Research Article

Distributed Robust H_∞ Consensus Control of Multiagent Systems with Communication Errors Using Dynamic Output Feedback Protocol

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This paper studies robust consensus problem for multiagent systems modeled by an identical linear time-invariant system under a fixed communication topology. Communication errors in the transferred data are considered, and only the relative output information between each agent and its neighbors is available. A distributed dynamic output feedback protocol is proposed, and sufficient conditions for reaching consensus with a prescribed H_∞ performance are presented. Numerical examples are given to illustrate the theoretical results.

1. Introduction

Consensus problem of multiagent systems has been a popular subject in system and control theory due to its widespread applications such as satellite formation flying, cooperative unmanned air vehicles, and mobile robots [1–3]. The study of consensus problem focuses on designing a distributed protocol using information which can only be obtained and shared locally to ensure that the resulting closed-loop system has the desired characteristics. A number of solutions that are based on relative states between each agent and its neighbors to the consensus problem have been proposed up to now. The theoretical framework of solving consensus for multiagent system was suggested by [4], providing the convergence analysis of a consensus protocol for a network of single integrators with directed fixed/switching topologies. Later, under different cases of communication topologies such as fixed, switching, and with communication delays, many different types of protocols have been proposed for different types of agent dynamics to reach global asymptotical consensus [2, 3, 5–16].

Recently, solving consensus problem for the multiagent systems by using output information has attracted particular attention due to its theoretical significance and wide applications. Reference [17] constructed a dynamic output feedback

protocol based on an observer for the synchronization of a network of identical linear state space models under a possibly time-varying and directed interconnection, where each agent needs to obtain all the observer's state information of its neighbors. Based on the low gain approach, [18] proposed a consensus protocol which only used the relative outputs for N identical linear dynamics with fixed directed communication topologies. Consensus problem with $\mathcal{L}_2$ external disturbance under switching undirected communication topologies was studied by [19], where a dynamic output feedback protocol was proposed for subjecting the external disturbances. Reference [20] studied the output consensus problem for a class of heterogeneous uncertain linear SISO multiagent systems, where each agent's output information and the relative outputs with its neighbors were used to design the controller. Reference [21] designed robust static output feedback controllers to achieve consensus for undirected networks of heterogeneous agents modeled as nonlinear systems of relative degree two.

It can be seen that there is a common assumption in the literatures mentioned above that each agent can receive accurate measurements of relative states or outputs between its neighbors and itself all the time. However, in some practical situations, agents cannot perfectly sense their neighbors due

to the existence of sensor failures or some other communication constraints. In view of this, we consider the consensus problems for the multiagent systems with communication errors. It is required to point out that the measurement for communication errors we considered is limited to some errors in the transferred data not including loss of communication. The robustness analysis of first-/second-order leader-follower consensus with communication errors is studied by [22, 23]. Some robustness issues for systems with external disturbances or model uncertainties are investigated by some other researchers [19, 24–28], which are different from the robust consensus problem stated in this paper.

Motivated by the above-mentioned works, we study the consensus problem for linear multiagent systems to attenuate the communication errors by using dynamic output feedback controller. The agent dynamics considered here are general stabilizable and detectable linear systems, and a dynamic consensus protocol is proposed which uses only the relative output information between each agent and its neighbors. The main contributions of this paper can be summarized as two aspects. Firstly, in order to describe the effects of communication errors on consensus, a concept called consensus with H_∞ performance is introduced which can characterize the effects of communication errors on the difference between the state of each agent and the average of states of all agents. The problem of consensus with H_∞ performance is transmitted into an H_∞ control problem of another reduced-order system. It is shown that consensus with H_∞ performance can be achieved if there exists a common dynamic output feedback controller which can be realized by solving H_∞ problem for $N-1$ linear dynamic systems simultaneously, where N is the number of agents. Secondly, in terms of the $N-1$ linear systems, a sufficient condition based on linear matrix inequalities for the existence of the controller is provided, and the approach to construct the corresponding controller is given.

The rest of this paper is organized as follows. Section 2 introduces basic notations and reviews some useful results on graph theory and robust H_∞ control theory. Section 3 formulates the problem and conditions for reaching consensus with H_∞ performances that are derived. The existence for a dynamic output feedback protocol and a method to construct such controller are proposed in Section 4. Numerical simulations are provided in Section 5. Section 6 concludes the paper.

Notations. Let $\mathbb{R}^{n \times n}$ and $\mathbb{C}^{n \times n}$ be the set of $n \times n$ real matrices and $n \times n$ complex matrices, respectively. Matrices, if not explicitly stated, are assumed to have compatible dimensions. I_n and $0_{n \times m}$ are the $n \times n$ identity matrix and the $n \times m$ zero matrix, respectively. For a matrix A , $\|A\|_2$ is the induced 2-norm of the vector norm, and $\|A\|_2 \triangleq \sigma(A)$. The notation $\sigma(A)$ is the maximal singular value of matrix A . Notations A^T , A^{-1} , and A^* represent the transpose, the inverse, and the complex conjugate transpose of matrix A , respectively. Let $\text{Im}(A)$ and $\text{ker}(A)$ be the image space and kernel of A . $A < 0$ ($A > 0$) means that the matrix A is negative (positive) definite. $\text{Span}\{v_1, v_2, \dots, v_m\}$ is a subspace of $\mathbb{R}^n$ spanned by $\{v_1, v_2, \dots, v_m\}$, where $v_i \in \mathbb{R}^n$, $i = 1, 2, \dots, m$.

The notation $\otimes$ represents the Kronecker product. For a vector $x \in \mathbb{C}^n$, $|x| \triangleq \sqrt{x^*x}$ is the Euclidean norm. $\mathbf{1}_N$ denotes the $N \times 1$ column vector whose elements are all ones. The space of piecewise continuous functions in $\mathbb{R}^m$ that are square integrable over $[0, +\infty)$ is denoted by $\mathcal{L}_2^m[0, \infty)$, for any $v(t) \in \mathcal{L}_2^m[0, \infty)$, and its normalized energy is defined by $\|v(t)\|_{\mathcal{L}_2} = (\int_0^\infty |v(t)|^2 dt)^{1/2}$. Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a state space realization of $G(s) = C(sI - A)^{-1}B + D$.

