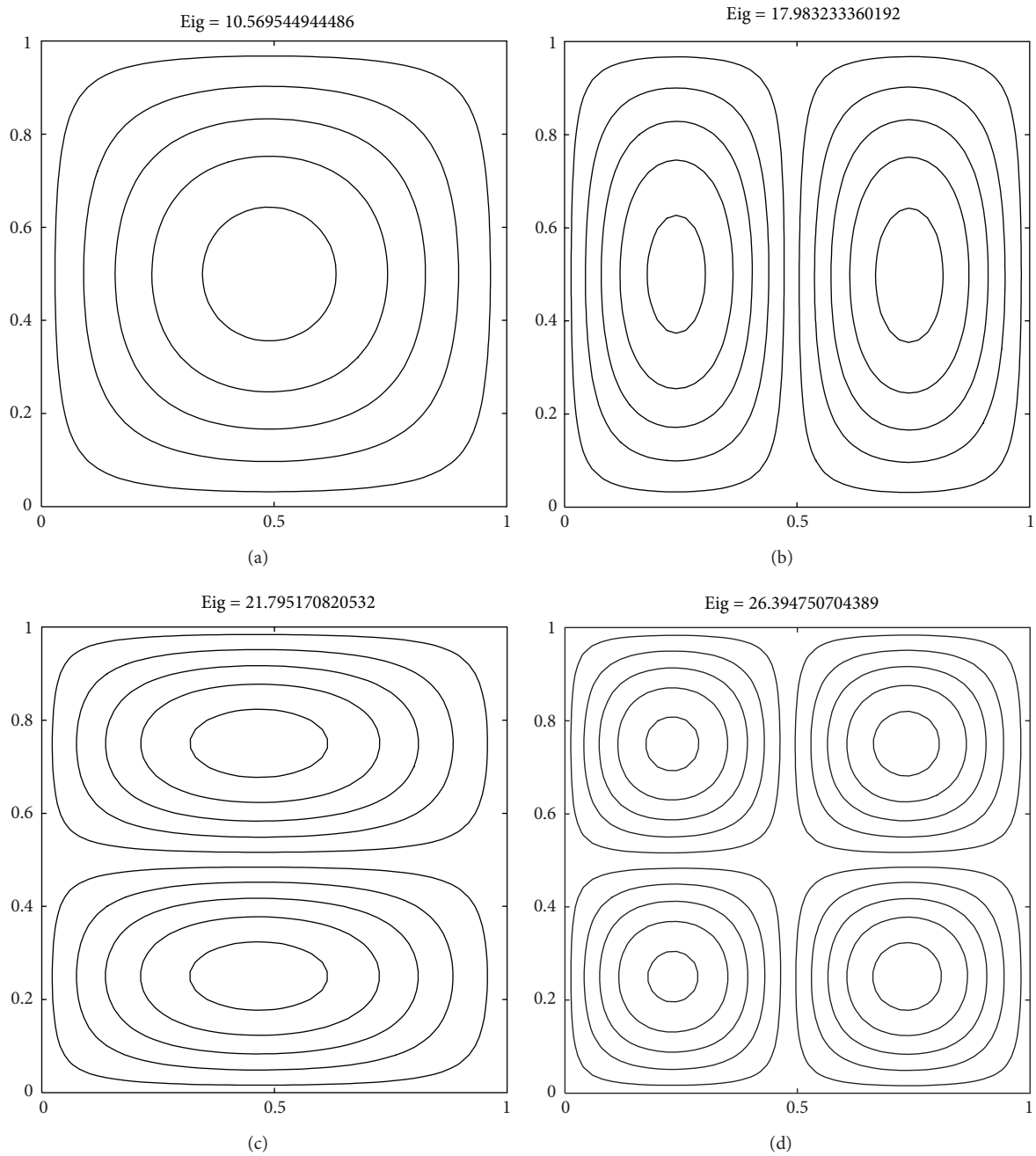

Figure 2: First 4 modes of vibration of four-layer $[0^\circ/90^\circ/90^\circ/0^\circ]$ simply supported laminated plate ($\overline{w} = (wa^2/h)\sqrt{\rho/E_2}$, $h/a = 0.2$), $E_1/E_2 = 30$, grid 19×19 points.

