Research Article

Event-Triggered Average Consensus for Multiagent Systems with Time-Varying Delay

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Received 6 December 2013; Accepted 13 March 2014; Published 4 May 2014

Academic Editor: Huaicheng Yan

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The paper investigates average consensus for multiagent systems with time-varying delay. A reducing dimension multiagent systems model is presented firstly. Using event-triggered mechanism to reduce network load, a comprehensive model is then proposed, which considers communication delay and triggered issue. Furthermore, the event-triggered average consensus stability of multiagent systems with fixed directed/undirected graph is analyzed, and sufficient conditions are provided. Moreover, the upper bound of time-varying delay can be obtained conveniently. Finally, simulation results confirm the feasibility and effectiveness of the proposed method.

1. Introduction

Recently distributed coordination of multiagent systems has attracted significant interest of many researchers. It is becoming increasingly important in multivehicles, sensor network, and formation flying [1–5]. Consensus (or synchronization) problem [1, 6–8] is one of the most important issues of multiagent systems. In practice, it is very important for multiagent systems to achieve average consensus [9].

With the development of industrial large-scale dynamical multiagent systems, the agent is responsible for collecting a lot of information from its neighboring nodes and transmission signals in the shared wireless communication network are massive. However, computing resources of each agent and the bandwidth of wireless network are limited. Therefore, it is necessary to study communication mechanism of multiagent systems [10], which is one of the most important issues in the implementation of average consensus algorithms. Moreover, communication networks are not always reliable, and communication delay is inevitably introduced [11–14]. It is well known that time delay may degrade the system performance or even cause the system instability [15–19]. Therefore, it is necessary to investigate the average consensus

problem with time delay. Some researchers have investigated the average consensus problem for multiagent systems with time delay and communication scheme, respectively.

Considering the average consensus problem with time delay, there are some research results that have been reported [9, 20–23]. For an undirected network with fixed/switching graphs, an upper bound of communication time delays was obtained [20, 21]. Average consensus problem for directed networked multiagent systems with fixed/switching graphs and constant/varying time delays was investigated based on the LMI method [22, 23]. A necessary and sufficient condition was derived for multiagent systems with heterogeneous time delays to achieve average consensus [9]. The average consensus problems have been investigated with time delays and other issues such as noise [24–26]. Moreover, some researches have investigated consensus or average consensus for high-order multiagent systems with time-varying delays [27–30]. However, how to get the quantitative relationship between maximum time delay and system stability for directed networks with fixed/switching graphs is still an open issue.

For communication schemes, the event-triggered mechanism has been proposed [31–34]. Compared with traditional time-triggered mechanism (i.e., the fixed sampling period),

the event-triggered mechanism can reduce the communication burden. This is because the signal can be transmitted only when the triggered condition is satisfied. Event-triggered mechanism has been employed in multiagent systems [10, 35–37] without time delay, which was an effective methodology for multiagent systems with limited computational and communicational resources. Furthermore, based on event-triggered mechanism, a tracking control problem of leader-follower multiagent continuous-time systems with communication delays was investigated [38], and the consensus problem of discrete-time multiagent systems with random communication delays was studied [39].

In this paper, the event-triggered average consensus problem for continuous-time multiagent systems in a directed network with time-varying delay is studied. The main contributions include the following. Firstly, a reducing dimension multiagent systems model with event-triggered average consensus protocol and time-varying delay is provided. Secondly, a Lyapunov-Krasovskii functional is constructed, and sufficient conditions are obtained, and all the agents can achieve the average consensus asymptotically. Different from [38, 39], an upper bound of time-varying communication delay is derived in this paper.

The paper is organized as follows. Section 2 presents the background and system model. The proposed approach is provided in Section 3. Section 4 gives the main results of the paper. Simulation results are described in Section 5. Section 6 concludes the paper.

2. Background and System Model

2.1. Preliminaries of Graph Theory. Let $G(V, E, A)$ be a weighted digraph of order n , with the set of nodes $V = \{v_1, \dots, v_n\}$, and set of edges $E = \{e_{ij} = (v_i, v_j)\} \subseteq V \times V$, where $e_{ij} = (v_i, v_j)$ is ordered, v_i is the edge's tail, and v_j is the head. The set of neighbors of node v_i is denoted as $N_i = \{v_j \in V \mid (v_j, v_i) \in E\}$. The node indexes of G belong to a finite index set $I = \{1, 2, \dots, n\}$. $A = [a_{ij}]$ is a weighted communication adjacency matrix with nonnegative adjacency elements a_{ij} , where $a_{ii} = 0$ and $a_{ij} > 0$ if and only if $v_i \in N_j$. The in-degree and out-degree of v_i are defined as

$$d_{\text{in}}(v_i) = \sum_{j=1}^n a_{ji}, \quad d_{\text{out}}(v_i) = \sum_{j=1}^n a_{ij}. \quad (1)$$

The degree matrix $D = [d_{ij}]_{n \times n}$ is a diagonal matrix with

$$d_{ij} = \begin{cases} d_{\text{out}}(v_i), & i = j, \\ 0, & i \neq j. \end{cases} \quad (2)$$

The graph Laplacian of G is defined as $L = D - A = [l_{ij}]_{n \times n}$. It can be easily obtained that the element l_{ij} of L satisfies

$$l_{ij} = \begin{cases} \sum_{j=1}^n a_{ij}, & j = i, \\ -a_{ij}, & i \neq j. \end{cases} \quad (3)$$

By definition, every row sum of L is zero and thus $1_n = [1 \ 1 \ \dots \ 1]^T \in \mathbb{R}^n$ is an eigenvector of L associated with the eigenvalue $\lambda = 0$. This therefore means that $\text{Rank}(L) \leq n - 1$.

Definition 1 (balanced graph [20]). We say the node v_i of a graph $G(V, E, A)$ is balanced if and only if $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$. A graph $G(V, E, A)$ is called balanced if and only if all of its nodes are balanced. Obviously any connected undirected graph is balanced.

Definition 2 (balanced matrix [22]). A square matrix $M \in \mathbb{R}^{n \times n}$ is said to be a balanced matrix if and only if $1_n^T M = 0$ and $M 1_n = 0$.

It is easy to know that the Laplacian L of a balanced graph G satisfies $1_n^T L = 0$ and $L 1_n = 0$.

Lemma 3 (see [22]). Consider the matrix

$$A = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix}. \quad (4)$$

The following statements hold.

- (1) The eigenvalues of A are n with multiplicity $n - 1$ and 0 with multiplicity 1. The vectors 1_n^T and 1_n are left and the right eigenvectors of A associated with the zero eigenvalue, respectively.
- (2) There exists an orthogonal matrix O such that

$$O^T A O = \begin{bmatrix} n I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix}, \quad (5)$$

and O is the matrix of eigenvectors of A . For any balanced matrix $B \in \mathbb{R}^{n \times n}$

$$O^T B O = \begin{bmatrix} * & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix}. \quad (6)$$

Remark 4. For the Laplacian L of a connected graph G , it is easy to have the following equation by Lemma 3:

$$U^T L U = \begin{bmatrix} U_1^T L U_1 & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix}, \quad (7)$$

where U is an orthogonal matrix of eigenvectors of L , and we define it as $[U_1 \ (1_n/\sqrt{n})]$, in which $U_1 \in \mathbb{R}^{n \times (n-1)}$ is the first $n - 1$ columns of U .