2. Preliminaries

2.1. Graph Theory. Directed graphs are used to model the information interaction among agents. Let $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N, \mathcal{A}_N)$ be a directed weighted graph, where $\mathcal{V}_N = \{1, 2, \dots, N\}$ is the node set, $\mathcal{E}_N \subseteq \mathcal{V}_N \times \mathcal{V}_N$ is the edge set, and $\mathcal{A}_N \in \mathbb{R}^{N \times N}$ is a weighted adjacency matrix with nonnegative elements a_{ij} . An edge of $\mathcal{G}_N$ is denoted by (i, j) which means that agent j can directly get information from agent i . $(j, i) \in \mathcal{E}_N$ if and only if $a_{ij} > 0$, otherwise $a_{ij} = 0$. If $(i, j) \in \mathcal{E}_N \Leftrightarrow (j, i) \in \mathcal{E}_N$, then $\mathcal{G}_N$ is said to be an undirected graph. In this paper, we assume that there are no self-cycles in $\mathcal{G}_N$; that is, $a_{ii} = 0$, $i = 1, 2, \dots, N$. The in-degree and out-degree of the i th agent are, respectively, defined as $d_{\text{in}}(i) = \sum_{j=1}^N a_{ij}$ and $d_{\text{out}}(i) = \sum_{j=1}^N a_{ji}$. Let $d_{\text{max}} = \max_i \{d_{\text{in}}(i)\}$. Correspondingly, the Laplacian matrix of graph $\mathcal{G}_N$ is denoted by $L_N = \Delta_N - \mathcal{A}_N \in \mathbb{R}^{N \times N}$, where $\Delta_N = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ii} = d_{\text{in}}(i)$.

A sequence $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k)$ of edges is called a directed path from node i_1 to node i_k . $\mathcal{G}_N$ is called a strongly connected digraph if for any $i, j \in \mathcal{V}_N$, there is a directed path from i to j . $\mathcal{G}_N$ has a directed spanning tree if there exists a node $r \in \mathcal{V}_N$ (a root) such that all other nodes can be linked to r via a directed path. A directed graph is called balanced if $\sum_{j \neq i} a_{ij} = \sum_{i \neq j} a_{ji}$ for all $i \in \mathcal{V}$.

Below are well-known results for the Laplacian matrix.

Lemma 1 (see [3]). *The Laplacian matrix L_N of a directed graph has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}_N$.*

Lemma 2 (see [3]). *The Laplacian matrix L_N of a directed graph has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_N$, and all of the other eigenvalues have positive real parts if and only if the directed graph has a directed spanning tree.*

Lemma 3 (see [24]). *Let L_N be the Laplacian matrix of a directed graph $\mathcal{G}_N$, then there exists an orthogonal matrix $\bar{U} = [\mathbf{1}_N / \sqrt{N} \ U] \in \mathbb{R}^{N \times N}$ such that*

$$\bar{U}^T L_N \bar{U} = \begin{bmatrix} 0 & \frac{1}{\sqrt{N}} L_N U \\ 0 & U^T L_N U \end{bmatrix}. \quad (1)$$

Furthermore, if $\mathcal{G}$ is a balanced graph, then

$$\bar{U}^T L_N \bar{U} = \begin{bmatrix} 0 & 0 \\ 0 & U^T L_N U \end{bmatrix}. \quad (2)$$

2.2. Robust H_∞ Control Theory. Consider that the n th-order linear time-invariant (LTI) system is described as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1\omega(t) + B_2u(t), \\ z(t) &= C_1x(t) + D_{11}\omega(t) + D_{12}u(t), \\ y(t) &= C_2x(t) + D_{21}\omega(t) + D_{22}u(t),\end{aligned}\quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state, $\omega(t) \in \mathcal{L}_2^q[0, \infty)$ is the external disturbance, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^p$ is the regulated output, $y(t) \in \mathbb{R}^l$ is the measured output, and A, B_i, C_i , and D_{ij} , for $i, j = 1, 2$, are known real constant matrices of appropriate dimensions. Without loss of generality, we assume that $D_{22} = 0$, (A, B_2) is stabilizable and (C_2, A) is detectable.

The n_c th-order dynamic output feedback (DOF) controller is described as follows:

$$\begin{aligned}\dot{x}_c(t) &= A_k x_c(t) + B_k y(t), \\ u(t) &= C_k x_c(t) + D_k y(t),\end{aligned}\quad (4)$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the controller state and A_k, B_k, C_k , and D_k are constant matrices with appropriate dimensions.

Let $S_{pc}(s) = \left[\frac{A_c}{C_c} \middle| \frac{B_c}{D_c} \right]$ be the transfer function from $\omega(t)$ to $z(t)$ of the closed-loop system obtained from (3) and (4), where

$$\begin{aligned}A_c &= A_0 + \bar{B}K\bar{C}, & B_c &= B_0 + \bar{B}K\bar{D}_{21}, \\ C_c &= C_0 + \bar{D}_{12}K\bar{C}, & D_c &= D_{11} + \bar{D}_{12}K\bar{D}_{21}, \\ A_0 &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, & B_0 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, & C_0 &= [C_1 \ 0], \\ \bar{B} &= \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, & \bar{C} &= \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, & \bar{D}_{12} &= [0 \ D_{12}], \\ \bar{D}_{21} &= \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, & K &= \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}.\end{aligned}\quad (5)$$

The H_∞ problem for the given LTI system (3) is to find a DOF controller (4) such that the closed-loop system is internally stable and $\|S_{pc}(s)\|_\infty < \gamma$ for some constant $\gamma > 0$.

To facilitate the consensus protocol design and stability analysis, several results of the H_∞ problem are recalled as follows.

Lemma 4 (see [29]). *Given $\gamma > 0$, there exists a DOF controller (4) which can solve the H_∞ problem for the LTI system (3) if and only if there exist symmetric matrices $X > 0$ and $Y > 0$ such that*

$$(a) \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} XA + A^*X & XB_1 & C_1^* \\ B_1^*X & -\gamma I & D_{11}^* \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$(b) \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} YA^* + AY & YC_1^* & B_1 \\ C_1Y & -\gamma I & D_{11} \\ B_1^* & D_{11}^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$(c) \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0,$$
(6)

where N_X and N_Y are full-rank matrices whose images satisfy

$$\begin{aligned}\text{Im}(N_X) &= \ker([C_2 \ D_{21}]), \\ \text{Im}(N_Y) &= \ker([B_2^* \ D_{12}^*]).\end{aligned}\quad (7)$$

As the results shown by [29], the DOF controller (4) solving the H_∞ problem for the LTI system (3) can be constructed as follows.

- (i) Find $X > 0$ and $Y > 0$ which satisfy Lemma 4.
- (ii) Let $X_c = \begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix}$, where $X_2 \in \mathbb{R}^{n \times n_c}$ satisfying $X - Y^{-1} = X_2 X_2^*$.
- (iii) Solve the the following inequality:

$$H_{X_c} + Q^* K^* P_{X_c} + P_{X_c}^* K Q < 0. \quad (8)$$

For a feasible solution K , where

$$P_{X_c} = [\bar{B}^* X_c \ 0 \ \bar{D}_{12}^*], \quad Q = [\bar{C} \ \bar{D}_{21} \ 0],$$

$$H_{X_c} = \begin{bmatrix} X_c A_0 + A_0^* X_c & X_c B_0 & C_0^* \\ B_0^* X_c & -\gamma_0 I & D_{11}^* \\ C_0 & D_{11} & -\gamma_0 I \end{bmatrix}, \quad (9)$$

and $A_0, B_0, C_0, \bar{B}, \bar{C}, \bar{D}_{12}, \bar{D}_{21}$ are defined by (5). The solution K provides the state space realization for a feasible controller (4) which can solve H_∞ problem for system (3).