3.1. Grid Distributions. As it has been widely proven [34–37], the GDQ numerical accuracy strongly depends on the grid distribution choice for a certain numerical problem. On the one hand, the natural and simplest choice of uniform grids does not lead to stable and accurate results for any GDQ computation; on the other hand, the Chebyshev-Gauss-Lobatto (C-G-L) grid point distribution gives accurate results in most of cases. In the numerical results proposed in this paper a C-G-L grid distribution is considered for both directions x and y. Consider

$$x_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\right)\pi\right),$$
$$y_j = \frac{1}{2}\left(1 - \cos\left(\frac{j-1}{M-1}\right)\pi\right), \tag{26}$$

where N, M are the total number of points in the given directions x, y, respectively.

3.2. The Static Problem. The partial differential system of equations developed in Section 2 has been discretised and numerically solved using the GDQ method. Since GDQ allows turning any derivative in an algebraic expression, the fundamental system of equations is solved in its strong form directly, within its boundary conditions. A static boundary problem can be indicated as follows:

$$\begin{aligned} \ell(u(\mathbf{x})) + q(\mathbf{x}) &= 0 \quad \text{in } \Omega, \\ \ell_B(u(\mathbf{x})) &= \mathbf{g} \quad \text{on } \partial\Omega, \end{aligned} \tag{27}$$

where ℓ is a partial differential operator and contains all the spatial derivatives of the problem with respect to a Cartesian coordinate system $\mathbf{x}$. The model degrees of freedom are $u(\mathbf{x})$, and the external forces acting on the domain Ω are indicated by $q(\mathbf{x})$. It is obvious that the differential problem cannot be solved without its boundary conditions, where ℓ_B is the differential operator related to the boundary conditions of the given system of equations and $\mathbf{g}$ are the boundary loads.

Once the boundary-value problem (27) is discretised by GDQ the following set of algebraic equations is obtained:

$$\begin{aligned} \mathscr{L}\mathbf{U} + \mathbf{Q} &= \mathbf{0}, \\ \mathscr{L}_B\mathbf{U} &= \mathbf{G}, \end{aligned} \tag{28}$$

where $\mathbf{U}$ is a vector representing a set of unknown functional values at all the interior points, $\mathscr{L}$ is a matrix carried out from the GDQ method, and $\mathbf{Q}$, $\mathbf{G}$ are the known vectors arising from the functions $q(\mathbf{x})$ and $\mathbf{g}$.

In order to obtain the global stiffness matrix of the physical problem under study, the discretised boundary conditions must be embedded into the discrete fundamental system of equations. For the sake of conciseness, the global static system (28) is reported in matrix form as follows:

$$\mathbf{KU} = \overline{\mathbf{Q}}, \tag{29}$$

where $\overline{\mathbf{Q}}$ is the external boundary and domain forces and $\mathbf{K}$ is the global stiffness matrix which can be inverted leading to the problem solution. The interested reader can refer to the paper by Viola et al. [39] for further details about static problem resolutions of plates and shells.

3.3. The Eigenproblem. Given a generic set of equations of motion, the linear eigenvalue problem can be achieved eliminating the external forces, as it is shown in the following:

$$\ell(u(\mathbf{x}, t)) = \rho\ddot{u}(\mathbf{x}, t), \tag{30}$$

where the right-hand term represents the inertia term, given by the product of density ρ and the acceleration term $\ddot{u}(\mathbf{x}, t)$. It is noted that the symbol ¨ stands for the second derivative respect to time of a given variable. In general, an eigenproblem seeks particular numbers, called eigenvalues (λ), and certain vectors, called eigenvectors $U(\mathbf{x})$. Mathematically speaking, the eigenvalue problem is obtained from (30) setting as follows:

$$u(\mathbf{x}, t) = U(\mathbf{x})\, e^{i\lambda t}. \tag{31}$$

Equation (31) indicates that variable separation technique is employed. In fact, the problem degree of freedom u which is function of space and time is given by the product of two functions, which are functions of space and time separately. Hence, substituting relation (31) into (30), a generalized eigenvalue problem can be achieved. For the sake of completeness, it is presented with its boundary conditions as

$$\begin{aligned} \ell(U(\mathbf{x})) + \rho\lambda^2 U(\mathbf{x}) &= 0 \quad \text{in } \Omega, \\ \ell_B U(\mathbf{x}) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{32}$$

