Research Article
The Extended Multiple \( (G'/G) \)-Expansion Method and Its Application to the Caudrey-Dodd-Gibbon Equation

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An extended multiple \( (G'/G) \)-expansion method is used to seek the exact solutions of Caudrey-Dodd-Gibbon equation. As a result, plentiful new complexiton solutions consisting of hyperbolic functions, trigonometric functions, rational functions, and their mixture with arbitrary parameters are effectively obtained. When some parameters are properly chosen as special values, the known double solitary-like wave solutions are derived from the double hyperbolic function solutions.

1. Introduction

A trouble and tedious but very important problem is to find exact solutions of nonlinear evolution equations (NLEEs). To solve this problem, a number of powerful and efficient methods have been established for obtaining exact traveling wave solutions, such as the inverse scattering method [1], the Backlund transform method [2, 3], the Darboux transform method [4], the Hirota’s bilinear transformation method [5], the Exp-function method [6], the tanh-function method [7], the Weierstrass elliptic function expansion method [8], the Jacobi elliptic function expansion method [9], the simplest equation method [10–12], and the modified method of simplest equation [13, 14]. With the development of computer science, directly searching for exact traveling wave solutions of NLEEs has attracted much attention. This is due to the availability of symbolic computation systems like Mathematica or Maple which enable us to perform the complex and tedious computation on computers. Recently, Wang et al. [15] proposed a new direct method called the \( (G'/G) \)-expansion method to look for traveling wave solutions of NLEEs. This method is straightforward, concise, and capable of producing new applications. Moreover, the solutions obtained by this method are of general nature and a number of specific solutions can be deduced by putting conditions on arbitrary constants present in the general solutions. Aslan reported the relationship between the \( (G'/G) \)-expansion method and the simplest equation method [16]. He told us that the former one is a specific form of the later one. However, \( (G'/G) \)-expansion method has become widely used to search for various exact solutions of NLEEs [17–20]. Lately, the further developed methods named the generalized \( (G'/G) \)-expansion method [21], the modified \( (G'/G) \)-expansion method [22], the extended \( (G'/G) \)-expansion method [23], the improved \( (G'/G) \)-expansion method [24], the \( (G'/G, 1/G) \)-expansion method [25], and the multiple \( (G'/G) \)-expansion method [26] have been proposed for constructing exact solutions to NLEEs. The aim of this paper is to find new exact complexiton solutions and double solitary-like wave solutions of the Caudrey-Dodd-Gibbon equation by using the extended multiple \( (G'/G) \)-expansion method.

The organization of this paper is as follows. In Section 2, we briefly describe the extended multiple \( (G'/G) \)-expansion method. In Section 3, for illustration, we restrict our attention to the Caudrey-Dodd-Gibbon equation and successfully construct many complexiton solutions and double solitary-like wave solutions. In Section 4, some conclusions are given.

2. Description of the Extended Multiple \( (G'/G) \)-Expansion Method

In this section, we will give the detailed description of the extended \( (G'/G) \)-expansion method for seeking the exact traveling wave solutions of NLEEs.
Suppose that a nonlinear partial differential equation (PDE), say in two independent variables \(x\) and \(t\), is given by

\[
F(u, u_x, u_t, u_{xx}, u_{xxx}, \ldots) = 0,
\]

where \(u = u(x, t)\) is an unknown function and \(F\) is a polynomial with respect to \(u(x, t)\) and its partial derivatives which involve the highest order derivatives and the nonlinear terms. In the following, we give the main steps of the extended \((G'/G)\)-expansion method.

**Step 1.** The traveling wave transformation is

\[
u(x, t) = u(\xi_1, \xi_2), \quad \xi_1 = k_1 x + c_1 t + l_1, \quad \xi_2 = k_2 x + c_2 t + l_2,
\]

where \(k_1, c_1, l_1\) and \(k_2, c_2, l_2\) are constants to be determined later.