Lemma 5 (see [30]). *Let $\gamma > 0$, $G(s) = \left[\frac{A}{C} \middle| \frac{B}{D} \right]$ with A is Hurwitz stable, and*

$$H \triangleq \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C^*)^* \end{bmatrix}, \quad (10)$$

where $R = \gamma^2 I - D^*D$. Then, the following conditions are equivalent:

- (1) $\|G(s)\|_\infty < \gamma$,
- (2) $\sigma(D) < \gamma$ and H have no eigenvalues on the imaginary axis,
- (3) there exists a matrix $X = X^* > 0$ such that

$$\begin{bmatrix} XA + A^*X & XB & C^* \\ B^*X & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (11)$$

3. H_∞ Consensus Problem

Consider a multiagent system consisting of N identical agents with linear dynamics described by

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t)\end{aligned}\quad i = 1, \dots, N, \quad (12)$$

where $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^m$ is the control input, $y_i(t) \in \mathbb{R}^p$ is the measured output, and A , B , and C are constant matrices with compatible dimensions. It is assumed that (A, B) is stabilizable and (C, A) is detectable, and without loss of generality, B is of full column rank. We say that the control input $u_i(t)$ solves the consensus problem for the multiagent system (12) if the states of the agents satisfy

$$\lim_{t \rightarrow +\infty} [x_i(t) - x_j(t)] = 0, \quad \forall i, j \in \mathcal{V}_N \quad (13)$$

for any initial states.

Assume that the communication topology among the N agents is represented by a fixed directed graph $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N, \mathcal{A}_N)$. Based on the relative output information between the agents, the following dynamic output feedback (DOF) control protocol is used by [18]:

$$\begin{aligned}\dot{v}_i(t) &= A_k v_i(t) + B_k \sum_{j=0}^N a_{ij} [y_j(t) - y_i(t)] \\ u_i(t) &= C_k v_i(t) + D_k \sum_{j=0}^N a_{ij} [y_j(t) - y_i(t)]\end{aligned}\quad i = 1, \dots, N, \quad (14)$$

where $v_i(t) \in \mathbb{R}^{n_c}$, n_c is a preassigned dimension of the coordinating law, and a_{ij} is the element of the corresponding adjacency matrix $\mathcal{A}_N$. The system matrix

$$K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \quad (15)$$

of the DOF control protocol (14) need to be designed to make the multiagent system (12) achieve consensus. A general method for constructing the system matrix K was presented by [18].

However, if there exist communication errors between the i th agent and the j th agent, $i, j = 1, 2, \dots, N$, then the performance of consensus will be affected by these errors, as illustrated by the example given below.

Example 6. We consider double-integrator systems given by

$$\begin{aligned}\ddot{\xi}_i(t) &= u_i(t), \\ y_i(t) &= \xi_i(t),\end{aligned}\quad i = 1, 2, \dots, N, \quad (16)$$

where $\xi_i(t), y_i(t) \in \mathbb{R}$. Let $\dot{\xi}_i(t) = \zeta_i(t)$. Then, the above system can be rewritten as the form of (12) with

$$\begin{aligned}x_i(t) &= \begin{bmatrix} \xi_i(t) \\ \zeta_i(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C &= [1 \ 0].\end{aligned} \quad (17)$$

FIGURE 1: Communication topology.

The weighted communication topology with 6 agents is shown in Figure 1. Using the results presented in [18], the DOF control protocol (14) can be constructed with

$$\begin{aligned}A_k &= \begin{bmatrix} -10 & -1 \\ -50.1039 & -0.485 \end{bmatrix}, \quad B_k = \begin{bmatrix} 10 \\ 50 \end{bmatrix}, \\ C_k &= [0.1039 \ 0.485], \quad D_k = 0.\end{aligned} \quad (18)$$

It is known that the consensus is asymptotically achieved when there are no communication errors with the designed protocol (see Figure 2(a)). However, communication errors are inevitable. Assume that a 1% error appears in all of the communication channels. Simulation results show that, under the same protocol, the system diverges in the sense that the position state of each agent is far away from the position state of leader (node 1) as can be seen in Figure 2(b).

Example 6 implies that, under the influence of communication errors, consensus cannot be achieved for each agent with the given control protocol. This provides motivation to design an appropriate DOF control protocol to attenuate the effects of communication errors on the consensus performance. In this paper, we assume that there exist communication errors in the transferred data; that is, the DOF control protocol takes the following form:

$$\begin{aligned}\dot{v}_i(t) &= A_k v_i(t) + B_k \sum_{j=0}^N a_{ij} [y_j(t) - y_i(t) + \omega_{ij}(t)], \\ u_i(t) &= C_k v_i(t) + D_k \sum_{j=0}^N a_{ij} [y_j(t) - y_i(t) + \omega_{ij}(t)]\end{aligned}\quad i = 1, \dots, N, \quad (19)$$

FIGURE 2: The disagreement states between x_i and x_1 without/with communication errors, $i = 2, 3, \dots, 6$.

where $\omega_{ij} \in \mathcal{L}_2^p[0, +\infty)$ represents the communication error when the i th agent gets information from the j th agent. For convenience, denote

$$\begin{aligned}
 \mathbf{x}(t) &= [\mathbf{x}_1^T(t), \dots, \mathbf{x}_N^T(t)]^T \in \mathbb{R}^{N(n+n_c)}, \\
 \mathbf{x}_i(t) &= [x_i^T(t), v_i^T(t)]^T \in \mathbb{R}^{n+n_c}, \quad i = 1, 2, \dots, N, \\
 \mathbf{y}(t) &= [y_1^T(t), y_2^T(t), \dots, y_N^T(t)]^T \in \mathbb{R}^{Np}, \\
 \mathbf{d}(t) &= [\mathbf{d}_1^T(t), \mathbf{d}_2^T(t), \dots, \mathbf{d}_N^T(t)]^T \in \mathbb{R}^{N(N-1)p}, \\
 \mathbf{d}_i(t) &= [\omega_{i1}^T(t), \dots, \omega_{i(i-1)}^T(t), \omega_{i(i+1)}^T(t), \dots, \omega_{iN}^T(t)]^T \\
 &\in \mathbb{R}^{(N-1)p}, \quad i = 1, 2, \dots, N.
 \end{aligned} \tag{20}$$

Then, the overall dynamics result in the system (12) with the DOF control protocol (19) can be written as

$$\dot{\mathbf{x}}(t) = (I_N \otimes \bar{A} - L_N \otimes \bar{B}\bar{C})\mathbf{x}(t) + (D_N \otimes \bar{B})\mathbf{d}(t), \tag{21}$$

where

$$\begin{aligned}
 \bar{A} &= \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} BD_k \\ B_k \end{bmatrix}, \quad \bar{C} = [C \ 0], \\
 D_N &= \begin{bmatrix} \delta_1 & 0_{1 \times N-1} & \cdots & 0_{1 \times N-1} \\ 0_{1 \times N-1} & \delta_2 & \cdots & 0_{1 \times N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{1 \times N-1} & 0_{1 \times N-1} & \cdots & \delta_N \end{bmatrix} \in \mathbb{R}^{N \times N(N-1)}, \\
 \delta_i &= [l_{i1} \ \cdots \ l_{i(i-1)} \ l_{i(i+1)} \ \cdots \ l_{iN}], \quad i = 1, \dots, N,
 \end{aligned} \tag{22}$$

and l_{ij} is the element of the Laplacian matrix L_N .