Applying GDQ rule, the eigenvalue problem (32) can be written in discretised form as follows:

$$\begin{aligned} \mathscr{L}\mathbf{U} + \lambda^2\mathbf{MU} &= \mathbf{0}, \\ \mathscr{L}_B\mathbf{U} &= 0, \end{aligned} \tag{33}$$

where $\mathbf{M}$ indicates the mass matrix of the structural system under study. Solving the discretised system (33), the eigenvalues λ and eigenvectors $\mathbf{U}$ of the given system can be obtained. Finally, the reader can refer to [38] for solutions of eigenvalue problems regarding plates and shells.

4. Numerical examples

4.1. Static Problems of Cross-Ply Laminated Plates. A simply supported square laminated plate of side a and thickness h is composed of four equal layers oriented at $[0^\circ/90^\circ/90^\circ/0^\circ]$. The plate is subjected to a sinusoidal vertical pressure of the form $p_z = P\sin(\pi x/a)\sin(\pi y/a)$ with the origin of the coordinate system located at the lower left corner on the midplane and P the maximum load (at the plate center).

The orthotropic material properties are given by $E_1 = 25.0E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, and $\nu_{12} = 0.25$.

The in-plane displacements, the transverse displacements, the normal stresses, and the in-plane and transverse shear stresses are presented in normalized form as

$$\begin{aligned} \overline{w} &= \frac{10^2 w(a/2, a/2, 0)\, h^3 E_2}{Pa^4}, \\ \overline{\sigma}_{xx} &= \frac{\sigma_{xx}(a/2, a/2, h/2)\, h^2}{Pa^2}, \\ \overline{\sigma}_{yy} &= \frac{\sigma_{yy}(a/2, a/2, h/4)\, h^2}{Pa^2}, \qquad \overline{\tau}_{xz} = \frac{\tau_{xz(0,a/2,0)} h}{Pa}. \end{aligned} \tag{34}$$

In Table 1 for the present layer-wise theory, using 7×7 up to 19×19 grid points, the results with higher-order solutions by Akhras et al. [41] and Reddy [11], FSDT solutions by Reddy and Chao [42], and an exact solution by Pagano [43] are compared. Furthermore, the numerical computations by the authors using radial basis functions (RBFs) collocation, by the authors with Reddy's theory [44], and a layer-wise theory [45] are compared too. The present layer-wise displacements are in excellent agreement for thinner or thicker plates. Highly accurate normal stresses and transverse shear stresses are

Table 1: [0°/90°/90°/0°] square laminated plate under sinusoidal load, applied at ($z = h/2$).