2.2. System Model. Suppose that the network system consists of n agents. Each agent is regarded as a node in the graph G . Let $x_i(t) \in \mathbb{R}$ denote the state (or value) of node v_i . The value of a node might represent physical quantities including position, temperature, and voltage. Moreover, assume that each node is an agent with dynamics:

$$\dot{x}_i(t) = u_i(t), \quad \{i \in I = 1, 2, \dots, n\}, \quad (8)$$

where $u_i(t)$ is the control input (or protocol). The consensus control protocol without communication time delay in [20] was given by

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t) (x_j - x_i). \quad (9)$$

FIGURE 1: Agent with the consensus protocol.

Here with the consensus protocol, each agent consists of a controller and dynamics as shown in Figure 1.

We say the nodes of a network have reached a consensus if and only if $x_i = x_j$ for all $i, j \in I, i \neq j$. Whenever the nodes of a network are all in agreement, the value of all nodes is called the group decision value. Particularly, for all $i, j \in I$, if there exists

$$\lim_{t \rightarrow \infty} x_i(t) = \frac{1}{n} \sum_{j=1}^n x_j(0), \quad (10)$$

protocol $u_i(t)$ asymptotically solves the average consensus problem.

Further, the consensus control law with communication time delay was given by [9]

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t) (x_j(t - \tau_{ij}^c) - x_i(t - \tau_{ij}^s)), \quad (11)$$

where $\tau_{ij}^c \geq 0$ is the communication time delay with which the state of node v_i passes through a channel e_{ij} before getting to node v_j . $\tau_{ij}^s \geq 0$ is the self-delay, which occurs when node v_i compares its self-information and information of node v_j over the network.

In this paper, we consider the case that the self-delay τ_{ij}^s is equal to the communication delay τ_{ij}^c , that is, $\tau_{ij}^s = \tau_{ij}^c = \tau_{ij}$, which is discussed in [9, 20, 22, 23]. Then the consensus control law (11) becomes

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t) (x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})). \quad (12)$$

Given protocol (9) and (11), the network dynamics of agent n is summarized as

$$\dot{x}(t) = -Lx(t), \quad (13)$$

$$\dot{x}(t) = -Lx(t - \tau_{ij}), \quad (14)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ denotes the value of all nodes and L is the Laplacian of graph G .

The system (13) without communication time delay under event-triggered strategy has been investigated in [36, 37].

In the following section, we will redefine protocol (12) and model (14) to take into account event-triggered mechanism. The consensus control formulation and problem statement for centralized event-triggered cooperative control are described in the following section.

FIGURE 2: The relationship between M_1 , M_2 , and M_3 .

3. The Proposed Approach

3.1. Event-Triggered Mechanism and Modeling of Multiagents Systems with Communication Time Delay. Each agent of multiagent systems is equipped with a small embedded microprocessor and regularly sampled with period h by microprocessor in practice, where the monotone increasing sampling sequence is described by the set $M_1 = \{0, h, 2h, \dots, kh\}$, $k \in \mathbb{N}$. Using event-triggered mechanism, only the successful transmitted signals are available, where the successful transmitted instants sequences $M_2 = \{0, t_1h, t_2h, \dots, t_kh\}$ satisfy $0 = t_0 < t_1 < \dots < t_{k-1} < t_k < \dots$, $t_k \in \mathbb{N}$. Considering the bounded delay $\tau_{t_k} \in [0, \bar{\tau}]$, which denotes the time delay at successful transmitted instant t_kh , the transmitted finishing instant is described by the set $M_3 = \{0, t_1h + \tau_{t_1}, t_2h + \tau_{t_2}, \dots, t_kh + \tau_{t_k}, \dots\}$. Moreover, the control input is generated by a zero-order holder (ZOH) with the holding time $t \in [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}})$. The relationship between M_1 , M_2 , and M_3 is shown in Figure 2.

Remark 5. Specially, if $M_1 = M_2$, it means that all sampled data are transmitted, that is, a time-triggered transmission scheme. If $M_1 \subset M_2$, it means that not all sampled data are transmitted; that is, the numbers of sampled data are reduced. Therefore, the event-triggered scheme can reduce communication burden.

3.1.1. Centralized Event-Triggered Mechanism. The state measurement error is defined by

$$e(kh) = x(kh) - x(t_kh), \quad (15)$$

where $e(kh)$ is the error between the current sampled value $x(kh)$ and the latest transmitted sampled value $x(t_kh)$.

The event-triggered transmission mechanism in [40, 41] is designed as

$$f(kh, t_kh) = e^T(kh) \Phi e(kh) - \gamma x^T(t_kh) \Phi x(t_kh) > 0, \quad (16)$$

where Φ is a positive definite matrix and $0 < \gamma < 1$ is a given and bounded positive scalar parameter.

When the local state measurement error signal exceeds the given threshold, that is, condition (16) is satisfied, the current sampled information $x(kh)$ is transmitted. Obviously,

FIGURE 3: Agent with event-triggered consensus protocol.

the current successful transmitted state value $x(kh)$ is the subsequences of the latest transmitted sampled value, which is denoted by $x(t_{k+1}h)$. Then each agent with event-triggered consensus protocol is shown in Figure 3.

Remark 6. The event-triggered mechanism (16) is characterized by the parameters γ , Φ , and h . Specially, if $\gamma = 0$ in (16), this leads to $t_{k+1}h = t_kh + h$, and it becomes a time-triggered transmission mechanism.

3.1.2. Modeling of Hybrid Event-Triggered Multiagent Systems. Here we considered the system (12) with fixed topology G and time-varying delay. We know that the input $u_i(t)$ is held constant in a control period; that is,

$$u_i(t) = u_i(t_kh), \quad t \in [t_kh, t_{k+1}h). \quad (17)$$

The proposed event-triggered control law of (12) in the centralized case is defined as

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t_kh) (x_j(t_kh - \tau_k) - x_i(t_kh - \tau_k)), \quad (18)$$

so that

$$u(t) = -Lx(t_kh), \quad t \in [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}}). \quad (19)$$

The network dynamics of system (14) is then given by

$$\dot{x}(t) = -Lx(t_kh), \quad t \in [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}}). \quad (20)$$

Obviously, $t_{k+1}h + \tau_{t_{k+1}} \geq t_kh + h + \tau_{t_k}$ and $t_kh + h + \bar{\tau} \geq t_kh + h + \tau_{t_{k+1}}$.

Next, two cases are discussed.

(1) If $t_{k+1}h + \tau_{t_{k+1}} \geq t_kh + h + \bar{\tau}$, there exists $l \geq 1 \in \mathbb{N}$, satisfying

$$t_kh + lh + \bar{\tau} \leq t_{k+1}h + \tau_{t_{k+1}} < t_kh + (l+1)h + \bar{\tau}. \quad (21)$$

For a detailed timing analysis, consider the following interval:

$$\begin{aligned} \Omega &= [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}}) \\ &= [t_kh + \tau_{t_k}, t_kh + h + \bar{\tau}) \\ &\cup \left\{ \bigcup_{i=1}^{l-1} [t_kh + ih + \bar{\tau}, t_kh + ih + h + \bar{\tau}) \right\} \\ &\cup [t_kh + lh + \bar{\tau}, t_{k+1}h + \tau_{t_{k+1}}) \\ &= \bigcup_{i=0}^l \Omega_i, \end{aligned} \quad (22)$$

where some sets Ω_i are defined as follows:

$$\begin{aligned} \Omega_0 &= [t_kh + \tau_{t_k}, t_kh + h + \bar{\tau}), \quad i = 0, \\ \Omega_i &= [i_kh + \bar{\tau}, i_kh + h + \bar{\tau}), \quad i = 1, 2, \dots, l-1 \\ \Omega_l &= [i_kh + \bar{\tau}, t_{k+1}h + \tau_{t_{k+1}}), \quad i = l, \end{aligned} \quad (23)$$

and $i_kh = t_kh + ih$, $i = 0, 1, 2, \dots, l$.