In order to characterize the effects of the communication errors on consensus performance, we need to define a controlled output for the multiagent system (12) as follows.

Assume that the fixed directed communication graph $\mathcal{G}_N$ has a spanning tree, and according to Lemma 2, the Laplacian matrix L_N of graph $\mathcal{G}_N$ has a simple zero, and all of the other eigenvalues are in the right half-plane. Let $e_1 = (1/\sqrt{N})\mathbf{1}_N$.

By Lemma 3, there exists an orthogonal matrix $[e_1 \ U] \in \mathbb{R}^{N \times N}$ such that

$$\begin{bmatrix} e_1^T \\ U^T \end{bmatrix} L_N [e_1 \ U] = \begin{bmatrix} 0 & e_1^T L_N U \\ 0 & \bar{L}_N \end{bmatrix}, \quad (23)$$

where $\bar{L}_N = U^T L_N U$. It is obvious that the eigenvalues of $\bar{L}_N$ are equal to the nonzero eigenvalues of L_N , which means that all of the eigenvalues of $\bar{L}_N$ are in the right half-plane. Here, the matrix U satisfies $U^T \mathbf{1}_N = 0$, $U^T U = I_{N-1}$, and $U U^T = I_N - (1/N) \mathbf{1}_N \mathbf{1}_N^T$ according to $[e_1 \ U]$ being an orthogonal matrix.

Let $C_0 = [I_n \ 0]$. Define an output vector $\mathbf{z}(t)$ as

$$\mathbf{z}(t) = (U^T \otimes C_0) \mathbf{x}(t) = (U^T \otimes I_n) \mathbf{x}(t), \quad (24)$$

where $\mathbf{z}(t) = [\mathbf{z}_1^T(t), \dots, \mathbf{z}_{N-1}^T(t)]^T \in \mathbb{R}^{(N-1)n}$, $\mathbf{z}_i(t) \in \mathbb{R}^n$, and $\mathbf{x}(t) = [x_1^T(t), x_2^T(t), \dots, x_N^T(t)]^T$. Then,

$$|\mathbf{z}(t)|^2 = |(U^T \otimes C_0) \mathbf{x}(t)|^2 = |x(t) - \mathbf{1}_N \otimes x_0(t)|^2, \quad (25)$$

where $x_0(t) = (1/N) \sum_{i=1}^N x_i(t)$, which means that $\mathbf{z}(t)$ can measure the difference between the state of each agent and the average state of all agents.

Let $\hat{\mathbf{x}}(t) = (U^T \otimes I_{n+n_c}) \mathbf{x}(t)$. Using the DOF control protocol (19), the system dynamics with the output $\mathbf{z}(t)$ can be represented as

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \tilde{A} \hat{\mathbf{x}}(t) + \tilde{B} \mathbf{d}(t), \\ \mathbf{z}(t) &= \tilde{C} \hat{\mathbf{x}}(t), \end{aligned} \quad (26)$$

where $\tilde{A} = I_{N-1} \otimes \bar{A} - \bar{L}_N \otimes \bar{B} \bar{C}$, $\tilde{B} = U^T D_N \otimes \bar{B}$, and $\tilde{C} = I_{N-1} \otimes C_0$.

From the fact that the null-space of matrix $U^T \otimes I_n$ is $\text{Span}\{\mathbf{1}_N \otimes I_n\}$, we know that

$$\lim_{t \rightarrow +\infty} \mathbf{z}(t) = \lim_{t \rightarrow +\infty} (U^T \otimes I_n) \mathbf{x}(t) = 0 \quad (27)$$

if and only if there exist $x_0(t) \in \mathbb{R}^n$ such that $\lim_{t \rightarrow +\infty} x_i(t) = x_0(t)$, which implies that consensus of the multiagent system (12) can be achieved asymptotically. However, it is obvious that $\mathbf{z}(t)$ cannot approach zero as t tending to infinity due to the existence of communication error $\mathbf{d}(t)$, which indicates that consensus cannot be achieved for the system (12) with the DOF control protocol (19). Inspired by the analysis above, it is reasonable to evaluate the effects of communication error on consensus of the system (12) with the DOF control protocol (19) by using the effects of communication error $\mathbf{d}(t)$ on the output $\mathbf{z}(t)$ of system (26). Notice that the latter can be quantitatively measured by the H_∞ norm of the transfer function matrix $G_{dz}(s)$ from $\mathbf{d}(t)$ to $\mathbf{z}(t)$, which is defined by $\|G_{dz}(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \{\|G_{dz}(j\omega)\|_2\} = \sup_{\omega \in \mathbb{R}} \{\sigma(G_{dz}(j\omega))\}$, that results in the following definition.

Definition 7. Given a scalar $\gamma > 0$. The system (12) with the DOF control protocol (19) is called to achieve consensus with H_∞ performance if the following conditions hold.

- (1) It can reach consensus when $\mathbf{d}(t) = 0$;
- (2) $\|G_{dz}(s)\|_\infty < \gamma$, where $G_{dz}(s)$ is the transfer function matrix of system (26) from $\mathbf{d}(t)$ to $\mathbf{z}(t)$ and the output $\mathbf{z}(t)$ is defined by (24).

A sufficient condition is given in the following theorem to ensure that the multiagent system (12) with the DOF control protocol (19) can achieve consensus with H_∞ performance.

Theorem 8. Given a scalar $\gamma > 0$. Assume that the fixed communication topology $\mathcal{G}_N$ has a spanning tree. The system (12) with the DOF control protocol (19) achieves consensus with H_∞ performance if there exists a matrix $X_c = X_c^* > 0$ such that

$$\begin{bmatrix} X_c (\bar{A} - \lambda_i \bar{B} \bar{C}) + (\bar{A} - \lambda_i \bar{B} \bar{C})^* X_c & X_c \bar{B} & C_0^* \\ \bar{B}^* X_c & -\gamma_0 I & 0 \\ C_0 & 0 & -\gamma_0 I \end{bmatrix} < 0, \quad (28)$$

where $\gamma_0 = d_{\max}^{-1} \gamma$ and λ_i is the nonzero eigenvalue of Laplacian matrix L_N , $i = 2, 3, \dots, N$.