a/h	Method	$\overline{w}$	$\overline{\sigma}_{xx}$	$\overline{\sigma}_{yy}$	$\overline{\tau}_{zx}$
4	HSDT finite strip method [41]	1.8939	0.6806	0.6463	0.2109
	HSDT [11]	1.8937	0.6651	0.6322	0.2064
	FSDT [42]	1.7100	0.4059	0.5765	0.1398
	Elasticity [43]	1.954	0.720	0.666	0.270
	Ferreira et al. [44] ($N = 21$)	1.8864	0.6659	0.6313	0.1352
	Ferreira (layer-wise) [45] ($N = 21$)	1.9075	0.6432	0.6228	0.2166
	Present (7×7 grid)	1.9087	0.6423	0.6258	0.2173
	Present (11×11 grid)	1.9091	0.6429	0.6265	0.2173
	Present (15×15 grid)	1.9091	0.6429	0.6265	0.2173
	Present (19×19 grid)	1.9091	0.6429	0.6265	0.2173
10	HSDT finite strip method [41]	0.7149	0.5589	0.3974	0.2697
	HSDT [11]	0.7147	0.5456	0.3888	0.2640
	FSDT [42]	0.6628	0.4989	0.3615	0.1667
	Elasticity [43]	0.743	0.559	0.403	0.301
	Ferreira et al. [44] ($N = 21$)	0.7153	0.5466	0.4383	0.3347
	Ferreira (layer-wise) [45] ($N = 21$)	0.7309	0.5496	0.3956	0.2888
	Present (7×7 grid)	0.7300	0.5481	0.3963	0.2993
	Present (11×11 grid)	0.7303	0.5487	0.3966	0.2993
	Present (15×15 grid)	0.7303	0.5487	0.3966	0.2993
	Present (19×19 grid)	0.7303	0.5487	0.3966	0.2993
100	HSDT finite strip method [41]	0.4343	0.5507	0.2769	0.2948
	HSDT [11]	0.4343	0.5387	0.2708	0.2897
	FSDT [42]	0.4337	0.5382	0.2705	0.1780
	Elasticity [43]	0.4347	0.539	0.271	0.339
	Ferreira et al. [44] ($N = 21$)	0.4365	0.5413	0.3359	0.4106
	Ferreira (layer-wise) [45] ($N = 21$)	0.4374	0.5420	0.2697	0.3232
	Present (7×7 grid)	0.4345	0.5385	0.2708	0.3359
	Present (11×11 grid)	0.4348	0.5391	0.2711	0.3359
	Present (15×15 grid)	0.4348	0.5391	0.2711	0.3359
	Present (19×19 grid)	0.4348	0.5391	0.2711	0.3359

Table 2: Square laminated plate under uniform load for $R = 5$.

R	Method	$\overline{w}$	$\overline{\sigma}_x^1$	$\overline{\sigma}_x^2$	$\overline{\sigma}_x^3$	$\overline{\tau}_{xz}^1$	$\overline{\tau}_{xz}^2$
5	HSDT [47]	256.13	62.38	46.91	9.382	3.089	2.566
	FSDT [47]	236.10	61.87	49.50	9.899	3.313	2.444
	CLT	216.94	61.141	48.623	9.783	4.5899	3.386
	Ferreira and Barbosa [48]	258.74	59.21	45.61	9.122	3.593	3.593
	Ferreira ($N = 15$) [21]	257.38	58.725	46.980	9.396	3.848	2.839
	Exact [46]	258.97	60.353	46.623	9.340	4.3641	3.2675
	HSDT [44] ($N = 11$)	253.6710	59.6447	46.4292	9.2858	3.8449	1.9650
	HSDT [44] ($N = 15$)	256.2387	60.1834	46.8581	9.3716	4.2768	2.2227
	HSDT [44] ($N = 21$)	257.1100	60.3660	47.0028	9.4006	4.5481	2.3910
	Present (7×7 grid)	257.3663	59.8307	46.4128	9.2826	3.8840	2.6959
	Present (11×11 grid)	258.1305	60.0447	46.4237	9.2847	4.0603	2.1679
	Present (15×15 grid)	258.1720	60.0772	46.4008	9.2802	4.0916	2.0593
	Present (19×19 grid)	258.1794	60.0889	46.3925	9.2785	4.0979	2.0386

Table 3: Square laminated plate under uniform load for $R = 10$.