Define

$$\eta(t) = t - i_kh, \quad t \in \Omega_i, \quad i = 0, 1, 2, \dots, l. \quad (24)$$

From (23) and (24):

$$\begin{aligned} \tau_{t_k} &\leq \eta(t) \leq h + \bar{\tau}, \quad t \in \Omega_0, \quad i = 0, \\ \bar{\tau} &\leq \eta(t) \leq h + \bar{\tau}, \quad t \in \Omega_i, \quad i = 1, 2, \dots, l-1, \\ \bar{\tau} &\leq \eta(t) \leq h + \bar{\tau}, \quad t \in \Omega_l, \quad i = l. \end{aligned} \quad (25)$$

From (25), it can be obtained that

$$\eta_1 < \eta(t) \leq \eta_3, \quad t \in \Omega_i, \quad i = 0, 1, 2, \dots, l, \quad (26)$$

where $\eta_1 = \inf\{\tau_{t_k}\} > 0$ and $\eta_3 \triangleq h + \bar{\tau}$.

Define the state measurement error

$$e(i_kh) = x(i_kh) - x(t_kh), \quad t \in \Omega_i, \quad i = 0, 1, \dots, l. \quad (27)$$

(2) If $t_{k+1}h + \tau_{t_{k+1}} < t_kh + h + \bar{\tau}$, define $\eta(t) = t - t_kh$, $t \in [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}})$.

It is clear that $\tau_{t_k} \leq \eta(t) \leq h + \bar{\tau}$; that is, $\eta_1 < \eta(t) \leq \eta_3$, $t \in [t_kh + \tau_{t_k}, t_{k+1}h + \tau_{t_{k+1}})$. Then there exists $e(i_kh) = x(t_kh) - x(t_kh) = 0$.

According to the definition and analysis of the aforementioned two cases, it is seen that (19) and (20) can be equivalently written as

$$\begin{aligned} u(t) &= -Lx(t - \eta(t)) + Le(i_kh), \\ \dot{x}(t) &= -Lx(t - \eta(t)) + Le(i_kh), \\ t &\in \Omega_i, \quad i = 0, 1, 2, \dots, l, \end{aligned} \quad (28)$$

where $i_kh = t_kh + ih$ and $0 < \eta_1 < \eta(t) \leq \eta_3$. The initial conditions of system (28) are assumed to satisfy $x(t) = \phi(t)$,

$t \in [-\eta_3, 0]$, $\phi(0) = x(0)$. Moreover, between $t_k h$ and $t_k h + ih$, no control signal is triggered; that is,

$$\begin{aligned} f(i_k h, t_k h) \\ = e^T(i_k h) \Phi e(i_k h) - \gamma x^T(t_k h) \Phi x(t_k h) \leq 0. \end{aligned} \quad (29)$$

Therefore, in the event-triggered formulation, the event time $t_{k+1} h$ is defined by

$$t_{k+1} h = t_k h + \min_i \{ih \mid e^T(i_k h) \Phi e(i_k h) \geq \delta \wp\}, \quad (30)$$

with $\wp = x^T(t_k h) \Phi x(t_k h)$, $0 < \gamma < 1$, $i \in \mathbb{N}$.

3.2. Problem Setting. The following task is to find the conditions for system (28) to reach average consensus with the event-triggered communication mechanism (29). Considering a directed network system (28) with fixed graph G that is strongly connected and balanced, denote $\alpha(t) = \text{Ave}(x) = (1/n) \sum_i x_i(t)$ by the average of the agents' states. Due to the balanced graph of G , the Laplacian L satisfies $1_n^T L = 0$ from Definition 2. For system (28), it implies $1_n^T \dot{x} = 1_n^T u = -1_n^T L[x(t) - e(i_k h)] = 0$; that is, $\sum_{i=1}^n \dot{x}_i(t) = \sum_{i=1}^n u_i(t) = 0$. Then the time derivative of $\alpha(t)$ is given by $\dot{\alpha}(t) = (1/n) \sum_i \dot{x}_i(t) = 0$. So that $\alpha = \text{Ave}(x) = \text{Ave}(x(0)) = (1/n) \sum_i x_i(0)$; that is, $\alpha = \text{Ave}(x)$ is an invariant quantity. Then the state vector $x(t)$ can be decomposed as [20]

$$x(t) = \alpha 1_n + \delta(t), \quad (31)$$

where $\alpha = \text{Ave}(x)$ and $\delta(t) \in \mathfrak{R}^n$ satisfies $\sum_i \delta_i(t) = 0$. For a strongly connected digraph, the vector $\delta(t)$ belongs to an $(n-1)$ -dimensional subspace called the disagreement eigenspace of L .

Submitting (31) to (28), system (28) is equivalent to

$$\begin{aligned} \dot{\delta}(t) &= -L\delta(t - \eta(t)) + Le(i_k h), \\ 0 &< \eta_1 < \eta(t) \leq \eta_3. \end{aligned} \quad (32)$$

Moreover, between $t_k h$ and $t_k h + ih$ ($i = 1, 2, \dots, l$), no control signal is triggered; that is,

$$\begin{aligned} f(i_k h, t_k h) \\ = e^T(i_k h) \Phi e(i_k h) - \gamma x^T(t_k h) \Phi x(t_k h) \leq 0. \end{aligned} \quad (33)$$

Because Φ is a positive definite matrix, there exists invertible matrix W satisfying $\Phi = W^T W$. Substituting $\Phi = W^T W$ into (33) yields

$$\begin{aligned} e^T(i_k h) W^T W e(i_k h) - \gamma x^T(t_k h) W^T W x(t_k h) \leq 0, \\ \|We(i_k h)\| \leq \sqrt{\gamma} \|Wx(t_k h)\|; \end{aligned} \quad (34)$$

that is,

$$\begin{aligned} \|We(i_k h)\| &\leq \sqrt{\gamma} \|W(\alpha 1_n + \delta(t_k h))\|, \\ \|We(i_k h)\| &\leq \sqrt{\gamma} \|\alpha W 1_n\| + \sqrt{\gamma} \|W\delta(t_k h)\|. \end{aligned} \quad (35)$$

So when the condition satisfies

$$\begin{aligned} \|We(i_k h)\| &\leq \sqrt{\gamma} \|W\delta(t_k h)\|, \\ \|We(i_k h)\|^2 &\leq \gamma \|W\delta(t_k h)\|^2, \end{aligned} \quad (36)$$

$$e^T(i_k h) W^T W e(i_k h) - \gamma \delta^T(t_k h) W^T W \delta(t_k h) \leq 0,$$

one can get

$$e^T(i_k h) \Phi e(i_k h) - \gamma \delta^T(t_k h) \Phi \delta(t_k h) \leq 0. \quad (37)$$

