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*Research Article*

# $H_\infty$ Enhanced Control Design of Discrete-Time Takagi-Sugeno State-Multiplicative Noisy Systems

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Design conditions for existence of the $H_\infty$ state feedback control for Takagi-Sugeno fuzzy discrete-time stochastic systems with state-multiplicative noise, stabilizing the closed-loop in such way that the quadratic performance in the mean is satisfied, are presented in the paper. Using newly introduced enhanced form of the bounded real lemma for such stochastic systems, the LMI-based procedure is provided for computation of gain matrices of the state control law, realized in the parallel distributed compensation structure. The approach is illustrated on an example, demonstrating the validity of the proposed method.

## 1. Introduction

The control design for systems with stochastic uncertainties is an area of study for several decades (see [1] and the references therein) and so far these practically and theoretically attractive fields derived numerous solutions. An approach, in which the parameter uncertainties were modeled as white noise processes in a linear system, was firstly applied for discrete-time systems [2] and continuous-time systems [3–5]. Unfortunately, this way of doing it on linear quadratic optimal control principle and reflecting the control input as a deterministic function of system state naturally led mainly to a nonstandard Riccati equation. Representing the system uncertainty by unknown disturbance signal in the system equations, $H_\infty$ stayed the most prominent method of dealing with the corresponding control problem of disturbance relaxation [6]. Combining both representations of uncertainty, adapting $H_\infty$ approach towards the disturbance attenuation for stochastic systems, and including those that ensure a performance bound in the $H_\infty$ sense, new accessions have been made in diverse practical problems for finite and infinite time horizons for continuous-time [7, 8] as well as for discrete-time stochastic systems [9–11], denoting later such systems as linear stochastic systems with multiplicative noise. The stochastic control for linear and nonlinear systems with multiplicative noises has founded its applications in many fields in control theory. Such models are encountered mainly in constrain control [12] and in gain scheduling when the scheduling parameters are corrupted with measurement noise [13].

In contrary to the linear framework, nonlinear systems are too complex to be represented by unified mathematical resources, and so a generic method has not been developed yet to design a controller valid for nonlinear stochastic systems with multiplicative noise [14]. An alternative is Takagi-Sugeno (TS) fuzzy approach [15], which benefits from the advantages of suitable linear approximation. Using the TS fuzzy model, each rule utilizes the local system dynamics by a linear model and the nonlinear system is represented by a collection of fuzzy rules. Recently, TS model based fuzzy control approaches are being fast and successfully used in nonlinear control frameworks. As a result, a range of stability analysis conditions [16], as well as control design methods [17–20], have been developed for TS fuzzy systems, relying mostly on the feasibility of an associated set of linear matrix inequalities (LMI) [21, 22]. An important fact is that the design problem is a standard feasibility problem with several LMIs. According to the T-S fuzzy model with multiplicative noise, there exist only few approaches investigated to solve the above-mentioned problems of control design and stability analysis for continuous-time [23, 24], as well as for discrete-time TS models [25–27] and these papers only present some results

on stability analysis, mean-square stabilization, and control for TS fuzzy stochastic systems with multiplicative noises.

Considering actual results in bounded real lemma forms for discrete-time stochastic systems [8, 14], new design conditions, based on an enhanced form of the bounded real lemma for TS fuzzy discrete-time stochastic systems with state-multiplicative noise and $H_\infty$ control of such systems, are derived in the paper. For some potential numerical problems, there are also posed viewpoints and clarification for discrete-time TS fuzzy model [28], used in simulation.

To present this, the paper is arranged in the next sections as follows. Following the introduction in Section 1, the control design task for discrete-time TS fuzzy stochastic systems with state-multiplicative noise is presented in Section 2. The preliminary results, focused in consequence on two rudimentary bounded real lemma forms for such defined stochastic systems, are presented in Section 3 and, subsequently, Section 4 provides the controller design conditions in equivalent forms of LMIs. Section 5 illustrates the control design task by a numerical solution and Section 6 draws some conclusions.