Proof. It is known that the system (12) with the DOF control protocol (19) achieves consensus with H_∞ performance if and only if $\tilde{A}$ is Hurwitz and

$$\|G_{dz}(s)\|_\infty = \|\tilde{C} (sI - \tilde{A})^{-1} \tilde{B}\|_\infty < \gamma, \quad (29)$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are defined by (26).

According to Lemma 5 and (28), we have $\bar{A} - \lambda_i \bar{B} \bar{C}$ is Hurwitz stable, and

$$\bar{H}_i \triangleq \begin{bmatrix} \bar{A} - \lambda_i \bar{B} \bar{C} & \gamma_0^{-2} \bar{B} \bar{B}^* \\ -C_0^* C_0 & -(\bar{A} - \lambda_i \bar{B} \bar{C})^* \end{bmatrix} \quad (30)$$

has no eigenvalues on the imaginary axis; that is, for any $\omega \in \mathbb{R}$,

$$\begin{aligned} (j\omega I - (\bar{A} - \lambda_i \bar{B} \bar{C})) \eta_1 - (\gamma_0^{-2} \bar{B} \bar{B}^*) \eta_2 &= 0, \\ C_0^* C_0 \eta_1 + (j\omega I + (\bar{A} - \lambda_i \bar{B} \bar{C})^*) \eta_2 &= 0, \end{aligned} \quad (31)$$

if and only if $\eta_1 = 0$ and $\eta_2 = 0$, where $\eta_1, \eta_2 \in \mathbb{C}^{n+n_c}$.

For matrix $\bar{L}_N$, there are two unitary matrices V_1 and V_2 such that

$$\begin{aligned} V_1^* \bar{L}_N V_1 &= T = [t_{ij}] \in \mathbb{R}^{(N-1) \times (N-1)}, \\ V_2^* \bar{L}_N V_2 &= S = [s_{ij}] \in \mathbb{R}^{(N-1) \times (N-1)}, \end{aligned} \quad (32)$$

where T and S are upper triangular, with diagonal entries $t_{ii} = \lambda_{i+1}$ and $s_{ii} = \lambda_{i+1}^*$, respectively, $i = 1, 2, \dots, N-1$. Now, suppose that $j\omega_0$ is an eigenvalue of

$$H \triangleq \begin{bmatrix} I_{N-1} \otimes \bar{A} - T \otimes \bar{B}\bar{C} & \gamma_0^{-2} I_{N-1} \otimes \bar{B}\bar{B}^* \\ -I_{N-1} \otimes C_0^* C_0 & -(I_{N-1} \otimes \bar{A} - S^* \otimes \bar{B}\bar{C})^* \end{bmatrix}. \quad (33)$$

Then, there exists a vector $\eta = [\eta_{1,1}^T, \eta_{1,2}^T, \dots, \eta_{1,(N-1)}^T, \eta_{2,1}^T, \eta_{2,2}^T, \dots, \eta_{2,(N-1)}^T]^T \neq 0$, where $\eta_{k,l} \in \mathbb{C}^{n+n_c}$, $k = 1, 2$, $l = 1, 2, \dots, N-1$, such that

$$\begin{aligned} & (j\omega_0 I - (\bar{A} - \lambda_{i+1} \bar{B}\bar{C})) \eta_{1,i} \\ & + \sum_{j=i+1}^{N-1} t_{ij} \bar{B}\bar{C} \eta_{1,j} - (\gamma_0^{-2} \bar{B}\bar{B}^*) \eta_{2,i} = 0, \\ & C_0^* C_0 \eta_{1,i} + (j\omega_0 I + (\bar{A} - \lambda_{i+1} \bar{B}\bar{C})^*) \eta_{2,i} \\ & - \sum_{j=i+1}^{N-1} s_{ij} (\bar{B}\bar{C})^* \eta_{2,j} = 0, \end{aligned} \quad (34)$$

where $i = 1, 2, \dots, N-1$. When $i = N-1$, from (31), it is easy to get that $\eta_{1,N-1} = \eta_{2,N-1} = 0$. Then, when $i = N-2$, (34) can be rewritten as

$$\begin{aligned} & (j\omega_0 I - (\bar{A} - \lambda_{N-1} \bar{B}\bar{C})) \eta_{1,N-2} - (\gamma_0^{-2} \bar{B}\bar{B}^*) \eta_{2,N-2} = 0, \\ & C_0^* C_0 \eta_{1,N-2} + (j\omega_0 I + (\bar{A} - \lambda_{N-1} \bar{B}\bar{C})^*) \eta_{2,N-2} = 0, \end{aligned} \quad (35)$$

which implies that $\eta_{1,N-2} = \eta_{2,N-2} = 0$. Similarly, it can be known that $\eta_{1,i} = \eta_{2,i} = 0$ for all $i = 1, 2, \dots, N-1$. This contradicts our assumption, and hence matrix H has no eigenvalues on the imaginary axis.

Notice that

$$\begin{aligned} \tilde{H} & \triangleq \begin{bmatrix} I_{N-1} \otimes \bar{A} - \bar{L}_N \otimes \bar{B}\bar{C} & \gamma_0^{-2} V_1 V_2^* \otimes \bar{B}\bar{B}^* \\ -V_2 V_1^* \otimes C_0^* C_0 & -(I_{N-1} \otimes \bar{A} - \bar{L}_N \otimes \bar{B}\bar{C})^* \end{bmatrix} \\ & = \begin{bmatrix} V_1 \otimes I_{n+n_c} & 0 \\ 0 & V_2 \otimes I_{n+n_c} \end{bmatrix} H \begin{bmatrix} V_1^* \otimes I_{n+n_c} & 0 \\ 0 & V_2^* \otimes I_{n+n_c} \end{bmatrix}, \end{aligned} \quad (36)$$

which means that matrix $\tilde{H}$ has no pure imaginary eigenvalues. Moreover, matrices $\bar{A} - \lambda_i \bar{B}\bar{C}$ are Hurwitz stable, $i = 2, 3, \dots, N$, which implies that matrix $\tilde{A} = I_{N-1} \otimes \bar{A} - \bar{L}_N \otimes \bar{B}\bar{C}$ is Hurwitz stable. Noting that V_1 and V_2 are unitary matrices, then there must exist two matrices Y_1 and Y_2 such that $Y_1 Y_1^* = V_1 V_1^*$, $Y_2^* Y_2 = V_2 V_2^*$, and $\|Y_1\|_2 = \|Y_2\|_2 = 1$. Thus, according to Lemma 5, it can be obtained that

$$\begin{aligned} & \|(Y_2 \otimes C_0) (sI - (I_{N-1} \otimes \bar{A} - \bar{L}_N \otimes \bar{B}\bar{C}))^{-1} (Y_1 \otimes \bar{B})\|_\infty \\ & = \|\tilde{C}(sI - \tilde{A})^{-1} (I_{N-1} \otimes \bar{B})\|_\infty < \gamma_0. \end{aligned} \quad (37)$$

