Research Article

Exact Solutions of Fractional Burgers and Cahn-Hilliard Equations Using Extended Fractional Riccati Expansion Method

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Received 13 February 2014; Accepted 18 May 2014; Published 10 June 2014

Academic Editor: Miguel A. F. Sanjuan

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Based on a general fractional Riccati equation and with Jumarie's modified Riemann-Liouville derivative to an extended fractional Riccati expansion method for solving the time fractional Burgers equation and the space-time fractional Cahn-Hilliard equation, the exact solutions expressed by the hyperbolic functions and trigonometric functions are obtained. The obtained results show that the presented method is effective and appropriate for solving nonlinear fractional differential equations.

1. Introduction

During recent years, fractional differential equations (FDEs) have attracted much attention because of their potential applications in engineering and applied sciences such as signal processing, materials and mechanics, biology systems, anomalous diffusion, and medical [1–4]. Thus, it is very important to find some proper methods for solving fractional differential equations. Many effective methods have been established to obtain the solutions of FDEs, such as the variational iteration method [5, 6], the finite difference method [7], the fractional complex transform [8, 9], the exponential function method [10], the fractional subequation method [11], the (G'/G) -expansion method [12, 13], and the first integral method [14]. However, the solutions of most fractional differential equations cannot be obtained explicitly in general. As a consequence, it is necessary to extend the existing methods for the fractional differential equations.

Recently, based on Jumarie's modified Riemann-Liouville derivative and the fractional Riccati equation $D_{\xi}^{\alpha}\phi(\xi) = \sigma + \phi^2(\xi)$, S. Zhang and H.-Q. Zhang [15] first proposed a new direct method called fractional subequation method in solving nonlinear time fractional biological population model and (4+1)-dimensional space-time fractional Fokas equation. Soon, the method was further improved by Guo et al. [16] and they obtained the analytical solutions of the space-time fractional Whitham-Broer-Kaup and generalized

Hirota-Satsuma coupled KdV equations by introducing a new general ansatz. In a similar way, Lu [17] modified the method to derive the rational formal solutions of the space-time fractional Whitham-Broer-Kaup, the foam drainage equation with time and space-fractional derivatives, and nonlinear time fractional biological population model. Very lately, by extending the fractional Riccati equation in [15–17] to the more general form $D_{\xi}^{\alpha}F(\xi) = A + BF^2(\xi)$, Abdel-Salam and Yousif [18] presented the fractional Riccati expansion method to obtain exact solutions of the space-time fractional Korteweg-de Vries equation, the space-time fractional RLW equation, the space-time fractional Boussinesq equation, and the space-time fractional Klein-Gordon equation. In this paper, based on Jumarie's modified Riemann-Liouville derivative and the general fractional Riccati equation $D_{\xi}^{\alpha}F(\xi) = A + BF(\xi) + CF^2(\xi)$, we will introduce an extended fractional Riccati expansion method to the time fractional Burgers equation and the space-time fractional Cahn-Hilliard equation [19]

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \varepsilon u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, 0 < \alpha \leq 1, \quad (1)$$

$$D_t^{\alpha}u - \gamma D_x^{\alpha}u - 6u(D_x^{\alpha}u)^2 - (3u^2 - 1)D_x^{2\alpha}u + D_x^{4\alpha}u = 0, \quad 0 < \alpha \leq 1, \quad (2)$$

to construct new exact solutions. This paper is organized as follows. In Section 2, some basic properties of Jumarie's modified Riemann-Liouville derivative are given. In Section 3, the main steps of the extended fractional Riccati expansion method are given. In Sections 4 and 5, we construct the exact solutions of (1) and (2) by the proposed method. Some conclusions are given in Section 6.