Throughout the paper, the notations are narrowly standard in such way that $\mathbf{x}^T$, $\mathbf{X}^T$ denote the transpose of the vector $\mathbf{x}$ and matrix $\mathbf{X}$, respectively, $\mathbf{X} = \mathbf{X}^T > 0$, $(\geq 0)$, means that $\mathbf{X}$ is a symmetric positive definite (semidefinite) matrix, diag$[\cdot]$ designates a block diagonal matrix, the symbol $\mathbf{I}_n$ indicates the $n$th-order identity matrix, $\mathcal{Z}_+$ is the set of all positive integers, $\mathbb{R}$ indicates the set of real numbers, $E_o\{\cdot\}$ remits expectation with respect to the variable $o$, $\mathbb{R}^n$, $\mathbb{R}^{n\times r}$ refer to the set of all $n$-dimensional real vectors and $n \times r$ real matrices, respectively, $L_2\langle 0, +\infty)$ entails the space of square summable functions over $\langle 0, +\infty)$, and $\delta_{ij}$ is the Kronecker delta function [29].

## 2. Problem Formulation

Through the paper, the task is concerned with state feedback design to control a TS fuzzy stochastic discrete-time dynamic system, given by the set of equations

$$\mathbf{q}(i+1) = \sum_{h=1}^{s} \mu_h(\boldsymbol{\theta}(i)) \left( \left( \mathbf{F}_h + \mathbf{V}_h o(i) \right) \mathbf{q}(i) + \mathbf{G}_h \mathbf{u}(i) + \mathbf{W}_h \mathbf{d}(i) \right), \tag{1}$$

$$\mathbf{y}(i) = \mathbf{C}\mathbf{q}(i), \tag{2}$$

where the initial condition $\mathbf{q}(0)$ is known, $\mathbf{q}(i) \in \mathbb{R}^n$, $\mathbf{u}(i) \in \mathbb{R}^r$, and $\mathbf{y}(i) \in \mathbb{R}^m$ are the state, input, and output vectors, respectively, $\mathbf{d}(i) \in \mathbb{R}^{r_w}$ is an exogenous disturbance vector, $\mathbf{F}_h \in \mathbb{R}^{n\times n}$, $\mathbf{G}_h \in \mathbb{R}^{n\times r}$, $\mathbf{W}_h \in \mathbb{R}^{n\times r_v}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{V}_h \in \mathbb{R}^{n\times n}$ are real matrices and $i \in \mathcal{Z}_+$. The scalar function $\mu_h(\boldsymbol{\theta}(i))$ is the weight for $h$th fuzzy rule, satisfying, by definition, the property

$$0 \leq \mu_h(\boldsymbol{\theta}(t)) \leq 1, \quad \sum_{h=1}^{s} \mu_h(\boldsymbol{\theta}(i)) = 1, \quad \forall h \in \langle 1, 2, \ldots, s \rangle, \tag{3}$$

$$\boldsymbol{\theta}(t) = \left[ \theta_1(t) \;\; \theta_2(t) \;\; \cdots \;\; \theta_v(t) \right] \tag{4}$$

which is the vector of the premise variables, where $s$, $v$ are the numbers of fuzzy rules and premise variables, respectively. It is supposed in the next that all premise variables are measurable and independent of $\mathbf{u}(t)$. More details can be found, for example, in [18, 21].

It is assumed that $o(i)$, $0 \leq i \leq j$, satisfies the condition

$$E_o\{o(i)\} = 0, \qquad E_o\{o(i)\, o(j)\} = 1\delta_{ij} \tag{5}$$

and disturbance is a random vector sequence $\{\mathbf{d}(i) \in L_2(\langle 0, \infty); \mathbb{R}^{r_w})$.

The problem of interest is to design in the mean square sense stable closed-loop system by the overall fuzzy state feedback controller in the parallel distributed compensation (PDF) form

$$\mathbf{u}(i) = -\sum_{l=1}^{s} \mu_l(\boldsymbol{\theta}(i)) \mathbf{K}_l \mathbf{q}(i), \tag{6}$$

which uses the same set of normalized membership functions (3) and $\mathbf{K}_l \in \mathbb{R}^{r\times n}$, $l = 1, 2, \ldots, s$ is the set of feedback controller gain matrices. It is supposed that all couples $(\mathbf{F}_h, \mathbf{G}_h)$ are controllable.