**Step 2.** Suppose that the solution of (1) can be written as follows:

\[
u(\xi_1, \xi_2) = A_0 + \sum_{i+j=0}^{n} \sum_{i=j}^{n} A_{ij} \left( \frac{G'_1(\xi_1)}{G_1(\xi_1)} \right)^i \left( \frac{G'_2(\xi_2)}{G_2(\xi_2)} \right)^j, \quad i, j = 0, 1, 2, \ldots \tag{3}
\]

where \(A_{0}, A_{ij} (i, j = 1, 2, \ldots, n)\) are constants to be determined later, \(n\) is an undetermined integer, and \(G_i(\xi_j)(i = 1, 2)\) satisfy the auxiliary linear ordinary differential equations:

\[
G''_i(\xi_j) + \lambda_i G'_i(\xi_j) + \mu_i G_i(\xi_j) = 0, \quad (i = 1, 2), \tag{4}
\]

where \(G'_i(\xi_j) = dG_i(\xi_j)/d\xi_j, G''_i(\xi_j) = d^2G_i(\xi_j)/d\xi_j^2\) and \(\lambda_i, \mu_i\) \((i = 1, 2)\) are constants.

**Step 3.** Determine the positive integer \(n\) by balancing the highest order derivatives and nonlinear terms in (1).

**Step 4.** Substituting (3) along with (4) into (1) yields a partial differential equation. Since terms \((G'_i(\xi_j)/G_1(\xi_1))^i (G'_2(\xi_j)/G_2(\xi_2))^j (i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots)\) in the partial differential equation are linearly independent, a set of overdetermined ordinary differential equations (ODEs) for \(A_0, A_{ij} (i, j = 1, 2, \ldots, n), k_1, k_2, c_1, c_2, l_1,\) and \(l_2\) can be obtained by vanishing all the coefficients of the terms \((G'_i(\xi_j)/G_1(\xi_1))^i (G'_2(\xi_j)/G_2(\xi_2))^j (i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots)\).

**Step 5.** Assuming that \(A_0, A_{ij} (i, j = 1, 2, \ldots, n), k_1, k_2, c_1, c_2, l_1,\) and \(l_2\) can be obtained by solving the ODEs in Step 4, then by substituting them into (3), we can obtain the exact solutions of (1) immediately.

### 3. Application of the Method to the Caudrey-Dodd-Gibbon Equation

In this section, we apply the extended multiple \((G'/G)\)-expansion method to construct the traveling wave solutions of the Caudrey-Dodd-Gibbon equation which is given by

\[
u_t + 3\nu
\]

where \(\nu\) is a real scalar function of the two independent variables \(x\) and \(t\).

For simplicity, we uniformly denote \(\Delta_1 = \lambda_1^2 - 4 \mu_1\) and \(\Delta_2 = \lambda_2^2 - 4 \mu_2\) in the following paper.

According to the homogeneous balance procedure, balancing the highest order nonlinear term \(\nu^3\) and the highest order derivative term \(\nu_{xxxx}\) in (5), we get \(3n + 1 = n + 5\). Hence we find \(n = 2\) and suppose that the solution of (5) is in the form

\[
u = A_0 + A_1 \left( \frac{G'_1(\xi_1)}{G_1(\xi_1)} \right) + A_2 \left( \frac{G'_1(\xi_1)}{G_1(\xi_1)} \right)^2 + B_1 \left( \frac{G'_2(\xi_2)}{G_2(\xi_2)} \right) + B_2 \left( \frac{G'_2(\xi_2)}{G_2(\xi_2)} \right)^2 + B_3 \left( \frac{G'_2(\xi_2)}{G_2(\xi_2)} \right)^3, \tag{6}
\]

Substituting (6) along with (4) into (5) yields a partial differential equation. Then by vanishing all the coefficients of the terms \((G'_1(\xi_1)/G_1(\xi_1))^i (G'_2(\xi_2)/G_2(\xi_2))^j (i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots)\) of the partial differential equation, we obtain 36 algebraic equations for \(A_j (i = 0, 1, 2), B_j (j = 0, 1, 2), k_1, k_2, c_1, c_2, l_1,\) and \(l_2\). Solving the system of algebraic equations with the aid of Maple 14, we obtain the following results:

\[
k_1 = F_1, \quad k_2 = F_2,
\]

\[
A_0 = \frac{1}{6} \left( F_1^2 \Delta_1 + F_2^2 \Delta_2 \right) - \frac{1}{4} \left( F_1^2 \lambda_1^2 + F_2^2 \lambda_2^2 \right), \quad B_0 = 0,
\]

\[
A_1 = -F_1^2 \lambda_1, \quad A_2 = -F_1^2, \quad B_1 = -F_2^2 \lambda_2, \quad B_2 = -F_2^2, \quad l_1 = F_3, \quad l_2 = F_4,
\]