2. Preliminaries

There are several definitions for fractional differential equations. These definitions include Caputo's fractional derivative [20], Kolwankar-Gangal derivative [21], and Jumarie's modified Riemann-Liouville derivative [22, 23]. In this, we give the modified Riemann-Liouville derivative defined by Jumarie:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, \quad 0 < \alpha < 1,$$

$$D_x^\alpha f(x) = \left(f^{(n)}(x)\right)^{(\alpha-n)}, \quad n \leq \alpha \leq n+1, n \geq 1, \quad (3)$$

where $f: R \rightarrow R$, $x \rightarrow f(x)$ denote a continuous (but not necessarily differentiable) function. Some useful formulas of Jumarie's modified Riemann-Liouville derivative were summarized in [22, 23]; three of them are

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0,$$

$$D_x^\alpha (u(x) v(x)) = v(x) D_x^\alpha u(x) + u(x) D_x^\alpha v(x),$$

$$D_x^\alpha f[u(x)] = f'_u[u(x)] D_x^\alpha u(x) = D_u^\alpha f[u(x)] (u'(x))^\alpha, \quad (4)$$

which will be used in the following sections.

3. Extended Fractional Riccati Expansion Method

In this section, we give the description of the extended fractional Riccati expansion method for solving the nonlinear FDE as

$$P(u, u_t, u_x, D_t^\alpha u, D_x^\alpha u, \dots) = 0, \quad 0 < \alpha \leq 1, \quad (5)$$

where u is an unknown function and P is a polynomial of u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Using the traveling wave transformation

$$u(x, t) = U(\xi), \quad \xi = x + \omega t, \quad (6)$$

where ω is a constant to be determined later, the nonlinear FDE (5) is reduced to a nonlinear fractional ordinary differential equation

$$Q(U(\xi), U'(\xi), U''(\xi), D_\xi^\alpha U(\xi), \dots) = 0, \quad (7)$$

where ${}^{\alpha t} = d/d\xi$.

Step 2. Suppose that the solution of ODE (7) can be written as follows:

$$U = a_0 + \sum_{j=1}^n (a_j F^j(\xi) + b_j F^{-j}(\xi)), \quad (8)$$

where a_0, a_j, b_j ($j = 1, 2, \dots, n$) are constants to be determined later and n is a positive integer that is given by the homogeneous balance principle, and $F = F(\xi)$ satisfies the Riccati equation

$$D_\xi^\alpha F(\xi) = A + BF(\xi) + CF^2(\xi), \quad 0 < \alpha \leq 1, \quad (9)$$

where A, B , and C are constants. Using the Mittag-Leffler function in one parameter $E_\alpha(x) = \sum_{k=0}^{\infty} (x^k / \Gamma(1 + k\alpha))$, ($\alpha > 0$), we obtain the following solutions of (9).

Type 1. When $B = 0$ and $A/C < 0$, the solutions of (9) are

$$F(\xi) = -\sqrt{-\frac{A}{C}} \tanh_\alpha \left(C \sqrt{-\frac{A}{C}} \xi \right), \quad (10)$$

$$F(\xi) = -\sqrt{-\frac{A}{C}} \coth_\alpha \left(C \sqrt{-\frac{A}{C}} \xi \right), \quad (11)$$

$$F(\xi) = -\sqrt{-\frac{A}{C}} \times \left[\coth_\alpha \left(2C \sqrt{-\frac{A}{C}} \xi \right) \pm \operatorname{csch}_\alpha \left(2C \sqrt{-\frac{A}{C}} \xi \right) \right]. \quad (12)$$

When $B = 0$ and $A/C > 0$, the solutions of (9) are

$$F(\xi) = \sqrt{\frac{A}{C}} \tanh_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right), \quad (13)$$

$$F(\xi) = -\sqrt{\frac{A}{C}} \cot_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right), \quad (14)$$

$$F(\xi) = \sqrt{\frac{A}{C}} \left[\tanh_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm \sec_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \right], \quad (15)$$

$$F(\xi) = -\sqrt{\frac{A}{C}} \left[\cot_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm \csc_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \right]. \quad (16)$$

Type 2. When $A = 0$ and $BC \neq 0$, the solution of (9) is

$$F(\xi) = -\frac{Bd}{C[d + \cosh_\alpha(B\xi) - \sinh_\alpha(Bd)]}, \quad (17)$$

where d is an arbitrary constant.