Between $t_k h$ and $t_k h + ih$ ($i = 1, 2, \dots, l$), no control signal is triggered, where

$$e(i_k h) = \delta(i_k h) - \delta(t_k h), \quad t \in \Omega_i, \quad i = 0, 1, 2, \dots, l. \quad (38)$$

For system (32), it follows that, from Remark 4,

$$\begin{aligned} U^T \dot{\delta}(t) &= -U^T L U U^T \delta(t - \eta(t)) + U^T L U U^T e(i_k h) \\ &= - \begin{bmatrix} U_1^T L U_1 & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix} U^T \delta(t - \eta(t)) \\ &\quad + \begin{bmatrix} U_1^T L U_1 & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix} U^T e(i_k h) \\ &= - \begin{bmatrix} \tilde{L} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix} U^T \delta(t - \eta(t)) \\ &\quad + \begin{bmatrix} \tilde{L} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 \end{bmatrix} U^T e(i_k h), \end{aligned} \quad (39)$$

where U is an orthogonal matrix of eigenvectors of L , and we define it as $[U_1 \quad (1_n/\sqrt{n})]$, in which $U_1 \in \mathfrak{R}^{n \times (n-1)}$ is the first $n-1$ columns of U . Denote $\tilde{L} = U_1^T L U_1$.

Noting that $\sum_i \delta_i(t - \eta(t)) = 0$, $\sum_i \delta_i(t) = 0$, we have

$$\begin{aligned} U^T \dot{\delta}(t) &= [* \dots * 0]^T = \left[\tilde{\delta}^T(t) 0 \right]^T, \\ U^T \delta(t - \eta(t)) &= [\tilde{\delta}^T(t - \eta(t)) 0]^T, \\ U^T \delta(t_k h) &= [* \dots * 0]^T = [\tilde{\delta}^T(t_k h) 0]^T, \\ U^T e(i_k h) &= [* \dots * 0]^T = [\tilde{e}^T(i_k h) 0]^T. \end{aligned} \quad (40)$$

Then, system (32) is equivalent to

$$\begin{aligned} \dot{\tilde{\delta}}(t) &= -\tilde{L} \tilde{\delta}(t - \eta(t)) + \tilde{L} \tilde{e}(i_k h), \\ t &\in \Omega_i, \quad i = 0, 1, 2, \dots, l, \\ 0 &< \eta_1 < \eta(t) \leq \eta_3, \end{aligned} \quad (41)$$

where $\tilde{L} \in \mathfrak{R}^{(n-1) \times (n-1)}$ and $\text{Rank}(\tilde{L}) = n-1$.

Condition (37) is equivalent to

$$\begin{aligned} [\tilde{e}^T(i_k h) 0] U^T \Phi U [\tilde{e}^T(i_k h) 0]^T \\ - \gamma [\tilde{\delta}^T(t_k h) 0] U^T \Phi U [\tilde{\delta}^T(t_k h) 0]^T \leq 0; \end{aligned} \quad (42)$$

that is,

$$\begin{aligned} & \tilde{e}^T(i_k h) \tilde{\Phi} \tilde{e}^T(i_k h) \\ & - \gamma \tilde{\delta}^T(t_k h) \tilde{\Phi} \tilde{\delta}^T(t_k h) \leq 0, \end{aligned} \quad (43)$$

where $0 < \tilde{\Phi} \in \mathfrak{R}^{(n-1) \times (n-1)}$ and $\tilde{\Phi} = U_1^T \Phi U_1$.

Lemma 7. If $\lim_{t \rightarrow \infty} \|\tilde{\delta}(t)\| = 0$, then $\lim_{t \rightarrow \infty} \|\delta(t)\| = 0$.

Proof. From $\sum_i \delta_i(t) = 0$, we have $U^T \delta(t) = [\tilde{\delta}^T(t) 0]^T$, and then $\delta(t) = U[\tilde{\delta}^T(t) 0]^T$. Therefore, Lemma 7 is obtained. This completes the proof. $\square$

Lemma 8 (see [42]). Let G be a balanced digraph; then G is strongly connected if and only if G is weakly connected.

Remark 9. The requirement of graph G that we discuss is strongly connected, but it can be obtained that it should be weakly connected based on Lemma 8.

Definition 10 (see [43]). Let $\Phi_1, \Phi_2, \dots, \Phi_N : \mathfrak{R}^m \mapsto \mathfrak{R}^n$ be a given finite number of functions such that they have positive values in an open subset D of $\mathfrak{R}^m$. Then, a reciprocally convex combination of these functions over D is a function of the form

$$\frac{1}{\alpha_1} \Phi_1 + \frac{1}{\alpha_2} \Phi_2 + \dots + \frac{1}{\alpha_N} \Phi_N : D \mapsto \mathfrak{R}^n, \quad (44)$$

where the real numbers α_i satisfy $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

For a reciprocally convex combination of scalar positive functions $\Phi_i = f_i$, Lemma 11 is obtained by Definition 10.

Lemma 11 (see [43]). Let $f_1, f_2, \dots, f_N : \mathfrak{R}^m \mapsto \mathfrak{R}$ have positive values in an open subset D of $\mathfrak{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies

$$\min_{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1} \sum_i \frac{1}{\alpha_i} f_i = \sum_i f_i + \max_{g_{i,j}(t)} \sum_{i \neq j} g_{i,j}(t) \quad (45)$$

subject to

$$\begin{aligned} & \min_{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1} \sum_i \frac{1}{\alpha_i} f_i = \sum_i f_i + \max_{g_{i,j}(t)} \sum_{i \neq j} g_{i,j}(t), \\ & \left\{ g_{i,j} : \mathfrak{R}^m \mapsto \mathfrak{R}, g_{j,i}(t) \triangleq g_{i,j}(t), \right. \\ & \left. \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}. \end{aligned} \quad (46)$$

Lemma 11 can be applied to handle the double integral terms of the Lyapunov-Krasovskii functional for time delay systems.

4. Main Results

Theorem 12. Considering a network system of n agents, the network has a balanced and weakly connected weighted

digraph G with uncertain time-varying communication delay. For some given positive constants η_1, η_3 , and γ , under the event-triggered mechanism (43), the system (41) is asymptotically stable, if there exist positive definite matrixes $P, \tilde{\Phi}, S, R_i$ ($i = 1, 2, 3$) $\in \mathfrak{R}^{(n-1) \times (n-1)}$, matrix Q_i ($i = 1, 2, 3$), U_j ($j = 2, 3$) with $(n-1)$ dimensions and $\begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0$, $\begin{bmatrix} R_j & * \\ U_j & R_j \end{bmatrix} > 0$ ($j = 2, 3$), such that

$$\begin{bmatrix} \Gamma_{11}^j & * \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} < 0 \quad (j = 2, 3), \quad (47)$$

where

$$\Gamma_{11}^2 = \begin{bmatrix} \Pi_{11} & * & * & * & * & * \\ \Pi_{21} & \Pi_{22} & * & * & * & * \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & * & * \\ 0 & \Pi_{42} & \Pi_{43} & \Pi_{44} & * & * \\ 0 & 0 & \Pi_{53} & \Pi_{54} & \Pi_{55} & * \\ \Pi_{61} & 0 & \Pi_{63} & 0 & 0 & \Pi_{66} \end{bmatrix},$$