In addition, it is easy to know that

$$\|U^T D_N\|_2 \leq \|D_N\|_2 = \max_i \left\{ \sqrt{\delta_i \delta_i^T} \right\} \leq d_{\max}. \quad (38)$$

Then, we have

$$\begin{aligned} \|G_{dz}(s)\|_\infty & = \|\tilde{C}(sI - \tilde{A})^{-1} \tilde{B}\|_\infty \\ & < d_{\max} \|\tilde{C}(sI - \tilde{A})^{-1} (I_{N-1} \otimes \bar{B})\|_\infty < \gamma. \end{aligned} \quad (39)$$

This completes the proof. $\square$

Remark 9. If there are no communication errors, that is, $\mathbf{d}(t) = 0$, then from Theorem 8, it is only required that $\bar{A} - \lambda_i \bar{B}\bar{C}$ is Hurwitz for all nonzero eigenvalues λ_i of the Laplacian matrix L_N . In this case, consensus can be achieved asymptotically for the multiagent system (12) with the DOF control protocol (19), which is the result shown by [18].

Remark 10. In fact, from Lemma 5, there exists $X_c > 0$ such that the inequality (28) holds, $i = 2, 3, \dots, N$, if and only if there exists a common n_c -th-order DOF controller

$$\mathcal{K}(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right] \quad (40)$$

solving H_∞ problem with performance γ_0 for $N-1$ n th-order LTI systems

$$\Sigma_i(s) = \left[\begin{array}{c|cc} A & 0 & B \\ \hline I_n & 0 & 0 \\ -\lambda_i C & D_{21} & 0 \end{array} \right], \quad (41)$$

where A , B , and C are the state space matrices for system (12), $D_{21} = I_p$, and λ_i is the nonzero eigenvalue of Laplacian matrix L_N , $i = 2, 3, \dots, N$.

4. Dynamic Output Feedback Design for H_∞ Consensus

In this section, we determine the system matrix K of the DOF control protocol (19) for the multiagent system (12) to achieve consensus with H_∞ performance. According to Remark 10, it is required to design a common DOF controller (40) to solve H_∞ problem for $N-1$ LTI systems. Notice that the common DOF controller is difficult to obtain; thus, we firstly consider the H_∞ problem for the systems $\Sigma_i(s)$, $i = 2, 3, \dots, N$.

Let N_X and N_Y be full-rank matrices whose images satisfy

$$\text{Im}(N_X) = \ker([C \ I_p]), \quad \text{Im}(N_Y) = \ker([B^* \ 0]). \quad (42)$$

Denote $N_X = [N_c^* \ N_I^*]^*$, where N_c and N_I have n and p rows, respectively.

Lemma 11. *If there exist matrices $X = X^* > 0$ and $Y = Y^* > 0$ such that*

$$(i) \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} XA + A^*X & I_n \\ I_n & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \gamma_m N_I^* N_I & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad (43)$$

$$(ii) N_Y^* \begin{bmatrix} YA^* + AY & Y \\ Y & -\gamma_0 I \end{bmatrix} N_Y < 0, \quad (44)$$

$$(iii) \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad (45)$$

where $\gamma_m = \gamma_0 \{\min_i |\lambda_i|^2\}$, then there exists a DOF controller $\mathcal{K}_i(s) = \begin{bmatrix} A_k^i & B_k^i \\ C_k^i & D_k^i \end{bmatrix}$ which can solve the H_∞ problem with a given performance $\gamma_0 > 0$ for the LTI system $\Sigma_i(s)$ given by (41), $i = 2, 3, \dots, N$.

Proof. According to Lemma 4, we have that there exists a DOF controller $\mathcal{K}_i(s)$ to solve the H_∞ problem for the LTI system $\Sigma_i(s)$ with a given index $\gamma_0 > 0$ if and only if there exist matrices $X_i = X_i^* > 0$ and $Y = Y^* > 0$ such that

$$(a) \begin{bmatrix} N_X^i & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i A + A^* X_i & 0 & I_n \\ 0 & -\gamma_0 I & 0 \\ I_n & 0 & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_X^i & 0 \\ 0 & I \end{bmatrix} < 0, \quad (46)$$

$$(b) \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} YA^* + AY & Y & 0 \\ Y & -\gamma_0 I & 0 \\ 0 & 0 & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$(c) \begin{bmatrix} X_i & I \\ I & Y \end{bmatrix} \geq 0,$$

where N_X^i spans the kernel of $[-\lambda_i C \ I_p]$. Notice the following facts:

$$[-\lambda_i C \ I_p] \begin{bmatrix} N_c \\ -\lambda_i N_I \end{bmatrix} = -\lambda_i (CN_c + I_p N_I) = 0, \quad (47)$$

and, for all nonzero λ_i ,

$$\text{rank} \begin{bmatrix} N_c \\ -\lambda_i N_I \end{bmatrix} = \text{rank} \begin{bmatrix} N_c \\ N_I \end{bmatrix}. \quad (48)$$

Thus, we can choose $N_X^i = \begin{bmatrix} N_c \\ -\lambda_i N_I \end{bmatrix}$. Let $X_i = X$ for all $i = 2, 3, \dots, N$. Then, it is easy to obtain that

$$\begin{aligned} & \begin{bmatrix} N_X^i & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i A + A^* X_i & 0 & I_n \\ 0 & -\gamma_0 I & 0 \\ I_n & 0 & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_X^i & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} XA + A^*X & I_n \\ I_n & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} \\ &\quad - \begin{bmatrix} \gamma_0 |\lambda_i|^2 N_I^* N_I & 0 \\ 0 & 0 \end{bmatrix} \\ &\leq \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} XA + A^*X & I_n \\ I_n & -\gamma_0 I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} \\ &\quad - \begin{bmatrix} \gamma_m N_I^* N_I & 0 \\ 0 & 0 \end{bmatrix} < 0 \end{aligned} \quad (49)$$

and (b), (c) naturally hold from (44), (45), respectively. This completes the proof. $\square$

Remark 12. Lemma 11 gives a sufficient condition for the existence of the controller $\mathcal{K}_i(s)$ which can solve the H_∞ problem for the system $\Sigma_i(s)$, $i = 2, 3, \dots, N$. As the results stated in Section 2.2, if there exist $X = X^* > 0$ and $Y = Y^* > 0$ satisfying Lemma 11, then the DOF controller $\mathcal{K}_i(s)$ can be obtained by solving the following inequality:

$$H_{X_c} + Q(\lambda_i)^* K_i^* P_{X_c} + P_{X_c}^* K_i Q(\lambda_i) < 0, \quad (50)$$

for K_i , $i = 2, 3, \dots, N$, where

$$H_{X_c} = \begin{bmatrix} X_c A_0 + A_0^* X_c & 0 & C_0^* \\ 0 & -\gamma_0 I & 0 \\ C_0 & 0 & -\gamma_0 I \end{bmatrix}, \quad (51)$$

$$P_{X_c} = [\widehat{B}^* X_c \ 0 \ 0], \quad Q(\lambda_i) = [\widehat{C}(\lambda_i) \ \widehat{D} \ 0],$$