Type 3. When $ABC \neq 0$ and $\Delta = B^2 - 4AC > 0$, the solutions of (9) are

$$F(\xi) = \frac{2A \cosh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right)}{\sqrt{\Delta} \sinh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right) - B \cosh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right)}, \quad (18)$$

$$F(\xi) = -\frac{2A \sinh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right)}{B \sinh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right) - \sqrt{\Delta} \cosh_\alpha \left(\left(\sqrt{\Delta}/2 \right) \xi \right)}. \quad (19)$$

When $ABC \neq 0$ and $\Delta = B^2 - 4AC < 0$, the solutions of (9) are

$$F(\xi) = -\frac{2A\cos_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{\sqrt{-\Delta}\sin_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right) + B\cos_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}, \quad (20)$$

$$F(\xi) = -\frac{2A\sin_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}{B\sin_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right) - \sqrt{-\Delta}\cos_\alpha\left(\left(\sqrt{-\Delta}/2\right)\xi\right)}. \quad (21)$$

The generalized hyperbolic and trigonometric functions in the above solutions (10)–(21) are defined as

$$\begin{aligned} \sinh_\alpha(\xi) &= \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2}, \\ \cosh_\alpha(\xi) &= \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2}, \\ \sin_\alpha(\xi) &= \frac{E_\alpha(i\xi^\alpha) - E_\alpha(-i\xi^\alpha)}{2i}, \\ \cos_\alpha(\xi) &= \frac{E_\alpha(i\xi^\alpha) + E_\alpha(-i\xi^\alpha)}{2i}, \\ \tanh_\alpha(\xi) &= \frac{\sinh_\alpha(\xi)}{\cosh_\alpha(\xi)}, \quad \coth_\alpha(\xi) = \frac{\cosh_\alpha(\xi)}{\sinh_\alpha(\xi)}, \\ \operatorname{sech}_\alpha(\xi) &= \frac{1}{\cosh_\alpha(\xi)}, \quad \operatorname{csch}_\alpha(\xi) = \frac{1}{\sinh_\alpha(\xi)}, \\ \tan_\alpha(\xi) &= \frac{\sin_\alpha(\xi)}{\cos_\alpha(\xi)}, \quad \cot_\alpha(\xi) = \frac{\cos_\alpha(\xi)}{\sin_\alpha(\xi)}, \\ \sec_\alpha(\xi) &= \frac{1}{\cos_\alpha(\xi)}, \quad \csc_\alpha(\xi) = \frac{1}{\sin_\alpha(\xi)}. \end{aligned} \quad (22)$$

Step 3. Substituting (8) into (7) and using (9), collecting all terms with the same order of $F(\xi)$ together, and then setting coefficients of the polynomial to zero yield a set of algebraic equations for a_0, a_j, b_j ($j = 1, 2, \dots, n$), and ω .

Step 4. Solving the algebraic equations obtained in Step 3, the constants a_0, a_j, b_j ($j = 1, 2, \dots, n$), and ω can be expressed. Substituting these values into expression (8), we can obtain the general form of the exact solution of (7).

Step 5. Substituting the exact solutions of (9) into the general form of exact solution obtained in Step 4, then we can obtain the exact solutions of (5).

Remark 1. If we take $A = \sigma$, $B = 0$, and $C = 1$, this agrees with the results obtained by S. Zhang and H.-Q. Zhang [15].

4. Exact Solutions of (1)

In this section, we use the above extended fractional Riccati expansion method to explore the exact solutions of time fractional Burgers equation (1). The study of Burgers equation

is important since it arises in the approximate theory of flow through a shock wave propagating in a viscous fluid [24] and in the modeling of turbulence [25]. In [26], Inc used the variational iteration method to obtain the numerical solutions of (1) for $\varepsilon = 1$. The approximate solution of (1) for $\varepsilon = 1$ based on Von-Neumann method is recently considered by El-Danaf and Hadhoud [27]. Bekir et al. [19] solved (1) using the exponential function method based on fractional complex transform to convert fractional differential equations into ordinary differential equations.