$$\Gamma_{11}^3 = \begin{bmatrix} \tilde{\Pi}_{11} & * & * & * & * & * \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & * & * & * & * \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} & * & * & * \\ 0 & \tilde{\Pi}_{42} & \tilde{\Pi}_{43} & \tilde{\Pi}_{44} & * & * \\ 0 & 0 & \tilde{\Pi}_{53} & \tilde{\Pi}_{54} & \tilde{\Pi}_{55} & * \\ \tilde{\Pi}_{61} & 0 & \tilde{\Pi}_{63} & 0 & 0 & \tilde{\Pi}_{66} \end{bmatrix},$$

$$\Gamma_{21} = \begin{bmatrix} 0 & 0 & -\eta_1 R_1 \tilde{L} & 0 & 0 & \eta_1 R_1 \tilde{L} \\ 0 & 0 & -\eta_{12} R_2 \tilde{L} & 0 & 0 & \eta_{12} R_2 \tilde{L} \\ 0 & 0 & -\eta_{23} R_3 \tilde{L} & 0 & 0 & \eta_{23} R_3 \tilde{L} \end{bmatrix},$$

$$\Gamma_{22} = \begin{bmatrix} -R_1 & * & * \\ 0 & -R_2 & * \\ 0 & 0 & -R_3 \end{bmatrix},$$

$$\Pi_{11} = \tilde{\Pi}_{11} = S - R_1, \quad \Pi_{21} = \tilde{\Pi}_{21} = R_1,$$

$$\Pi_{22} = \tilde{\Pi}_{22} = -S - R_1 - R_2 + Q_1,$$

$$\Pi_{31} = \tilde{\Pi}_{31} = -\tilde{L}^T P, \quad \Pi_{32} = R_2 - U_2,$$

$$\Pi_{33} = U_2 + U_2^T - 2R_2 + \gamma \tilde{\Phi},$$

$$\tilde{\Pi}_{33} = U_3 + U_3^T - 2R_3 + \gamma \tilde{\Phi},$$

$$\Pi_{42} = Q_3 + U_2, \quad \tilde{\Pi}_{42} = Q_3 + R_2,$$

$$\Pi_{43} = R_2 - U_2, \quad \tilde{\Pi}_{43} = R_3 - U_3^T,$$

$$\Pi_{44} = \tilde{\Pi}_{44} = Q_2 - R_2 - R_3 - Q_1, \quad \tilde{\Pi}_{53} = R_3 - U_3,$$

$$\begin{aligned}
\Pi_{54} &= R_3 - Q_3, & \tilde{\Pi}_{54} &= U_3 - Q_3, \\
\Pi_{55} &= \tilde{\Pi}_{55} = -R_3 - Q_2, \\
\Pi_{61} &= \tilde{\Pi}_{61} = \tilde{L}^T P, & \Pi_{63} &= \tilde{\Pi}_{63} = -\gamma \tilde{\Phi}, \\
\Pi_{66} &= \tilde{\Pi}_{66} = \gamma \tilde{\Phi} - \tilde{\Phi},
\end{aligned} \tag{48}$$

and $\tilde{L} = U_1^T L U_1$, $U_1 \in \mathfrak{R}^{n \times (n-1)}$ is the first $n-1$ columns of U , and U is an orthogonal matrix of eigenvectors of L .

The proof of Theorem 12 is presented in the appendix.

Corollary 13. Consider system (28) with the conditions (29), and assume that the network has a balanced and weakly weighted digraph G with uncertain time-varying communication delay. Then, the system asymptotically achieves average consensus; that is,

$$\lim_{t \rightarrow \infty} x_i(t) = \alpha = \frac{1}{n} \sum_i x_i(0). \tag{49}$$

Proof. Using Theorem 12, we have that $\lim_{t \rightarrow \infty} \tilde{\delta}(t) = 0$. According to Lemma 7, $\lim_{t \rightarrow \infty} x(t) = \alpha 1_n + \lim_{t \rightarrow \infty} \delta(t) = \alpha 1_n$, so we have $\lim_{t \rightarrow \infty} x_i(t) = \alpha = (1/n) \sum_i x_i(0)$. This completes the proof. $\square$

From Remark 6, when $\gamma = 0$ in (29), the event-triggered mechanism becomes a time-triggered sampling mechanism. The following Corollary 14 provides the results for this case.

Corollary 14. Considering a network system of n agents, the network has a balanced and weakly connected weighted digraph G with time-varying communication delay. For some given positive constants η_1, η_2 , the system (28) asymptotically achieves average consensus, if there exist positive definite

matrixes P, R_i ($i = 1, 2, 3$), $S, \tilde{\Phi} \in \mathfrak{R}^{(n-1) \times (n-1)}$, $\begin{bmatrix} R_j & * \\ U_j & R_j \end{bmatrix} > 0$ ($j = 2, 3$), $\begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0$ and matrix Q_i ($i = 1, 2, 3$), U_j ($j = 2, 3$) with $(n-1)$ dimensions, such that

$$\begin{bmatrix} \Xi_{11}^j & * \\ \Xi_{21} & \Xi_{22} \end{bmatrix} < 0, \quad (j = 2, 3), \tag{50}$$

where

$$\Xi_{11}^j = \begin{bmatrix} \Theta_{11} & * & * & * & * \\ \Theta_{21} & \Theta_{22} & * & * & * \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & * & * \\ 0 & \Theta_{42} & \Theta_{43} & \Theta_{44} & * \\ 0 & 0 & \Theta_{53} & \Theta_{54} & \Theta_{55} \end{bmatrix},$$

TABLE 1: Comparison of maximum allowable delays.

γ	0.00	0.10	0.30	0.50	0.80
$\tau_{\max}$	0.3645	0.3452	0.3044	0.2582	0.1244

FIGURE 4: A directed graph.

$$\Xi_{21} = \begin{bmatrix} 0 & 0 & -\eta_1 R_1 \tilde{L} & 0 & 0 \\ 0 & 0 & -\eta_{12} R_2 \tilde{L} & 0 & 0 \\ 0 & 0 & -\eta_{23} R_3 \tilde{L} & 0 & 0 \end{bmatrix},$$

$$\Xi_{22} = \begin{bmatrix} -R_1 & * & * \\ 0 & -R_2 & * \\ 0 & 0 & -R_3 \end{bmatrix},$$

$$\Theta_{11} = S - R_1, \quad \Theta_{21} = R_1,$$

$$\Theta_{22} = -S - R_1 - R_2 + Q_1,$$

$$\Theta_{31} = -\tilde{L}^T P, \quad \Theta_{32} = (3-j)(R_2 - U_2),$$

$$\Theta_{33} = U_j + U_j^T - 2R_j,$$

$$\Theta_{42} = Q_3 + (3-j)U_2 + (j-2)R_2,$$

$$\Theta_{43} = R_j - (3-j)U_2 - (j-2)U_3^T,$$

$$\Theta_{44} = Q_2 - R_2 - R_3 - Q_1,$$

$$\Theta_{53} = (j-2)(R_3 - U_3),$$

$$\Theta_{54} = (3-j)R_3 + (j-2)U_3 - Q_3,$$

$$\Theta_{55} = -R_3 - Q_2, \tag{51}$$

and $\tilde{L} = U_1^T L U_1$, $U_1 \in \mathfrak{R}^{n \times (n-1)}$ is the first $n-1$ columns of U , and U is an orthogonal matrix of eigenvectors of L .