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad C_0 = [I \ 0], \quad \widehat{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad (52)$$

$$\widehat{C}(\lambda_i) = \begin{bmatrix} 0 & I \\ -\lambda_i C & 0 \end{bmatrix}, \quad \widehat{D} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$K_i = \begin{bmatrix} A_k^i & B_k^i \\ C_k^i & D_k^i \end{bmatrix}, \quad X_c = \begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix}, \quad (53)$$

$$X_2 X_2^* = X - Y^{-1}.$$

Obviously, if there exists a common K that makes the inequality (50) hold for all $i = 2, 3, \dots, N$, then there exists a common n_c -th-order DOF controller $\mathcal{K}(s)$ which can solve the H_∞ problem for the LTI systems Σ_i , $i = 2, 3, \dots, N$. Thus, we have the following result.

Theorem 13. *Given a scalar $\gamma > 0$, and let $\gamma_0 = d_{\max}^{-1} \gamma$. Assume that the fixed communication topology $\mathcal{G}_N$ has a spanning tree.*

Then, there exists a DOF control protocol (19) for the system (12) achieving consensus with H_∞ performance if

- (i) there exist matrices $X > 0$ and $Y > 0$ satisfying (43), (44), and (45),

- (ii) there exists a matrix K satisfying the following LMIs:

$$H_{X_c} + Q(\lambda_i)^* K^* P_{X_c} + P_{X_c}^* K Q(\lambda_i) < 0, \quad (54)$$

for all nonzero eigenvalues λ_i of Laplacian matrix L_N , where H_{X_c} , P_{X_c} , and $Q(\lambda_i)$ are given in (51) and X_c is defined by (53).

Proof. From the analysis above, the conditions (i) and (ii) hold which implies that there exist matrices K and $X_c > 0$ such that (28) holds due to the fact that (54) is exactly (28). The proof is completed by using Theorem 8 directly. $\square$

Remark 14. If we want controllers of order n_c less than n , it is only required to add the additional constraint

$$\text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_c, \quad (55)$$

to (i) of Theorem 13, which can be obtained by using Corollary 7.8 given by [29] and Theorem 13 directly.

Remark 15. Theorem 13 gives the sufficient conditions under which there exists a DOF control protocol such that the multiagent system (12) achieve consensus with a given H_∞ performance. When the conditions are satisfied, the procedure to construct the DOF control protocol is presented as follows.

Step 1: Solve LMIs (43), (44), and (45) for getting a solution: $X > 0$ and $Y > 0$.

Step 2: Construct $X_c > 0$ as (53).

Step 3: Solve the $N - 1$ LMIs (54) for a common feasible solution K .

Remark 16. Assume that there is no communication error in the system. As shown in Remark 9, in this case, the given problem is to design a stabilizing controller $\mathcal{K}(s)$ defined by (40) for the LTI systems (41) with $D_{21} = 0$. Using Theorem 5.8 given by [29] and the fact that the kernels of $\lambda_i C$ and C are exactly equal for all nonzero λ_i , reproducing the steps of the proof of Theorem 13, we have the following results. Assume that the fixed communication topology $\mathcal{G}_N$ has a spanning tree, then there exists a DOF control protocol (14) with order n_c for the system (12) achieving consensus if

- (1) there exist matrices $X = X^* > 0$ and $Y = Y^* > 0$ such that

$$N_X^* (A^* X + X A) N_X < 0, \quad N_Y^* (A Y + Y A^*) N_Y < 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad \text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_c, \quad (56)$$

where N_X and N_Y span the kernels of C and B^* , respectively,

- (2) there exists a matrix K satisfying the following LMIs:

$$A_0^* P + P A_0 + \widehat{C}(\lambda_i)^* K^* \widehat{B}^* P + P \widehat{B} K \widehat{C}(\lambda_i) < 0, \quad (57)$$

for all nonzero eigenvalues λ_i of Laplacian matrix L_N , where

$$P = \begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix} \quad (58)$$

and X_2 satisfies $X_2 X_2^* = X - Y^{-1}$ and A_0 , $\widehat{B}$, and $\widehat{C}(\lambda_i)$ are given in (51).

Moreover, using the method similar to that stated in Remark 15, we can construct the DOF controller for the multiagent system (12) reaching consensus.

Notice that condition (ii) in Theorem 13 implies that we need to solve $N - 1$ LMIs after constructing X_c , which increase the difficulty of the numerical calculation if the size of the multiagent system N is large. We give the following conditions, which can reduce the computational complexity for getting the DOF control protocol by solving four LMIs.

Denote that α_i and β_i are the real part and imaginary part of λ_i , respectively, where λ_i is the nonzero eigenvalue of Laplacian matrix L_N , $i = 2, 3, \dots, N$. Let $\alpha_0 = \min_i \{\alpha_i\}$, $\bar{\alpha}_0 = \max_i \{\alpha_i\}$, $\beta_0 = \min_i \{\beta_i\}$, and $\bar{\beta}_0 = \max_i \{\beta_i\} = -\beta_0$.

Theorem 17. Given a scalar $\gamma > 0$, and let $\gamma_0 = d_{\max}^{-1} \gamma$. Assume that the fixed communication topology $\mathcal{G}_N$ has a spanning tree. Then, there exists a DOF control protocol (19) for the system (12) achieving consensus with H_∞ performance if

- (i) there exist matrices $X > 0$ and $Y > 0$ satisfying (43), (44), and (45);

- (ii) there exists a matrix K such that

$$H_{X_c} + Q(\alpha_k + j\beta_k)^* K^* P_{X_c} + P_{X_c}^* K Q(\alpha_k + j\beta_k) < 0 \quad (59)$$

for $(\alpha_k, \beta_k) \in \{(\alpha_0, \beta_0), (\alpha_0, \bar{\beta}_0), (\bar{\alpha}_0, \beta_0), (\bar{\alpha}_0, \bar{\beta}_0)\}$, where H_{X_c} , P_{X_c} , and $Q(\alpha_k + j\beta_k)$ are given in (51) and X_c is defined by (53).

Proof. To complete the proof, we only need to show that if LMI (59) holds for $(\alpha_k, \beta_k) \in \{(\alpha_0, \beta_0), (\alpha_0, \bar{\beta}_0), (\bar{\alpha}_0, \beta_0), (\bar{\alpha}_0, \bar{\beta}_0)\}$, then LMI (54) holds for all nonzero eigenvalues $\lambda_i = \alpha_i + j\beta_i$ of Laplacian matrix L_N , $i = 2, 3, \dots, N$.