Let

$$u(x, t) = U(\xi), \quad \xi = x + \omega t, \quad (23)$$

where ω is nonzero constant, and substituting (23) into (1), we obtain

$$\omega^\alpha D_\xi^\alpha U + \varepsilon U U' - \nu U'' = 0. \quad (24)$$

Thus, the solution of (24) has the form

$$U = a_0 + a_1 F(\xi) + \frac{b_1}{F(\xi)}. \quad (25)$$

Substituting (25) into (24) and using (9), collecting the coefficients of $F(\xi)$, and then setting the coefficients of $F(\xi)$ to zero, we can obtain a set of algebraic equations about a_0, a_1, b_1, ω . Solving the algebraic equations obtained above, we can have the following solutions.

Type 1. When $B = 0$, we have the following cases.

Case 1. We have

$$a_0 = -\frac{\omega^\alpha}{\varepsilon}, \quad a_1 = \frac{2\nu C}{\varepsilon}, \quad b_1 = 0, \quad (26)$$

where ω is nonzero constants.

Substituting (26) into (25), then according to (10)–(16), we obtain the following hyperbolic function and trigonometric function solutions of (1), respectively. Consider

$$u_1(x, t) = -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{-\frac{A}{C}} \tanh_\alpha\left(C \sqrt{-\frac{A}{C}} \xi\right), \quad (27)$$

where $A/C < 0$, $\xi = x + \omega t$.

The profile for exact solution $u_1(x, t)$ of (27) is shown in Figure 1. Consider

$$u_2(x, t) = -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{-\frac{A}{C}} \coth_\alpha\left(C \sqrt{-\frac{A}{C}} \xi\right), \quad (28)$$

where $A/C < 0$, $\xi = x + \omega t$. Consider

$$\begin{aligned} u_3(x, t) = & -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{-\frac{A}{C}} \left[\coth_\alpha\left(2C \sqrt{-\frac{A}{C}} \xi\right) \right. \\ & \left. \pm \operatorname{csch}_\alpha\left(2C \sqrt{-\frac{A}{C}} \xi\right) \right], \end{aligned} \quad (29)$$

FIGURE 1: The solution $u_1(x, t)$ in (27) shows the 3D and 2D graphs for the given parameters $A = -1$, $C = 1$, $\nu = 0.6$, $\varepsilon = 0.1$, and $\omega = 0.2$.

where $A/C < 0$, $\xi = x + \omega t$. Consider

$$u_4(x, t) = -\frac{\omega^\alpha}{\varepsilon} + \frac{2\nu C}{\varepsilon} \sqrt{\frac{A}{C}} \tan_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right), \quad (30)$$

where $A/C > 0$, $\xi = x + \omega t$. Consider

$$u_5(x, t) = -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{\frac{A}{C}} \cot_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right), \quad (31)$$

where $A/C > 0$, $\xi = x + \omega t$. Consider

$$u_6(x, t) = -\frac{\omega^\alpha}{\varepsilon} + \frac{2\nu C}{\varepsilon} \sqrt{\frac{A}{C}} \left[\tan_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm \sec_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \right], \quad (32)$$

where $A/C > 0$, $\xi = x + \omega t$. Consider

$$u_7(x, t) = -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{\frac{A}{C}} \left[\cot_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm \csc_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \right], \quad (33)$$

where $A/C > 0$, $\xi = x + \omega t$.

Case 2. We have

$$a_0 = -\frac{\omega^\alpha}{\varepsilon}, \quad a_1 = 0, \quad b_1 = -\frac{2\nu A}{\varepsilon}, \quad (34)$$

where ω is nonzero constants.