R	Method	$\overline{w}$	$\overline{\sigma}_x^1$	$\overline{\sigma}_x^2$	$\overline{\sigma}_x^3$	$\overline{\tau}_{xz}^1$	$\overline{\tau}_{xz}^2$
	HSDT [47]	152.33	64.65	51.31	5.131	3.147	2.587
	FSDT [47]	131.095	67.80	54.24	4.424	3.152	2.676
	CLT	118.87	65.332	48.857	5.356	4.3666	3.7075
	Ferreira and Barbosa [48]	159.402	64.16	47.72	4.772	3.518	3.518
	Ferreira ($N = 15$) [21]	158.55	62.723	50.16	5.01	3.596	3.053
	Exact [46]	159.38	65.332	48.857	4.903	4.0959	3.5154
10	Third-order [44] ($N = 11$)	153.0084	64.7415	49.4716	4.9472	2.7780	1.8207
	Third-order [44] ($N = 15$)	154.2490	65.2223	49.8488	4.9849	3.1925	2.1360
	Third-order [44] ($N = 21$)	154.6581	65.3809	49.9729	4.9973	3.5280	2.3984
	Present (7×7 grid)	158.1023	64.5969	48.7204	4.8720	3.7227	3.1570
	Present (11×11 grid)	158.8400	64.9304	48.6523	4.8652	3.9394	2.4751
	Present (15×15 grid)	158.8945	64.9907	48.6005	4.8601	3.9772	2.3470
	Present (19×19 grid)	158.9036	65.0106	48.5834	4.8583	3.9841	2.3263

Table 4: Square laminated plate under uniform load for $R = 15$.

R	Method	$\overline{w}$	$\overline{\sigma}_x^1$	$\overline{\sigma}_x^2$	$\overline{\sigma}_x^3$	$\overline{\tau}_{xz}^1$	$\overline{\tau}_{xz}^2$
	HSDT [47]	110.43	66.62	51.97	3.465	3.035	2.691
	FSDT [47]	90.85	70.04	56.03	3.753	3.091	2.764
	CLT	81.768	69.135	55.308	3.687	4.2825	3.8287
	Ferreira and Barbosa [48]	121.821	65.650	47.09	3.140	3.466	3.466
	Ferreira ($N = 15$) [21]	121.184	63.214	50.571	3.371	3.466	3.099
	Exact [46]	121.72	66.787	48.299	3.238	3.9638	3.5768
15	Third-order [44] ($N = 11$)	113.5941	66.3646	49.8957	3.3264	2.1686	1.5578
	Third-order [44] ($N = 15$)	114.3874	66.7830	50.2175	3.3478	2.6115	1.9271
	Third-order [44] ($N = 21$)	114.6442	66.9196	50.3230	3.3549	3.0213	2.2750
	Present (7×7 grid)	120.5292	65.8690	48.2620	3.2175	3.6285	3.3561
	Present (11×11 grid)	121.2654	66.3110	48.1025	3.2068	3.8616	2.6168
	Present (15×15 grid)	121.3247	66.3973	48.0221	3.2015	3.9001	2.4901
	Present (19×19 grid)	121.3340	66.4233	47.9984	3.1999	3.9063	2.4731

Table 5: The normalized fundamental frequency of the simply supported cross-ply laminated square plate $[0^\circ/90^\circ/90^\circ/0^\circ]$ ($\overline{w} = (wa^2/h)\sqrt{\rho/E_2}$, $h/a = 0.2$).

Method	Grid	E_1/E_2			
		10	20	30	40
Liew et al. [50]		8.2924	9.5613	10.320	10.849
Exact (Khdeir and Librescu) [49]		8.2982	9.5671	10.326	10.854
	11×11	8.5846	9.8384	10.5695	11.0649
Present	15×15	8.5846	9.8384	10.5696	10.0649
	19×19	8.5846	9.8384	10.5695	10.0649

also obtained. It should be noted that the transverse shear stresses are directly obtained by the constitutive relations, not from the equations of equilibrium, so no recovery procedure has been employed. The thickness-stretching approach is important for thicker plates, as our approach produces the best results for transverse displacements, without shear correction.

4.2. Three-Layer Square Sandwich Plate in Bending, under Uniform Load. A simply supported sandwich square plate, under a uniform transverse load, is considered. This is the classical sandwich example of Srinivas [46]. The material properties of the sandwich core are expressed in the stiffness matrix, $\overline{Q}_{\text{core}}$, as

$$\overline{Q}_{\text{core}} = \begin{bmatrix} 0.999781 & 0.231192 & 0 & 0 & 0 \\ 0.231192 & 0.524886 & 0 & 0 & 0 \\ 0 & 0 & 0.262931 & 0 & 0 \\ 0 & 0 & 0 & 0.266810 & 0 \\ 0 & 0 & 0 & 0 & 0.159914 \end{bmatrix}. \tag{35}$$

Skins material properties are related to core properties by a factor R as $\overline{Q}_{\text{skin}} = R\overline{Q}_{\text{core}}$.