\[
c_1 = \frac{F_1}{4} \left( F_1^4 \Delta_1^2 - 5 F_2^4 \Delta_2^2 \right), \quad c_2 = \frac{F_2}{4} \left( F_2^4 \Delta_2^2 - 5 F_1^4 \Delta_1^2 \right), \tag{7}
\]

where \(F_1, F_2, F_3,\) and \(F_4\) are arbitrary constants.
Substituting (7) into (6), the general form of solution of (5) can be expressed by
\[
\begin{align*}
\begin{aligned}
\Delta_1 & = F_1 \Delta_1^2 + F_2 \Delta_2^2 - \frac{1}{4} \left( F_1^2 \lambda_1^2 + F_2^2 \lambda_2^2 \right) \\
\Delta_2 & = F_2 \Delta_2^2 - \frac{1}{4} \left( F_1^2 \lambda_1^2 + F_2^2 \lambda_2^2 \right)
\end{aligned}
\end{align*}
\]
(8)

where \( F_1, F_2, F_3, \) and \( F_4 \) are arbitrary constants.

From the general solutions of (4), which depend on different choices of \( \lambda_1, \lambda_2, \mu_1, \) and \( \mu_2, \) some complexiton solutions of (5) can be derived immediately.

Case 1. Setting \( \Delta_1 > 0 \) and \( \Delta_2 > 0, \) the complexiton solutions consisting of hyperbolic functions of (5) can be derived as
\[
\begin{align*}
\xi_1 & = F_1 x + \frac{F_1}{4} \left( F_1^2 \Delta_1^2 - 5F_2^2 \Delta_2^2 \right) t + F_3, \\
\xi_2 & = F_2 x + \frac{F_2}{4} \left( F_1^2 \Delta_1^2 - 5F_2^2 \Delta_2^2 \right) t + F_4,
\end{align*}
\]
(9)

where \( F_1, F_2, F_3, \) and \( F_4 \) are determined by using (9). In order to show the properties of the double solitary wave solution visually, as an example, we plot the 3D graphs of solution (10) for some fixed parameters, which are shown in Figure 1.

It is easy to see that the double solitary-like wave solution can be obtained at \( M_1^2 > N_1^2, \) \( M_2^2 > N_2^2 \) and \( M_1 \neq 0, M_2 \neq 0 \) as follows:
\[
\begin{align*}
\begin{aligned}
\xi_1 & = \frac{F_1 \Delta_1^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_1}}{2} \xi_1 + \xi_{10} \right) \\
\xi_2 & = \frac{F_2 \Delta_2^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_2}}{2} \xi_2 + \xi_{20} \right),
\end{aligned}
\end{align*}
\]
(11)

where \( \xi_{10} = \tanh^{-1}(N_1/M_1) \) and \( \xi_{20} = \tanh^{-1}(N_2/M_2). \)

Case 2. Setting \( \Delta_1 > 0 \) and \( \Delta_2 < 0, \) the complexiton solutions consisting of hyperbolic functions and trigonometric functions of (5) can be derived as
\[
\begin{align*}
\begin{aligned}
\xi_1 & = \frac{F_1 \Delta_1^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_1}}{2} \xi_1 + \xi_{10} \right) \\
\xi_2 & = \frac{F_2 \Delta_2^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_2}}{2} \xi_2 + \xi_{20} \right),
\end{aligned}
\end{align*}
\]
(12)

where \( F_1, F_2, M_1, M_2, N_1, \) and \( N_2 \) are arbitrary constants and \( \xi_1 \) and \( \xi_2 \) are determined by using (9). In order to show the properties of the double solitary wave solution visually, as an example, we plot the 3D graphs of solution (10) for some fixed parameters, which are shown in Figure 1.

The profiles of (12) are shown in Figure 2.

It is easy to see that the complexiton solutions can be rewritten at \( M_1^2 > N_1^2, M_2^2 > N_2^2, \) and \( M_1 \neq 0, M_2 \neq 0 \) as follows:
\[
\begin{align*}
\begin{aligned}
\xi_1 & = \frac{F_1 \Delta_1^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_1}}{2} \xi_1 + \xi_{10} \right) \\
\xi_2 & = \frac{F_2 \Delta_2^2}{4} \tanh^2 \left( \frac{\sqrt{\Delta_2}}{2} \xi_2 + \xi_{20} \right),
\end{aligned}
\end{align*}
\]
(13)
Case 3. Setting $\Delta_1 < 0$ and $\Delta_2 < 0$, the complexiton solutions consisting of trigonometric functions of (5) can be derived as