Substituting (34) into (25), then according to (12), (15), and (16), we obtain the following hyperbolic function and trigonometric function solutions of (1), respectively:

$$u_8(x, t) = \left(2\nu A \sinh_\alpha \left(2C \sqrt{-\frac{A}{C}} \xi \right) - \omega^\alpha \sqrt{-\frac{A}{C}} \cosh_\alpha \left(2C \sqrt{-\frac{A}{C}} \xi \right) \mp \omega^\alpha \sqrt{-\frac{A}{C}} \right) \times \left(\varepsilon \sqrt{-\frac{A}{C}} \left[\cosh_\alpha \left(2C \sqrt{-\frac{A}{C}} \xi \right) \pm 1 \right] \right)^{-1}, \quad (35)$$

where $A/C < 0$, $\xi = x + \omega t$. Consider

$$u_9(x, t) = -\left(\omega^\alpha \sqrt{\frac{A}{C}} \sin_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) + 2\nu A \cos_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm \omega^\alpha \sqrt{\frac{A}{C}} \right) \quad (36)$$

$$\times \left(\varepsilon \sqrt{\frac{A}{C}} \left[\sin_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm 1 \right] \right)^{-1},$$

where $A/C > 0$, $\xi = x + \omega t$. Consider

$$u_{10}(x, t) = \left(2\nu A \sin_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) - \omega^\alpha \sqrt{\frac{A}{C}} \cos_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \mp \omega^\alpha \sqrt{\frac{A}{C}} \right) \times \left(\varepsilon \sqrt{\frac{A}{C}} \left[\cos_\alpha \left(2C \sqrt{\frac{A}{C}} \xi \right) \pm 1 \right] \right)^{-1}, \quad (37)$$

where $A/C > 0$, $\xi = x + \omega t$.

FIGURE 2: The solution $u_{14}(x, t)$ in (44) shows the 3D and 2D graphs for the given parameters $A = -1$, $B = 2$, $C = 1$, $\nu = 0.6$, $\varepsilon = 0.1$, and $\omega = 0.1$.

Case 3. We have

$$a_0 = -\frac{\omega^\alpha}{\varepsilon}, \quad a_1 = \frac{2\nu C}{\varepsilon}, \quad b_1 = -\frac{2\nu A}{\varepsilon}, \quad (38)$$

where ω is nonzero constants.

Substituting (38) into (25), then according to (10) and (13), we obtain the following hyperbolic function and trigonometric function solutions of (1), respectively:

$$u_{11}(x, t) = -\frac{\omega^\alpha}{\varepsilon} - \frac{2\nu C}{\varepsilon} \sqrt{-\frac{A}{C}} \tanh_\alpha \left(C \sqrt{-\frac{A}{C}} \xi \right) + \frac{2\nu A}{\varepsilon \sqrt{-A/C}} \coth_\alpha \left(C \sqrt{-\frac{A}{C}} \xi \right), \quad (39)$$

where $A/C < 0$, $\xi = x + \omega t$. Consider

$$u_{12}(x, t) = -\frac{\omega^\alpha}{\varepsilon} + \frac{2\nu C}{\varepsilon} \sqrt{\frac{A}{C}} \tan_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right) - \frac{2\nu A}{\varepsilon \sqrt{A/C}} \cot_\alpha \left(C \sqrt{\frac{A}{C}} \xi \right), \quad (40)$$

where $A/C > 0$, $\xi = x + \omega t$.

Type 2. When $A = 0$ and $BC \neq 0$, we have

$$a_0 = \frac{\nu B - \omega^\alpha}{\varepsilon}, \quad a_1 = \frac{2\nu C}{\varepsilon}, \quad b_1 = 0, \quad (41)$$

where ω is nonzero constants.

Substituting (41) into (25), then according to (17), we obtain the following hyperbolic function solution of (1):

$$u_{13}(x, t) = \frac{\nu B - \omega^\alpha}{\varepsilon} - \frac{2\nu B d}{\varepsilon [d + \cosh_\alpha(B\xi) - \sinh_\alpha(Bd)]}, \quad (42)$$

where $\xi = x + \omega t$.

Type 3. When $ABC \neq 0$, we have

$$a_0 = \frac{\nu B - \omega^\alpha}{\varepsilon}, \quad a_1 = \frac{2\nu C}{\varepsilon}, \quad b_1 = 0, \quad (43)$$

where ω is nonzero constants.