## 3. Basic Preliminaries

**Lemma 1** (a TS sector approximation of the function sine). *The function sine can be approximated on the interval $-\pi < x < \pi$ as*

$$\sin(x) = \sum_{h=1}^{2} \mu_h(x) d_h, \tag{7}$$

*where $d_1$, $d_2$ are the sector bounds and the fuzzy membership functions take the forms*

$$\mu_1(x) = 1 - \frac{\sin(x)}{x}, \qquad \mu_2(x) = \frac{\sin(x)}{x}. \tag{8}$$

*Proof (compare [30]).* Since it can be considered that

$$0 \doteq d_1 \leq \sin(x) \leq d_2 = x \tag{9}$$

and (3) implies

$$\sum_{h=1}^{2} \mu_h(x) = 1, \quad 0 \leq \mu_h(x) \leq 1, \; h = 1, 2, \tag{10}$$

then (7) can be written as

$$\mu_2(x) d_2 + \left(1 - \mu_2(x)\right) d_1 = \sin(x). \tag{11}$$

Thus, evidently, the real fuzzy membership functions have to take the forms

$$\mu_2(x) = \frac{\sin(x) - d_1}{d_2 - d_1}, \qquad \mu_1(x) = 1 - \mu_2(x). \tag{12}$$

Approximating for the left bounded sector, that is, $d_1 = 0$, $d_2 = x$, then (12) implies (8). □

It is evident that if $d_1 = 0$ and $d_2 = x$, then (7) gives a trivial solution

$$\sin(x) = \mu_2(x)\, d_2 = \frac{\sin(x)}{x} x = \sin(x). \tag{13}$$

*Remark 2.* Very often the fuzzy membership functions in the above given task are defined as simple triangles [31]; that is,

$$\mu_1(x) = 1 - \frac{1}{\pi}|x|, \qquad \mu_2(x) = \frac{1}{\pi}|x|. \tag{14}$$

Moreover, the minimal sector value is often considered approximately as $d_1 = 10^{-2}/\pi$, or $d_1 = (10^{-2}/\pi)x$.

**Lemma 3.** *If* $\mathbf{M}$, $\mathbf{N}$ *are matrices of appropriate dimension and* $\mathbf{X}$ *is a symmetric positive definite matrix, then*

$$\mathbf{M}^T\mathbf{X}\mathbf{N} + \mathbf{N}^T\mathbf{X}\mathbf{M} \le \mathbf{N}^T\mathbf{X}\mathbf{N} + \mathbf{M}^T\mathbf{X}\mathbf{M}. \tag{15}$$

*Proof.* Since $\mathbf{X} = \mathbf{X}^T > 0$, then

$$\left(\mathbf{X}^{1/2}\mathbf{M} - \mathbf{X}^{1/2}\mathbf{N}\right)^T \left(\mathbf{X}^{1/2}\mathbf{M} - \mathbf{X}^{1/2}\mathbf{N}\right) \ge 0, \tag{16}$$

$$\mathbf{M}^T\mathbf{X}\mathbf{M} + \mathbf{N}^T\mathbf{X}\mathbf{N} - \mathbf{M}^T\mathbf{X}\mathbf{N} - \mathbf{N}^T\mathbf{X}\mathbf{M} \ge 0, \tag{17}$$

respectively. It is evident that (17) implies (15). □

**Lemma 4.** *If* $\mathbf{X}$ *is a symmetric positive definite matrix,* $\mathbf{L}_l$ *are matrices of appropriate dimension,* $\mu_l \ge 0$ *is a real scalar,* $l = 1, 2, \ldots, s$, *and* $s$ *is a positive integer, then*

$$\left(\sum_{h=1}^{s}\mu_h\mathbf{L}_h^T\right)\mathbf{X}\left(\sum_{k=1}^{s}\mu_k\mathbf{L}_k\right) \le \sum_{h=1}^{s}\mu_h\mathbf{L}_h^T\mathbf{X}\mathbf{L}_h\left(\sum_{k=1}^{s}\mu_k\right) \tag{18}$$