$$u_3 = \frac{1}{6} (F_1^2 \Delta_1 + F_2^2 \Delta_2) + \frac{F_2^2 \Delta_1}{4} \times \left( \frac{M_1 \sin\left(\left(\frac{\sqrt{-\Delta_1}}{2}\right) \xi_1\right) - N_1 \cos\left(\left(\frac{\sqrt{-\Delta_1}}{2}\right) \xi_1\right)}{M_1 \cos\left(\left(\frac{\sqrt{-\Delta_1}}{2}\right) \xi_1\right) + N_1 \sin\left(\left(\frac{\sqrt{-\Delta_1}}{2}\right) \xi_1\right)} \right)^2,$$

$$+ \frac{F_2^2 \Delta_2}{4} \times \left( \frac{M_2 \sin\left(\left(\frac{\sqrt{-\Delta_2}}{2}\right) \xi_2\right) - N_2 \cos\left(\left(\frac{\sqrt{-\Delta_2}}{2}\right) \xi_2\right)}{M_2 \cos\left(\left(\frac{\sqrt{-\Delta_2}}{2}\right) \xi_2\right) + N_2 \sin\left(\left(\frac{\sqrt{-\Delta_2}}{2}\right) \xi_2\right)} \right)^2,$$

(14)

where $F_1, F_2, M_1, M_2, N_1,$ and $N_2$ are arbitrary constants and $\xi_1$ and $\xi_2$ are determined by using (9).

The profiles of (14) are shown in Figure 3.

It is easy to see that the complexiton solutions can be rewritten at $M_2^2 > M_1^2, N_2^2 > N_1^2$ and $M_2 \neq 0, N_2 \neq 0$ as follows:

$$u_3 = \frac{1}{6} (F_1^2 \Delta_1 + F_2^2 \Delta_2) + \frac{F_2^2 \Delta_1}{4} \tan^2\left(\frac{\sqrt{-\Delta_1}}{2} \xi_1 - \xi_{10}\right)$$

$$+ \frac{F_2^2 \Delta_2}{4} \tan^2\left(\frac{\sqrt{-\Delta_2}}{2} \xi_2 - \xi_{20}\right),$$

(15)

where $\xi_{10} = \tan^{-1}(N_1/M_1)$ and $\xi_{20} = \tan^{-1}(N_2/M_2)$.

Case 4. Setting $\Delta_1 = 0$ and $\Delta_2 > 0$, the complexiton solutions consisting of rational functions and hyperbolic functions of (5) can be derived as

$$u_4 = \frac{1}{6} (F_1^2 \Delta_1 + F_2^2 \Delta_2) - \left( \frac{F_1 N_1}{M_1 + N_1 \xi_1} \right)^2 - \frac{F_2^2 \Delta_2}{4} \times \left( \frac{M_2 \sinh\left(\left(\frac{\sqrt{\Delta_2}}{2}\right) \xi_2\right) + N_2 \cosh\left(\left(\frac{\sqrt{\Delta_2}}{2}\right) \xi_2\right)}{M_2 \cosh\left(\left(\frac{\sqrt{\Delta_2}}{2}\right) \xi_2\right) + N_2 \sinh\left(\left(\frac{\sqrt{\Delta_2}}{2}\right) \xi_2\right)} \right)^2,$$

(16)

where $F_1, F_2, M_1, M_2, N_1,$ and $N_2$ are arbitrary constants and $\xi_1$ and $\xi_2$ are determined by using (9).

The profiles of (16) are shown in Figure 4.

It is easy to see that the complexiton solutions can be rewritten at $N_2^2 > N_1^2$ and $N_2 \neq 0$ as follows:

$$u_4 = \frac{1}{6} (F_1^2 \Delta_1 + F_2^2 \Delta_2) - \left( \frac{F_1 N_1}{M_1 + N_1 \xi_1} \right)^2$$

$$- \frac{F_2^2 \Delta_2}{4} \tanh^2\left(\frac{\sqrt{\Delta_2}}{2} \xi_2 + \xi_{20}\right),$$