Substituting (43) into (25), then according to (18)–(21), we obtain the following hyperbolic function and trigonometric function solutions of (1), respectively:

$$u_{14}(x, t) = \frac{\nu B - \omega^\alpha}{\varepsilon} + \left(4\nu C A \cosh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right) \times \left(\varepsilon \left[\sqrt{\Delta} \sinh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - B \cosh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right] \right)^{-1}, \quad (44)$$

where $\Delta = B^2 - 4AC > 0$, $\xi = x + \omega t$.

The profile for exact solution $u_{14}(x, t)$ of (44) is shown in Figure 2. Consider

$$u_{15}(x, t) = \frac{\nu B - \omega^\alpha}{\varepsilon} - \left(4\nu C A \sinh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right) \times \left(\varepsilon \left[B \sinh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - \sqrt{\Delta} \cosh_\alpha \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right] \right)^{-1}, \quad (45)$$

FIGURE 3: The solution $u_4(x, t)$ in (56) shows the 3D and 2D graphs for the given parameter $\gamma = 2.3$.

where $\Delta = B^2 - 4AC > 0$, $\xi = x + \omega t$. Consider

$$u_{16}(x, t) = \frac{\nu B - \omega^\alpha}{\varepsilon} - \left(4\nu C A \cos_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right) \times \left(\varepsilon \left[\sqrt{-\Delta} \sin_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) + B \cos_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right] \right)^{-1}, \quad (46)$$

where $\Delta = B^2 - 4AC < 0$, $\xi = x + \omega t$. Consider

$$u_{17}(x, t) = \frac{\nu B - \omega^\alpha}{\varepsilon} - \left(4\nu C A \sin_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right) \times \left(\varepsilon \left[B \sin_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) - \sqrt{-\Delta} \cos_\alpha \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right] \right)^{-1}, \quad (47)$$

where $\Delta = B^2 - 4AC < 0$, $\xi = x + \omega t$.

5. Exact Solutions of (2)

In this section, we use the above extended fractional Riccati expansion method to explore the exact solutions of Cahn-Hilliard equation (2). In the case of $\alpha = 1$, this equation is related to a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics [19]. The study of Cahn-Hilliard equation is important since it arises in material sciences [28, 29]. However we notice that this equation is very difficult to be solved and several articles investigated it [19, 30]. Jafari et al. [31] used the fractional subequation method to calculate the numerical solutions of (2). Bekir et al. [19] solved (2) using the

exponential function method based on fractional complex transform to convert fractional differential equations into ordinary differential equations.

Let

$$u(x, t) = U(\xi), \quad \xi = x + \omega t, \quad (48)$$

where ω is nonzero constant, and substituting (48) into (2), we obtain

$$(\omega^\alpha - \gamma) D_\xi^\alpha U - 6U(D_\xi^\alpha U)^2 - (3U^2 - 1) D_\xi^{2\alpha} U + D_\xi^{4\alpha} U = 0. \quad (49)$$

Thus, the solution of (49) has the form

$$U = a_0 + a_1 F(\xi) + \frac{b_1}{F(\xi)}. \quad (50)$$

Substituting (50) into (49) and using (9), collecting the coefficients of $F(\xi)$, and then setting the coefficients of $F(\xi)$ to zero, we can obtain a set of algebraic equations about a_0, a_1, b_1, ω . Solving the algebraic equations obtained above, we can have the following solutions.

Type 1. When $B = 0$, we have the following cases.

Case 1. We have

$$a_0 = 0, \quad a_1 = \frac{\sqrt{2}}{2A}, \quad b_1 = 0, \quad \gamma = \omega^\alpha, \quad C = -\frac{1}{2A}. \quad (51)$$

Substituting (51) into (50), then according to (10)–(12), we obtain the following hyperbolic function solutions of (2), respectively:

$$u_1(x, t) = \tanh_\alpha \left(\frac{\sqrt{2}}{2} \xi \right), \quad (52)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$. Consider

$$u_2(x, t) = \coth_\alpha \left(\frac{\sqrt{2}}{2} \xi \right), \quad (53)$$

FIGURE 4: The solution $u_5(x, t)$ in (58) shows the 3D and 2D graphs for the given parameter $\gamma = 2.3$.

where $\xi = x + \omega t$ and $\gamma = \omega^\alpha$. Consider

$$u_3(x, t) = \coth_\alpha(\sqrt{2}\xi) \pm \operatorname{csch}_\alpha(\sqrt{2}\xi), \quad (54)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$.