*and, if* $\sum_{k=1}^{s}\mu_k = 1$,

$$\left(\sum_{h=1}^{s}\mu_h\mathbf{L}_h^T\right)\mathbf{X}\left(\sum_{k=1}^{s}\mu_k\mathbf{L}_k\right) \le \sum_{h=1}^{s}\mu_h\mathbf{L}_h^T\mathbf{X}\mathbf{L}_h. \tag{19}$$

*Proof.* Considering (18), it can be written as

$$\left(\sum_{h=1}^{s}\mu_h\mathbf{L}_h^T\right)\mathbf{X}\left(\sum_{k=1}^{s}\mu_k\mathbf{L}_k\right) = \sum_{h=1}^{s}\sum_{k=1}^{s}\mu_h\mu_k\mathbf{L}_h^T\mathbf{X}\mathbf{L}_k; \tag{20}$$

then, using (15), we have

$$\begin{aligned}
&\left(\sum_{i=1}^{s}\mu_h\mathbf{L}_h^T\right)\mathbf{X}\left(\sum_{k=1}^{s}\mu_k\mathbf{L}_k\right)\\
&\quad= \sum_{h=1}^{s}\sum_{k=1}^{s}\mu_h\mu_k\left(\frac{1}{2}\mathbf{L}_h^T\mathbf{X}\mathbf{L}_k + \frac{1}{2}\mathbf{L}_k^T\mathbf{X}\mathbf{L}_k\right)\\
&\quad\le \sum_{h=1}^{s}\sum_{k=1}^{s}\frac{1}{2}\mu_h\mu_k\left(\mathbf{L}_h^T\mathbf{X}\mathbf{L}_h + \mathbf{L}_k^T\mathbf{X}\mathbf{L}_k\right)\\
&\quad= \sum_{h=1}^{s}\mu_h\left(\sum_{k=1}^{s}\frac{1}{2}\mu_k\mathbf{L}_k^T\mathbf{X}\mathbf{L}_k\right) + \sum_{k=1}^{s}\mu_k\left(\sum_{h=1}^{s}\frac{1}{2}\mu_h\mathbf{L}_h^T\mathbf{X}\mathbf{L}_h\right).
\end{aligned} \tag{21}$$

Evidently, (21) implies (18). □

**Lemma 5** (bounded real lemma). *The deterministically unforced system* (1) *and* (2) *is stable in the mean square sense and with the quadratic performance if there exist a positive definite matrix* $\mathbf{P} \in \mathbb{R}^{n\times n}$ *and a positive scalar* $\gamma \in \mathbb{R}$ *such that for all* $h = 1, 2, \ldots, s$

$$\mathbf{P} = \mathbf{P}^T > 0, \quad \gamma > 0, \tag{22}$$

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{P}\mathbf{F}_h & \mathbf{P}\mathbf{W}_h & -\mathbf{P} & * & * \\ \mathbf{P}\mathbf{V}_h & \mathbf{0} & \mathbf{0} & -\mathbf{P} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0. \tag{23}$$

*Here and hereafter,* $*$ *denotes the symmetric item in a symmetric matrix.*

*Proof.* Let the Lyapunov function candidate be

$$\begin{aligned}
v(\mathbf{q}(i)) &= \mathbf{q}^T(i)\,\mathbf{P}\mathbf{q}(i)\\
&\quad + \gamma^{-1}\sum_{j=0}^{i-1}\left(\mathbf{y}^T(j)\,\mathbf{y}(j) - \gamma^2\mathbf{d}^T(j)\,\mathbf{d}(j)\right),
\end{aligned} \tag{24}$$

where $\gamma \in \mathbb{R}$ is square of the $H_\infty$ norm of the transfer function matrix for the disturbance input $\mathbf{d}$ and the system output $\mathbf{y}$.