(17)

where $\xi_{20} = \tan^{-1}(N_2/M_2)$.
Case 5. Setting $\Delta_1 = 0$ and $\Delta_2 < 0$, the complexiton solutions consisting of rational functions and trigonometric functions of (5) can be derived as

$$
\begin{align*}
  u_5 &= \frac{1}{6} \left( F_1^2 \Delta_1 + F_2^2 \Delta_2 \right) - \left( \frac{F_1 N_1}{M_1 + N_1 \xi_1} \right)^2 + \left( \frac{F_2^2 \Delta_2}{4} \right) \\
  &\times \left( M_2 \sin \left( \frac{\sqrt{-\Delta_2}}{2} \xi_2 \right) - N_2 \cos \left( \frac{\sqrt{-\Delta_2}}{2} \xi_2 \right) \right)^2 ,
\end{align*}
$$

where $F_1, F_2, M_1, M_2, N_1$, and $N_2$ are arbitrary constants and $\xi_1$ and $\xi_2$ are determined by using (9).

It is easy to see that the complexiton solutions can be rewritten at $N_2^2 > N_1^2$ and $N_2 \neq 0$ as follows:

$$
\begin{align*}
  u_5 &= \frac{1}{6} \left( F_1^2 \Delta_1 + F_2^2 \Delta_2 \right) - \left( \frac{F_1 N_1}{M_1 + N_1 \xi_1} \right)^2 \\
  &\times \left( \frac{F_2^2 \Delta_2}{4} \tan^2 \left( \frac{\sqrt{-\Delta_2}}{2} \xi_2 - \xi_2 \right) \right) ,
\end{align*}
$$

where $\xi_2 = \tan^{-1}(N_2/M_2)$.

Case 6. Setting $\Delta_1 = 0$ and $\Delta_2 = 0$, the complexiton solutions consisting of rational functions of (5) can be derived as

$$
\begin{align*}
  u_6 &= \frac{1}{6} \left( F_1^2 \Delta_1 + F_2^2 \Delta_2 \right) - \left( \frac{F_1 N_1}{M_1 + N_1 \xi_1} \right)^2 - \left( \frac{F_2 N_2}{M_2 + N_2 \xi_2} \right)^2 ,
\end{align*}
$$

where $F_1, F_2, M_1, M_2, N_1$, and $N_2$ are arbitrary constants and $\xi_1$ and $\xi_2$ are determined by using (9).

Similarly, when setting $(\Delta_1 < 0$ and $\Delta_2 > 0), (\Delta_1 > 0$ and $\Delta_2 = 0)$, and $(\Delta_1 < 0$ and $\Delta_2 = 0)$ some other complexiton solutions of (5) can be obtained. We omit them here for convenience.

Remark 1. In this method, two independent variables $\xi_1$ and $\xi_2$ were introduced, and, therefore, double solitary-like wave solutions and some other complexiton solutions consisting of hyperbolic functions, trigonometric functions, rational functions, and their mixture with arbitrary parameters for the Caudrey-Dodd-Gibbon equation can be obtained.

Remark 2. Comparing our results with Naher et al.’s results in [20], it can be seen that the solutions we obtained are new and more plentiful.

Remark 3. The validity of the solutions we obtained is verified by using the mathematical software Maple 14.

Remark 4. The ansatz (6) can be expressed as another form:

$$
\begin{align*}
  u &= A_0 + A_1 \left( \frac{Y'_1(\xi_1)}{Y_1(\xi_1)} - \frac{\lambda_1}{2} \right) + A_2 \left( \frac{Y'_2(\xi_1)}{Y_2(\xi_1)} - \frac{\lambda_2}{2} \right)^2 \\
  &+ B_1 \left( \frac{Y'_2(\xi_2)}{Y_2(\xi_2)} - \frac{\lambda_2}{2} \right)^2 + B_2 \left( \frac{Y'_1(\xi_2)}{Y_1(\xi_2)} - \frac{\lambda_1}{2} \right)^2 ,
\end{align*}
$$

where $Y(\xi_i) (i = 1, 2)$ satisfy the simplest equations:

$$
Y''(\xi_i) + \frac{4\mu_i - \lambda_i^2}{4} Y(\xi_i) = 0 , \quad (i = 1, 2) .
$$

4. Conclusions

In the present work, an extended multiple $(G'/G)$-expansion method was applied to the Caudrey-Dodd-Gibbon equation, and we successfully obtained some exact complexiton solutions expressed by hyperbolic functions, the trigonometric functions, the rational functions, and their mixture. To the best of our knowledge, these solutions with arbitrary parameters are new. The results of [20] have been enriched.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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