Case 2. We have

$$\begin{aligned} a_0 &= 0, & a_1 &= 0, & b_1 &= \sqrt{2}A, \\ \gamma &= \omega^\alpha, & C &= -\frac{1}{2A}. \end{aligned} \quad (55)$$

Substituting (55) into (50), then according to (12), we obtain the following hyperbolic function solutions of (2):

$$u_4(x, t) = \frac{\sinh_\alpha(\sqrt{2}\xi)}{\cosh_\alpha(\sqrt{2}\xi) \pm 1}, \quad (56)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$.

The profile for exact solution $u_4(x, t)$ of (56) is shown in Figure 3.

Case 3. We have

$$\begin{aligned} a_0 &= 0, & a_1 &= \frac{\sqrt{2}}{8A}, & b_1 &= \sqrt{2}A, \\ \gamma &= \omega^\alpha, & C &= -\frac{1}{8A}. \end{aligned} \quad (57)$$

Substituting (57) into (50), then according to (10), we obtain the following hyperbolic function solution of (2):

$$u_5(x, t) = \frac{1}{2} \tanh_\alpha\left(\frac{\sqrt{2}}{4}\xi\right) + \frac{1}{2} \coth_\alpha\left(\frac{\sqrt{2}}{4}\xi\right), \quad (58)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$.

The profile for exact solution $u_5(x, t)$ of (58) is shown in Figure 4.

Case 4. We have

$$\begin{aligned} a_0 &= 0, & a_1 &= \frac{\sqrt{2}}{4A}, & b_1 &= \sqrt{2}A, \\ \gamma &= \omega^\alpha, & C &= \frac{1}{4A}. \end{aligned} \quad (59)$$

Substituting (59) into (50), then according to (13), (15), and (16), we obtain the following trigonometric function solutions of (2), respectively:

$$u_6(x, t) = \frac{\sqrt{2}}{2} \tan_\alpha\left(\frac{1}{2}\xi\right) + \frac{\sqrt{2}}{2} \cot_\alpha\left(\frac{1}{2}\xi\right), \quad (60)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$. Consider

$$u_7(x, t) = \pm \sqrt{2} \sec_\alpha(\xi), \quad (61)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$. Consider

$$u_8(x, t) = \mp \sqrt{2} \csc_\alpha(\xi), \quad (62)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$.

Type 2. When $A = 0$ and $BC \neq 0$, we have

$$\begin{aligned} a_0 &= -1, & a_1 &= \sqrt{2}C, & b_1 &= 0, \\ \gamma &= \omega^\alpha, & B &= -\sqrt{2}. \end{aligned} \quad (63)$$

Substituting (63) into (50), then according to (17), we obtain the following hyperbolic function solution of (2):

$$u_9(x, t) = \frac{d - \cosh_\alpha(\sqrt{2}\xi) - \sinh_\alpha(\sqrt{2}\xi)}{d + \cosh_\alpha(\sqrt{2}\xi) + \sinh_\alpha(\sqrt{2}\xi)}, \quad (64)$$

where $\xi = x + \omega t$, $\gamma = \omega^\alpha$.

Type 3. When $ABC \neq 0$, we have the following cases.

Case 2. We have

$$\begin{aligned} a_0 &= \frac{\sqrt{2}}{2}B, & a_1 &= \frac{\sqrt{2}(B^2 - 2)}{4A}, & b_1 &= 0, \\ \gamma &= \omega^\alpha, & C &= \frac{B^2 - 2}{4A}. \end{aligned} \quad (65)$$

Substituting (65) into (50), then according to (18) and (19), we obtain the following hyperbolic function solutions of (2), respectively:

$$u_{10}(x, t) = \frac{B \sinh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right) - \sqrt{2} \cosh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right)}{\sqrt{2} \sinh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right) - B \cosh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right)}, \quad (66)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = 2 > 0$. Consider

$$u_{11}(x, t) = \frac{\sqrt{2} \sinh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right) - B \cosh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right)}{B \sinh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right) - \sqrt{2} \cosh_{\alpha} \left(\left(\frac{\sqrt{2}}{2} \right) \xi \right)}, \quad (67)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = 2 > 0$.