Taking expectation with respect to $o(i)$, then it yields

$$\begin{aligned}
&E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\\
&\quad= E_o\left\{\mathbf{q}^T(i+1)\,\mathbf{P}\mathbf{q}(i+1)\right\} - \mathbf{q}^T(i)\,\mathbf{P}\mathbf{q}(i)\\
&\qquad+ \gamma^{-1}\mathbf{y}^T(i)\,\mathbf{y}(i) - \gamma\mathbf{d}^T(i)\,\mathbf{d}(i).
\end{aligned} \tag{25}$$

Solving for the deterministically unforced system (1) and (2), the following is obtained:

$$\begin{aligned}
&E_o\left\{\mathbf{q}^T(i+1)\,\mathbf{P}\mathbf{q}(i+1)\right\}\\
&\quad= \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\sum_{k=1}^{s}\mu_k(\boldsymbol{\theta}(i))\\
&\qquad\times\Big\{\mathbf{d}^T(i)\,\mathbf{W}_h^T\mathbf{P}\mathbf{W}_k\mathbf{d}(i)\\
&\qquad\quad+ \mathbf{q}^T(i)\left(\mathbf{F}_h^T\mathbf{P}\mathbf{F}_k + E_o\left\{o^2(i)\right\}\mathbf{V}_h^T\mathbf{P}\mathbf{V}_k\right)\mathbf{q}(i)\\
&\qquad\quad+ \mathbf{q}^T(i)\,\mathbf{F}_h^T\mathbf{P}\mathbf{W}_k\mathbf{d}(i) + \mathbf{d}^T(i)\,\mathbf{W}_h^T\mathbf{P}\mathbf{F}_k\mathbf{q}(i)\Big\}.
\end{aligned} \tag{26}$$

Considering (19), then the substitution of (5) and (26) in (25) gives

$$\begin{aligned}
&E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\\
&\quad\le \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\\
&\qquad\times\Big\{\mathbf{q}^T(i)\left(-\mathbf{P} + \mathbf{F}_h^T\mathbf{P}\mathbf{F}_h + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h + \gamma^{-1}\mathbf{C}^T\mathbf{C}\right)\mathbf{q}(i)\\
&\qquad\quad+ \mathbf{q}^T(i)\,\mathbf{F}_h^T\mathbf{P}\mathbf{W}_h\mathbf{d}(i) + \mathbf{d}^T(i)\,\mathbf{W}_h^T\mathbf{P}\mathbf{F}_h\mathbf{q}(i)\\
&\qquad\quad+ \mathbf{d}^T(i)\,\mathbf{W}_h^T\mathbf{P}\mathbf{W}_h\mathbf{d}(i) - \gamma\mathbf{d}^T(i)\,\mathbf{d}(i)\Big\}.
\end{aligned} \tag{27}$$

Defining the composite vector

$$\mathbf{q}^{\circ T}(i) = \left[\mathbf{q}^T(i) \quad \mathbf{d}^T(i)\right], \tag{28}$$

the next inequality has to be satisfied

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad \le \sum_{h=1}^{s} \mu_h(\boldsymbol{\theta}(i))\, \mathbf{q}^{\circ T}(i)\, \mathbf{P}_h^{\circ} \mathbf{q}^{\circ}(i) < 0 \end{aligned} \tag{29}$$

if

$$\mathbf{P}_h^{\circ} = \begin{bmatrix} -\mathbf{P} + \mathbf{F}_h^T\mathbf{P}\mathbf{F}_h + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h + \gamma^{-1}\mathbf{C}^T\mathbf{C} & \mathbf{F}_h^T\mathbf{P}\mathbf{W}_h \\ \mathbf{W}_h^T\mathbf{P}\mathbf{F}_h & -\gamma\mathbf{I}_{r_w} + \mathbf{W}_h^T\mathbf{P}\mathbf{W}_h \end{bmatrix} < 0. \tag{30}$$