Remark 15. It is known that any connected undirected graph is balanced by Definition 1. Therefore, the proposed Theorem 12 and Corollaries 13 and 14 can be applied to the connected undirected network.

FIGURE 5: State trajectories of the network with $\gamma = 0.1$, $\eta_1 = 0$, $\eta_3 = 0.30$.

$$\tilde{\Phi} = \begin{bmatrix} 0.6811 & -0.0578 & 0.0297 & -0.0494 & -0.0685 \\ -0.0578 & 0.7583 & 0.0240 & -0.0303 & -0.0035 \\ 0.0297 & 0.0240 & 0.8437 & -0.0933 & -0.1292 \\ -0.0494 & -0.0303 & -0.0933 & 0.7566 & 0.0349 \\ -0.0685 & -0.0035 & -0.1292 & 0.0349 & 0.8766 \end{bmatrix}, \quad (52)$$

$$\Phi = U_1 * \tilde{\Phi} * U_1^T = \begin{bmatrix} 0.6166 & -0.1657 & -0.1087 & -0.1499 & -0.1298 & -0.0625 \\ -0.1657 & 0.6262 & -0.0956 & -0.1338 & -0.1168 & -0.1143 \\ -0.1087 & -0.0956 & 0.6863 & -0.2101 & -0.1995 & -0.0724 \\ -0.1499 & -0.1338 & -0.2101 & 0.6497 & -0.0182 & -0.1378 \\ -0.1298 & -0.1186 & -0.1995 & -0.0182 & 0.7074 & -0.2432 \\ -0.0625 & -0.1143 & -0.0724 & -0.1378 & -0.2432 & 0.6302 \end{bmatrix}.$$

The state trajectories of the network are shown in Figure 5. It is seen obviously that average consensus is asymptotically achieved. Figure 6 shows the event-triggered time instant and time intervals. It is seen that the sampled data that need be transmitted reduce importantly. Evolution of the error norm is seen in Figure 7. It is seen that

5. Numerical Example

To verify the effectiveness of proposed method, numerical example was operated on.

Example. Considering a directed network of six agents as shown in Figure 4, it has balanced and weakly connected digraph with 0-1 weights. Set the initial condition $x(0) = [1; -0.5; 2; -2; 0.5; -1]$; then $\alpha = \text{Ave}(x(0)) = 0$.

Set $\gamma = 0, 0.10, 0.30, 0.50, 0.80$, using Theorem 12, and maximum allowable delay $\tau_{\max}$ can be easily gained. Simulation results are listed in Table 1. It is found that the maximum allowable delay decreases with the increasing of γ .

For the first case $\gamma = 0.1$, $h = 0.01$, $\eta_1 = 0$, $\eta_3 = 0.30$, using Theorem 12, the corresponding feasible solution is

the solid line represents the evolution of $e^T(t_k h) \Phi e(t_k h)$, which stays below the specified state-dependent threshold $\gamma x^T(t_k h) \Phi x(t_k h)$ which is represented by the dotted line.

For the another case $\gamma = 0.3$, $h = 0.01$, $\eta_1 = 0$, $\eta_3 = 0.30$, using Theorem 12, the corresponding feasible solution is

$$\tilde{\Phi} = \begin{bmatrix} 2.1939 & -0.7841 & -1.2138 & -1.2540 & 0.8663 \\ -0.7841 & 5.3061 & 0.5101 & -0.0073 & -0.5493 \\ -1.2138 & 0.5101 & 4.3940 & -1.8528 & -0.2296 \\ -1.2540 & -0.0073 & -1.8528 & 5.6745 & 0.7017 \\ 0.8663 & -0.5493 & -0.2296 & 0.7017 & 6.1132 \end{bmatrix}, \quad (53)$$

$$\Phi = \begin{bmatrix} 3.2067 & -1.3028 & -1.0731 & -1.4047 & 1.0093 & -0.4355 \\ -1.3028 & 4.2036 & -0.2639 & -0.7973 & -1.5763 & -0.2634 \\ -1.0731 & -0.2639 & 4.4228 & -2.2695 & -0.2068 & -0.6096 \\ -1.4047 & -0.7973 & -2.2695 & 5.0211 & 0.5312 & -1.0808 \\ 1.0093 & -1.5763 & -0.2068 & 0.5312 & 2.3404 & -2.0979 \\ -0.4355 & -0.2634 & -0.6096 & -1.0808 & -2.0979 & 4.4871 \end{bmatrix},$$

FIGURE 6: Event-triggered time and time intervals with $\gamma = 0.1$, $\eta_1 = 0$, $\eta_3 = 0.30$.

FIGURE 7: Evolution of $e^T(t_k h)\Phi e(t_k h)$ with $\gamma = 0.1$, $\eta_1 = 0$, $\eta_3 = 0.30$.

Similarly, Figure 5 shows the state trajectories of the network with $\gamma = 0.3$, $\eta_1 = 0$, $\eta_3 = 0.30$. The network also asymptotically achieved average consensus. Event-triggered time instant and time intervals are shown in Figure 8.

It is found from Figures 6 and 9 that the numbers of event-triggered time instant reduce and the average value of event-triggered time intervals increases with the increasing of γ . Simulation results are listed in Table 2. Therefore, the proposed event-triggered mechanism can reduce much signal transmission and thus reduce the multiagent network load.

6. Conclusions

This paper has mainly investigated the event-triggered average consensus problem in a directed/undirected network for

TABLE 2: Simulation results for different γ with $\eta_1 = 0$, $\eta_3 = 0.30$.

γ	Event numbers	Average value of event intervals	Max. (min.) value of event intervals
0.1	22	0.1890	0.26 (0.12)
0.3	13	0.3075	0.38 (0.18)

FIGURE 8: State trajectories of the topology G with $\gamma = 0.3$, $\eta_1 = 0$, $\eta_3 = 0.30$.

FIGURE 9: Event-triggered time and time intervals with $\gamma = 0.3$, $\eta_1 = 0$, $\eta_3 = 0.30$.

multiagent systems with fixed topology and time-varying delay. Sufficient conditions for average consensus are presented, and an upper bound of time-varying communication delay is derived. Furthermore, due to unreliable information channels and limited bandwidth, communication between agents may produce data packet dropout and out-of-order. Considering these issues, how to study the average consensus

problem is the further work. Moreover, how to solve decentralized event-triggered mechanism with networked-related nondeterministic issues is another important direction in the future.