Transverse displacement and stresses are normalized through factors

$$\overline{w} = w\left(\frac{a}{2}, \frac{a}{2}, 0\right)\frac{0.999781}{hq},$$

$$\overline{\sigma}_x^1 = \frac{\sigma_x^{(1)}(a/2, a/2, -h/2)}{q}, \qquad \overline{\sigma}_x^2 = \frac{\sigma_x^{(1)}(a/2, a/2, -2h/5)}{q},$$

$$\overline{\sigma}_x^3 = \frac{\sigma_x^{(2)}(a/2, a/2, -2h/5)}{q},$$

$$\overline{\sigma}_y^1 = \frac{\sigma_y^{(1)}(a/2, a/2, -h/2)}{q}, \qquad \overline{\sigma}_y^2 = \frac{\sigma_y^{(1)}(a/2, a/2, -2h/5)}{q},$$

$$\overline{\sigma}_y^3 = \frac{\sigma_y^{(2)}(a/2, a/2, -2h/5)}{q},$$

$$\overline{\tau}_{xz}^1 = \frac{\tau_{xz}^{(2)}(0, a/2, 0)}{q}, \qquad \overline{\tau}_{xz}^2 = \frac{\tau_{xz}^{(2)}(0, a/2, -2h/5)}{q}. \tag{36}$$

Transverse displacement and stresses for a sandwich plate are indicated in Tables 2, 3, and 4 and compared with various sources, for three R ratios. This present approach results are in excellent agreement both in transverse displacement and transverse shear stresses. In particular for higher values of R, the present formulation computes more accurate results than those of Pandya and kant [47]. Some other sources such as laminated shell finite elements [48], multiquadrics [21], and third-order formulation presented by Ferreira et al. [44] are compared. In all cases, the lesson learned is that layer-wise approach should be used instead of ESL theories in the analysis of sandwich soft-core plates, where the skin properties are much higher than core properties. The present technique shows that the thickness-stretching approach is important for thicker plates.

4.3. Free Vibration Problems of Cross-Ply Laminated Plates. In this example, all layers of the laminate are assumed to be of the same thickness and density and made of the same linearly elastic composite material. The following material parameters of a layer are used: $E_1/E_2 = 10$, 20, 30, or 40; $G_{12} = G_{13} = 0.6E_2$; $G_3 = 0.5E_2$; $\nu_{12} = 0.25$. The subscripts 1 and 2 denote the directions normal and transverse to the fiber direction in a lamina, which may be oriented at an angle to the plate axes. The ply angle of each layer is measured from the global x-axis to the fiber direction.

The example under consideration is a simply supported square plate with a cross-ply lamination scheme $[0^\circ/90^\circ/90^\circ/0^\circ]$. The thickness and length of the plate are denoted by h and a, respectively. The thickness-to-span ratio $h/a = 0.2$ is employed in the computation. Table 5 lists the fundamental frequency of the simply supported laminate made of various modulus ratios of E_1/E_2. Figure 2 illustrates the modes of vibration for $E_1/E_2 = 40$ with 15×15 grid points. It is found that the present meshless results are in very close agreement with the values of [49] and the meshfree results of Liew et al. [50] based on the FSDT. The small differences may be due to the consideration of the through-the-thickness deformations in the present formulation.

5. Conclusions

The first-order and the third-order shear deformation theories are laminate-wise, with laminate degrees of freedom, where all layers have the same rotations. Layer-wise formulations can better accommodate the deformation behaviour of some laminates, in particular for sandwich laminates, where core and skin materials are so different.

In this paper the static and free vibration analysis of composite laminated plates by the use of generalized differential quadrature and using a layer-wise theory with independent rotations in each layer is performed.

The equations of motion were derived and interpolated; moreover, boundary conditions interpolation was schematically formulated.

Finally, composite laminated plate and sandwich plate were considered for testing the present methodology. The obtained results show an excellent accuracy for all cases.

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