Notice that

$$\begin{aligned} & H_{X_c} + Q(\alpha_i + j\beta_i)^* K^* P_{X_c} + P_{X_c}^* K Q(\alpha_i + j\beta_i) \\ &= \begin{bmatrix} X_c (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C}) + (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C})^* X_c & X_c \bar{B} & C_0^* \\ \bar{B}^* X_c & -\gamma_0 I & 0 \\ C_0 & 0 & -\gamma_0 I \end{bmatrix} \\ &< 0 \end{aligned} \quad (60)$$

which, in virtue of the Schur Complement Lemma, is equivalent to

$$\begin{aligned} & X_c (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C}) + (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C})^* X_c \\ &+ \gamma_0^{-1} (X_c \bar{B} \bar{B}^* X_c + C_0^* C_0) < 0, \end{aligned} \quad (61)$$

FIGURE 3: The disagreement states between x_i and x_1 without communication error $\mathbf{d}(t)$, $i = 2, 3, \dots, 6$.

FIGURE 4: The state trajectories of the multiagent system with communication error $\mathbf{d}(t)$.

where $\bar{A}$, $\bar{B}$, and $\bar{C}$ are defined by (22). For convenience, we denote

$$\begin{aligned} \mathcal{H}(\alpha_i, \beta_i) &= X_c (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C}) \\ &\quad + (\bar{A} - (\alpha_i + j\beta_i) \bar{B} \bar{C})^* X_c \\ &\quad + \gamma_0^{-1} (X_c \bar{B} \bar{B}^* X_c + C_0 C_0^*) \\ &= H_0 + \alpha_i H_1 + j\beta_i H_2, \end{aligned} \quad (62)$$

where $H_0 = X_c \bar{A} + \bar{A}^* X_c + \gamma_0^{-1} X_c \bar{B} \bar{B}^* X_c + \gamma_0^{-1} C_0 C_0^*$, $H_1 = -X_c \bar{B} \bar{C} - \bar{C}^* \bar{B}^* X_c$, and $H_2 = -X_c \bar{B} \bar{C} + \bar{C}^* \bar{B}^* X_c$.

In fact, there must exist $s \in [0, 1]$, $r \in [0, 1]$ such that $\alpha_i = s\alpha_0 + (1-s)\bar{\alpha}_0$, $\beta_i = r\beta_0 + (1-r)\bar{\beta}_0$. When $r \geq s$,

$$\begin{aligned} \mathcal{H}(\alpha_i, \beta_i) &= H_0 + [s\alpha_0 + (1-s)\bar{\alpha}_0] H_1 \\ &\quad + j[r\beta_0 + (1-r)\bar{\beta}_0] H_2 \\ &= s\mathcal{H}(\alpha_0, \beta_0) + (r-s)\mathcal{H}(\bar{\alpha}_0, \beta_0) \\ &\quad + (1-r)\mathcal{H}(\bar{\alpha}_0, \bar{\beta}_0) < 0. \end{aligned} \quad (63)$$

Similarly, when $s \geq r$, we have

$$\begin{aligned} \mathcal{H}(\alpha_i, \beta_i) &= r\mathcal{H}(\alpha_0, \beta_0) + (s-r)\mathcal{H}(\alpha_0, \bar{\beta}_0) \\ &\quad + (1-s)\mathcal{H}(\bar{\alpha}_0, \bar{\beta}_0) < 0. \end{aligned} \quad (64)$$

This completes the proof. $\square$

5. Numerical Example

An example is shown to verify the results obtained in the above section. The agent dynamics and the communication

topology are given in Example 6, and the H_∞ performance index is chosen as $\gamma = 10$. According to the results presented in Section 4, we have

$$\begin{aligned} X &= \begin{bmatrix} 1.8218 & -1.2978 \\ -1.2978 & 3.6435 \end{bmatrix}, \quad Y = \begin{bmatrix} 1.2145 & -0.9109 \\ -0.9109 & 3.6435 \end{bmatrix}, \\ X_c &= \begin{bmatrix} 1.8218 & -1.2978 & -0.8688 & 0.2312 \\ -1.2978 & 3.6435 & 1.8148 & 0.1107 \\ -0.8688 & 1.8148 & 1.0000 & 0 \\ 0.2312 & 0.1107 & 0 & 1.0000 \end{bmatrix}. \end{aligned} \quad (65)$$

Then, by solving the LMIs (54), we can get a feasible controller (40) with

$$\begin{aligned} A_k &= \begin{bmatrix} -7.0377 & -5.6811 \\ -0.6005 & -5.2415 \end{bmatrix}, \quad B_k = \begin{bmatrix} -9.5464 \\ -0.5110 \end{bmatrix}, \\ C_k &= [3.4520 \quad 2.9461], \quad D_k = 4.9523, \end{aligned} \quad (66)$$

and the H_∞ norm of system (26) is $\|G_{dz}(s)\|_\infty = 3.0707$. With the designed DOF control protocol, the disagreement states between x_i and x_1 without communication error $\mathbf{d}(t)$ are shown in Figure 3, $i = 2, 3, \dots, 6$, which implies that the consensus can be reached when $\mathbf{d}(t) = 0$. The communication error $\mathbf{d}(t) \in \mathcal{L}_2[0, \infty)$ is supposed to be

$$\mathbf{d}(t) = \begin{cases} \theta \sin(t), & t \in [0, 40], \\ 0, & t > 40, \end{cases} \quad (67)$$

where $\theta = [1.11 \quad 0.58 \quad 1.48 \quad 1.59 \quad 2.12 \quad 2.53 \quad 0.42 \quad 1.31 \quad 4.0 \quad 0.14]^T$. Under zero initial condition, the state trajectories of the six agents are depicted in Figure 4, and the corresponding energy trajectories of $\mathbf{d}(t)$ and $\mathbf{z}(t)$ are given in Figure 5.

FIGURE 5: Energy trajectories of the controlled output $\mathbf{z}(t)$ and the communication error $\mathbf{d}(t)$.

It is noted obviously that $\|\mathbf{z}(t)\|_{\mathcal{L}_2} \leq \|\mathbf{d}(t)\|_{\mathcal{L}_2}$. Thus, the multi-agent system with the given DOF controller can achieve the consensus with the given H_∞ performance, which validates the effectiveness of the proposed protocol and demonstrates the correctness of the obtained theoretical results.

6. Conclusions

This paper is devoted to the consensus problem for multi-agent systems molded by linear time-invariant systems under fixed directed communication topologies and subject to communication errors in the transferred data. A dynamic output feedback control algorithm is proposed. The theoretical analysis shows that if there exists a common dynamic output feedback controller which can solve H_∞ problem for $N - 1$ linear time-invariant systems of order n , then the consensus with a desired H_∞ level can be reached. By using H_∞ theory, a sufficient condition in terms of linear matrix inequalities is given to ensure the existence for such a controller. A procedure for the controller design is presented.

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