Appendix

Proof of Theorem 12. The system (28) is equivalent to (41). Based on the system (41), we construct a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V(t) = & \tilde{\delta}^T(t) P \tilde{\delta}(t) + \int_{t-\eta_1}^t \tilde{\delta}^T(\alpha) S \tilde{\delta}(\alpha) d\alpha \\ & + \sum_{i=1}^3 (\eta_i - \eta_{i-1}) \int_{-\eta_i}^{-\eta_{i-1}} \int_{t+s}^t \dot{\tilde{\delta}}^T(v) R_i \dot{\tilde{\delta}}(v) dv ds \\ & + \int_{t-\rho}^t \zeta^T(\beta) Q \zeta(\beta) d\beta, \end{aligned} \quad (\text{A.1})$$

where P, R_i ($i = 1, 2, 3$), $S \in \mathfrak{R}^{(n-1) \times (n-1)}$ are positive definite matrices:

$$\begin{aligned} \rho & \triangleq \frac{\eta_3 - \eta_1}{2}, \quad \eta_0 = 0, \quad \eta_2 \triangleq \frac{\eta_1 + \eta_3}{2}, \\ \zeta(\beta) & \triangleq \begin{bmatrix} \tilde{\delta}(\beta - \eta_1) \\ \tilde{\delta}(\beta - \eta_2) \end{bmatrix}, \\ Q & \triangleq \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} > 0, \quad Q_i \ (i = 1, 2, 3) \in \mathfrak{R}^{(n-1) \times (n-1)}. \end{aligned} \quad (\text{A.2})$$

Define

$$\begin{aligned} V_1(t) & = \tilde{\delta}^T(t) P \tilde{\delta}(t), \\ V_2(t) & = \int_{t-\eta_1}^t \tilde{\delta}^T(\alpha) S \tilde{\delta}(\alpha) d\alpha, \\ V_3(t) & = \sum_{i=1}^3 (\eta_i - \eta_{i-1}) \int_{-\eta_i}^{-\eta_{i-1}} \int_{t+s}^t \dot{\tilde{\delta}}^T(v) R_i \dot{\tilde{\delta}}(v) dv ds, \\ V_4(t) & = \int_{t-\rho}^t \zeta^T(\beta) Q \zeta(\beta) d\beta. \end{aligned} \quad (\text{A.3})$$

Then $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)$.

The derivative of (A.1) with respect to t is

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t), \quad (\text{A.4})$$

where

$$\begin{aligned} \dot{V}_1(t) & = 2\tilde{\delta}^T(t) P \dot{\tilde{\delta}}(t), \\ \dot{V}_2(t) & = \tilde{\delta}(t)^T(t) S \tilde{\delta}(t) - \tilde{\delta}^T(t - \eta_1) S \tilde{\delta}(t - \eta_1), \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \dot{V}_3(t) & = \sum_{i=1}^3 \dot{\tilde{\delta}}^T(t) (\eta_i - \eta_{i-1})^2 R_i \dot{\tilde{\delta}}(t) \\ & \quad - \sum_{i=1}^3 (\eta_i - \eta_{i-1}) \int_{t-\eta_i}^{t-\eta_{i-1}} \dot{\tilde{\delta}}^T(v) R_i \dot{\tilde{\delta}}(v) dv, \end{aligned} \quad (\text{A.6})$$

$$\dot{V}_4(t) = \xi^T(t) Q \xi(t) - \xi^T(t - \rho) Q \xi(t - \rho). \quad (\text{A.7})$$

Define

$$\begin{aligned} \eta_{21} & \triangleq \eta_2 - \eta_1, \quad \eta_{32} \triangleq \eta_3 - \eta_2, \\ \chi(t) & = [\tilde{\delta}^T(t) \tilde{\delta}^T(t - \eta_1) \tilde{\delta}^T(t - \eta_2) \tilde{\delta}^T(t - \eta_3) \tilde{e}^T(i_k h)]^T, \\ & \quad t \in \Omega_i, \quad i = 0, 1, 2, \dots, l. \\ e_1 & \triangleq [I \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ e_2 & \triangleq [0 \ I \ 0 \ 0 \ 0 \ 0]^T, \\ e_3 & \triangleq [0 \ 0 \ I \ 0 \ 0 \ 0]^T, \\ e_4 & \triangleq [0 \ 0 \ 0 \ I \ 0 \ 0]^T, \\ e_5 & \triangleq [0 \ 0 \ 0 \ 0 \ I \ 0]^T, \\ e_6 & \triangleq [0 \ 0 \ 0 \ 0 \ 0 \ I]^T, \\ e_7 & \triangleq (-\tilde{L}e_3^T + \tilde{L}e_6^T)^T. \end{aligned} \quad (\text{A.8})$$

Then the system (41) is rewritten as $\dot{\tilde{\delta}}(t) = e_7^T \chi(t)$.

Using Jensens inequality [44] to handle the integral items in (A.6). Next, two cases are discussed.

(1) When $\eta_1 < \eta(t) \leq \eta_2$, $t \in \Omega_i$, $i = 0, 1, 2, \dots, l$, we can get

$$\begin{aligned} & \eta_1 \int_{t-\eta_1}^t \dot{\tilde{\delta}}^T(v) R_1 \dot{\tilde{\delta}}(v) dv \\ & \geq \eta_1 \times \frac{1}{\eta_1} \sum_{k=1}^n \dot{\tilde{\delta}}^T(\varepsilon) \Delta t_i R_1 \sum_{k=1}^n \dot{\tilde{\delta}}(\varepsilon) \Delta t_i \\ & = (\tilde{\delta}(t) - \tilde{\delta}(t - \eta_1))^T R_1 (\tilde{\delta}(t) - \tilde{\delta}(t - \eta_1)) \\ & = \chi^T(t) (e_1 - e_2) R_1 (e_1 - e_2)^T \chi(t), \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned}
& (\eta_2 - \eta_1) \int_{t-\eta_2}^{t-\eta_1} \dot{\delta}^T(v) R_2 \dot{\delta}(v) dv \\
&= \eta_{21} \int_{t-\eta(t)}^{t-\eta_1} \dot{\delta}^T(v) R_2 \dot{\delta}(v) dv \\
&+ \eta_{21} \int_{t-\eta_2}^{t-\eta(t)} \dot{\delta}^T(v) R_2 \dot{\delta}(v) dv \quad (\text{A.10})
\end{aligned}$$

$$\begin{aligned}
& \geq \frac{\eta_{21}}{\eta(t) - \eta_1} \chi^T(t) (e_2 - e_3) R_2 (e_2 - e_3)^T \chi(t) \\
&+ \frac{\eta_{21}}{\eta_2 - \eta(t)} \chi^T(t) (e_3 - e_4) R_2 (e_3 - e_4)^T \chi(t), \\
& (\eta_3 - \eta_2) \int_{t-\eta_3}^{t-\eta_2} \dot{\delta}^T(v) R_3 \dot{\delta}(v) dv \\
& \geq \frac{\eta_3 - \eta_2}{\eta_3 - \eta_2} \chi^T(t) (e_4 - e_5) R_3 (e_4 - e_5)^T \chi(t) \quad (\text{A.11}) \\
&= \chi^T(t) (e_4 - e_5) R_3 (e_4 - e_5)^T \chi(t).
\end{aligned}$$

Applying Lemma 11 to (A.10) yields

$$(\eta_2 - \eta_1) \int_{t-\eta_2}^{t-\eta_1} \dot{\delta}^T(v) R_2 \dot{\delta}(v) dv \geq \chi^T(t) \Lambda_2 \chi(t), \quad (\text{A.12})$$

where

$$\begin{aligned}
\Lambda_2 &= \begin{bmatrix} (e_2 - e_3)^T \\ (e_3 - e_4)^T \end{bmatrix}^T \begin{bmatrix} R_2 & U_2^T \\ U_2 & R_2 \end{bmatrix} \begin{bmatrix} (e_2 - e_3)^T \\ (e_3 - e_4)^T \end{bmatrix}, \\
& \begin{bmatrix} R_2 & * \\ U_2 & R_2 \end{bmatrix} > 0. \quad (\text{A.13})
\end{aligned}$$