Writing (30) as follows

$$\begin{aligned} &\begin{bmatrix} -\mathbf{P} + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h + \gamma^{-1}\mathbf{C}^T\mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{F}_h^T\mathbf{P} \\ \mathbf{W}_h^T\mathbf{P} \end{bmatrix} \mathbf{P}^{-1} \left[\mathbf{P}\mathbf{F}_h \quad \mathbf{P}\mathbf{W}_h\right] < 0 \end{aligned} \tag{31}$$

and using the Schur complement property, (31) can be rewritten in the next form:

$$\begin{bmatrix} -\mathbf{P} + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h + \gamma^{-1}\mathbf{C}^T\mathbf{C} & \mathbf{0} & \mathbf{F}_h^T\mathbf{P} \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & \mathbf{W}_h^T\mathbf{P} \\ \mathbf{P}\mathbf{F}_h & \mathbf{P}\mathbf{W}_h & -\mathbf{P} \end{bmatrix} < 0. \tag{32}$$

Applying twice again the Schur complement property, (32) implies (23). This concludes the proof. □

*Remark 6.* The inequality (23) represents the standard bounded real lemma form (compare, e.g., [32, page 66], for continuous-time linear systems). If it is necessary to reduce the range of setting (defined by $\gamma$), Lyapunov function candidate can be selected in the form

$$v(\mathbf{q}(i)) = \mathbf{q}^T(i)\,\mathbf{P}\mathbf{q}(i) + \sum_{j=0}^{i-1}\left(\mathbf{y}^T(j)\,\mathbf{y}(j) - \gamma\mathbf{d}^T(j)\,\mathbf{d}(j)\right). \tag{33}$$

Now the conditions that also satisfied $E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \le 0$ are

$$\mathbf{P} = \mathbf{P}^T > 0, \quad \gamma > 0, \tag{34}$$

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{P}\mathbf{F}_h & \mathbf{P}\mathbf{W}_h & -\mathbf{P} & * & * \\ \mathbf{P}\mathbf{V}_h & \mathbf{0} & \mathbf{0} & -\mathbf{P} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0. \tag{35}$$

Note, in the same way, the structure of (35) is used in the following lemmas and theorems.

**Lemma 7** (enhanced bounded real lemma). *The deterministically unforced system* (1) *and* (2) *is stable in the mean square sense and with the quadratic performance if there exist positive definite matrices* $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ *and a positive scalar* $\gamma \in \mathbb{R}$ *such that for all* $h = 1, 2, \ldots, s$

$$\mathbf{P} = \mathbf{P}^T > 0, \quad \mathbf{Q} = \mathbf{Q}^T > 0, \quad \gamma > 0, \tag{36}$$

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{Q}\mathbf{F}_h & \mathbf{Q}\mathbf{W}_h & \mathbf{P} - 2\mathbf{Q} & * & * \\ \mathbf{Q}\mathbf{V}_h & \mathbf{0} & \mathbf{0} & -0.5\mathbf{Q} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0. \tag{37}$$

*Proof.* Since $\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i)) = 1$, for the deterministically unforced system (1), it is implied that

$$\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\left(\mathbf{F}_h\mathbf{q}(i) + \mathbf{V}_h o(i)\,\mathbf{q}(i) + \mathbf{W}_h\mathbf{d}(i) - \mathbf{q}(i+1)\right) = \mathbf{0}. \tag{38}$$

Then, with an arbitrary symmetric positive definite matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, it is yielded that

$$\begin{aligned} &\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\left\{\mathbf{q}^T(i+1) + \mathbf{q}^T(i)\,\mathbf{V}_h^T o(i)\right\} \\ &\quad \times \mathbf{Q}\sum_{k=1}^{s}\mu_k(\boldsymbol{\theta}(i))\left\{\mathbf{F}_K\mathbf{q}(i) + \mathbf{V}_k o(i)\,\mathbf{q}(i)\right. \\ &\qquad \left. + \mathbf{W}_k\mathbf{d}(i) - \mathbf{q}(i+1)\right\} = \mathbf{0} \end{aligned} \tag{39}$$