Case 5. We have

$$\begin{aligned} a_0 &= \frac{\sqrt{2}}{2} B, & a_1 &= -\frac{\sqrt{2}(B^2 - 2)}{8A}, & b_1 &= \sqrt{2}A, \\ \gamma &= \omega^{\alpha}, & C &= -\frac{B^2 - 2}{8A}. \end{aligned} \quad (68)$$

Substituting (68) into (50), then according to (18) and (19), we obtain the following hyperbolic function solutions of (2), respectively:

$$\begin{aligned} u_{12}(x, t) &= \frac{\sqrt{2}}{2} B - \frac{\left((B^2 - 2) \sqrt{2}/4 \right) \cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)}{\sqrt{\Delta} \sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - B \cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)} \\ &+ \frac{\left(\sqrt{2}/2 \right) \left[\sqrt{\Delta} \sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - B \cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right]}{\cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)}, \end{aligned} \quad (69)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = (3B^2/2) - 1 > 0$. Consider

$$\begin{aligned} u_{13}(x, t) &= \frac{\sqrt{2}}{2} B + \frac{\left((B^2 - 2) \sqrt{2}/4 \right) \sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)}{B \sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - \sqrt{\Delta} \cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)} \\ &- \frac{\left(\sqrt{2}/2 \right) \left[B \sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) - \sqrt{\Delta} \cosh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right) \right]}{\sinh_{\alpha} \left(\left(\frac{\sqrt{\Delta}}{2} \right) \xi \right)}, \end{aligned} \quad (70)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = (3B^2/2) - 1 > 0$.

Substituting (68) into (50), then according to (20) and (21), we obtain the following trigonometric function solutions of (2), respectively:

$$\begin{aligned} u_{14}(x, t) &= \frac{\sqrt{2}}{2} B + \frac{\left((B^2 - 2) \sqrt{2}/4 \right) \cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)}{\sqrt{-\Delta} \sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) + B \cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)} \\ &- \frac{\left(\sqrt{2}/2 \right) \left[\sqrt{-\Delta} \sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) + B \cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right]}{\cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)}, \end{aligned} \quad (71)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = (3B^2/2) - 1 < 0$. Consider

$$\begin{aligned} u_{15}(x, t) &= \frac{\sqrt{2}}{2} B + \frac{\left((B^2 - 2) \sqrt{2}/4 \right) \sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)}{B \sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) - \sqrt{-\Delta} \cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)} \\ &- \frac{\left(\sqrt{2}/2 \right) \left[B \sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) - \sqrt{-\Delta} \cos_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right) \right]}{\sin_{\alpha} \left(\left(\frac{\sqrt{-\Delta}}{2} \right) \xi \right)}, \end{aligned} \quad (72)$$

where $\xi = x + \omega t$, $\gamma = \omega^{\alpha}$, and $\Delta = B^2 - 4AC = (3B^2/2) - 1 < 0$.

6. Conclusions

In this paper, we introduced an extended fractional Riccati expansion method and carried it out to obtain more new exact solutions of the time fractional Burgers equation and the space-time fractional Cahn-Hilliard equation. These exact solutions include hyperbolic function solutions and trigonometric function solutions. To the best of our knowledge, the solutions obtained in this Letter have not been reported in literature. Though some exact solutions of the aforementioned two equations have been obtained via the variational iteration method [26], the Von-Neumann method [27], the exponential function method [19], and fractional subequation method [31], they are different from the solutions obtained in this paper. The paper shows that this method is reliable and simple. We will investigate other types of nonlinear fractional differential equations and so on.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research was supported by the Natural Science Foundations of Yunnan Province, China (nos. 2011FZ193 and 2013FZ117).

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