Substituting (A.9)–(A.11) into (A.6) yields

$$\begin{aligned}
\dot{V}_3 &\leq \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 \chi^T(t) e_7 R_i e_7^T \chi(t) \\
&- \chi^T(t) (e_1 - e_2) R_1 (e_1 - e_2)^T \chi(t) \\
&- \chi^T(t) \Lambda_2 \chi(t) \\
&- \chi^T(t) (e_4 - e_5) R_3 (e_4 - e_5)^T \chi(t). \quad (\text{A.14})
\end{aligned}$$

Substituting (A.5), (A.7), and (A.14) into (A.4) yields

$$\begin{aligned}
\dot{V}(t) &\leq \chi^T(t) \left[e_7 P e_1^T + e_1 P e_7^T \right] \chi(t) \\
&+ \chi^T(t) e_1 S e_1^T \chi(t) - \chi^T(t) e_2 S e_2^T \chi(t) \\
&+ \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 \chi^T(t) e_7 R_i e_7^T \chi(t) \\
&- \chi^T(t) (e_1 - e_2) R_1 (e_1 - e_2)^T \chi(t) \\
&- \chi^T(t) \Lambda_2 \chi(t) \\
&- \chi^T(t) (e_4 - e_5) R_3 (e_4 - e_5)^T \chi(t) \\
&+ \chi^T(t) \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix} \chi(t) \\
&- \chi^T(t) \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix} \chi(t). \quad (\text{A.15})
\end{aligned}$$

Since, between $t_k h$ and $t_k h + ih$, no control signal is triggered, then condition (43) is satisfied; that is,

$$\begin{aligned}
& \tilde{e}^T(i_k h) \tilde{\Phi} \tilde{e}^T(i_k h) - \tilde{e}^T(i_k h) \tilde{\Phi} \tilde{e}^T(i_k h) \\
&\leq \gamma \tilde{\delta}^T(t_k h) \tilde{\Phi} \tilde{\delta}^T(t_k h) - \tilde{e}^T(i_k h) \tilde{\Phi} \tilde{e}^T(i_k h) \\
&= \gamma \chi^T(t) (e_3 - e_6) \tilde{\Phi} (e_3 - e_6)^T \chi(t) \\
&- \chi^T(t) e_6 \tilde{\Phi} e_6^T \chi(t). \quad (\text{A.16})
\end{aligned}$$

Substituting (A.16) into (A.15) yields

$$\begin{aligned}
\dot{V}(t) &\leq \chi^T(t) \left[e_7 P e_1^T + e_1 P e_7^T + e_1 S e_1^T - e_2 S e_2^T \right. \\
&- (e_1 - e_2) R_1 (e_1 - e_2)^T - \Lambda_2 \\
&- (e_4 - e_5) R_3 (e_4 - e_5)^T \\
&+ \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix} \\
&- \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix} \\
&+ \gamma (e_3 - e_6) \tilde{\Phi} (e_3 - e_6)^T - e_6 \tilde{\Phi} e_6^T \\
&\left. + \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 e_7 R_i e_7^T \right] \chi(t). \quad (\text{A.17})
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 e_7 R_i e_7^T \\
&= \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 (-\tilde{L}e_3^T + \tilde{L}e_6^T)^T \\
&\quad \times R_i (-\tilde{L}e_3^T + \tilde{L}e_6^T) \\
&= \Gamma_{21}^T \Gamma_{22}^{-1} \Gamma_{21},
\end{aligned} \tag{A.18}$$

based on the Lyapunov theory, the system (41) is asymptotically stable such that

$$\begin{aligned}
& e_7 P e_1^T + e_1 P e_7^T + e_1 S e_1^T - e_2 S e_2^T \\
&\quad - (e_1 - e_2) R_1 (e_1 - e_2)^T - \Lambda_2 \\
&\quad - (e_4 - e_5) R_3 (e_4 - e_5)^T \\
&\quad + \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix} \\
&\quad - \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix} \\
&\quad + \gamma (e_3 - e_6) \tilde{\Phi} (e_3 - e_6)^T - e_6 \tilde{\Phi} e_6^T \\
&\quad + \Gamma_{21}^T \Gamma_{22}^{-1} \Gamma_{21} < 0.
\end{aligned} \tag{A.19}$$

Finally, using Schur complement, (A.19) is equivalent to (47) ($j = 2$).

(2) Similarly, when $\eta_2 < \eta(t) \leq \eta_3$, $t \in \Omega_i$, $i = 0, 1, 2, \dots, l$, in the same way, we get

$$\begin{aligned}
\dot{V}(t) &\leq \chi^T(t) \left[e_7 P e_1^T + e_1 P e_7^T + e_1 S e_1^T - e_2 S e_2^T \right. \\
&\quad - (e_1 - e_2) R_1 (e_1 - e_2)^T \\
&\quad - (e_2 - e_4) R_3 (e_2 - e_4)^T \\
&\quad - \Lambda_3 + \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix} \\
&\quad - \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix} \\
&\quad + \gamma (e_3 - e_6) \tilde{\Phi} (e_3 - e_6)^T - e_6 \tilde{\Phi} e_6^T \\
&\quad \left. + \sum_{i=1}^3 (\eta_i - \eta_{i-1})^2 e_7 R_i e_7^T \right] \chi(t),
\end{aligned} \tag{A.20}$$

where

$$\Lambda_3 = \begin{bmatrix} (e_4 - e_3)^T \\ (e_3 - e_5)^T \end{bmatrix}^T \begin{bmatrix} R_3 & U_3^T \\ U_3 & R_3 \end{bmatrix} \begin{bmatrix} (e_4 - e_3)^T \\ (e_3 - e_5)^T \end{bmatrix}, \tag{A.21}$$

$$\begin{bmatrix} R_3 & * \\ U_3 & R_3 \end{bmatrix} > 0.$$

If $V(t) < 0$, the system (41) is asymptotically stable. So we have

$$\begin{aligned}
& e_7 P e_1^T + e_1 P e_7^T + e_1 S e_1^T - e_2 S e_2^T \\
&\quad - (e_1 - e_2) R_1 (e_1 - e_2)^T - \Lambda_2 \\
&\quad - (e_4 - e_5) R_3 (e_4 - e_5)^T \\
&\quad + \gamma (e_3 - e_6) \tilde{\Phi} (e_3 - e_6)^T - e_6 \tilde{\Phi} e_6^T \\
&\quad + \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix} \\
&\quad - \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix}^T \begin{bmatrix} Q_1 & * \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} e_4^T \\ e_5^T \end{bmatrix} \\
&\quad + \Gamma_{21}^T \Gamma_{22}^{-1} \Gamma_{21} < 0.
\end{aligned} \tag{A.22}$$

Finally, using Schur complement, (A.22) is equivalent to (47) ($j = 3$). This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (Grant no. 61104089), National Key Scientific Instrument and Equipment Development Project (Grant no. 2012YQ15008703), the National High Technology Research and Development Program of China (Grant no. 2011AA040103-7), Science and Technology Commission of Shanghai Municipality (11jc1404000), and Shanghai Rising-Star Program (13QA1401600).

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