and also

$$\begin{aligned} &E_o\left\{\mathbf{q}^T(i+1)\right\}\mathbf{Q}\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i)) \\ &\qquad \times \left(\mathbf{F}_h\mathbf{q}(i) + \mathbf{W}_h\mathbf{d}(i) - E_o\{\mathbf{q}(i+1)\}\right) \\ &\quad + \sum_{h=1}^{s}\sum_{k=1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_k(\boldsymbol{\theta}(i))\,\mathbf{q}^T(i)\,\mathbf{V}_h^T\mathbf{Q}\mathbf{V}_k\mathbf{q}(i) = \mathbf{0}, \end{aligned} \tag{40}$$

respectively. Adding (40) as well as its transposition to (25) gives

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad = E_o\left\{\mathbf{q}^T(i+1)\right\}\mathbf{P}E_o\{\mathbf{q}(i+1)\} \\ &\qquad + \gamma^{-1}\,\mathbf{q}^T(i)\,\mathbf{C}^T\mathbf{C}\mathbf{q}(i) - \gamma\mathbf{d}^T(i)\,\mathbf{d}(i) - \mathbf{q}^T(i)\,\mathbf{P}\mathbf{q}(i) \\ &\qquad + E_o\left\{\mathbf{q}^T(i+1)\right\}\mathbf{Q}\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\left(\mathbf{F}_h\mathbf{q}(i) + \mathbf{W}_h\mathbf{d}(i)\right. \\ &\qquad\qquad \left. - E_o\{\mathbf{q}(i+1)\}\right) \end{aligned}$$

$$+\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\left(\mathbf{F}_h\mathbf{q}(i)+\mathbf{W}_h\mathbf{d}(i)-E_o\{\mathbf{q}(i+1)\}\right)^T \times \mathbf{Q}E_o\{\mathbf{q}(i+1)\} + 2\mathbf{q}^T(i)\sum_{h=1}^{s}\sum_{k=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_k(\boldsymbol{\theta}(i))\mathbf{V}_h^T\mathbf{Q}\mathbf{V}_k\mathbf{q}(i) < 0. \tag{41}$$

Defining the composite vector

$$\mathbf{q}^{\bullet T}(i) = \left[\mathbf{q}^T(i) \quad \mathbf{d}^T(i) \quad E_o\left\{\mathbf{q}^T(i+1)\right\}\right] \tag{42}$$

and using (19), then (41) can be written as

$$E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \le \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i)\mathbf{P}_h^{\bullet}\mathbf{q}^{\bullet}(i) < 0, \tag{43}$$

where, for all $h$,

$$\mathbf{P}_h^{\bullet} = \begin{bmatrix} -\mathbf{P}+\mathbf{C}^T\mathbf{C}+2\mathbf{V}_h^T\mathbf{Q}\mathbf{V}_h & \mathbf{0} & \mathbf{F}_h^T\mathbf{Q} \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & \mathbf{W}_h^T\mathbf{Q} \\ \mathbf{Q}\mathbf{F}_h & \mathbf{Q}\mathbf{W}_h & \mathbf{P}-2\mathbf{Q} \end{bmatrix} < 0. \tag{44}$$

Using Schur complement property to construct the LMI form, (44) can be rewritten as (37). This concludes the proof. □

*Remark 8.* The inequality (37) is an enhanced representation of bounded real lemma for a given class of TS stochastic systems. The inequality is linear with respect to the system variables and does not involve any product of the Lyapunov matrix $\mathbf{P}$ and the system matrices $\mathbf{F}_h, \mathbf{W}_h, \mathbf{V}_h$. This offers its main possibility to be applied in systems with polytopic uncertainties.

*Remark 9.* To incorporate noise variance into design condition, the choice of (39) is not unique. Another, but more conservative, solution can be obtained if (39) is chosen, for example, as

$$\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\left\{\mathbf{q}^T(i+1)+\mathbf{q}^T(i+1)\mathbf{V}_h^T o(i)\right\} \times \mathbf{Q}\sum_{k=1}^{s}\mu_k(\boldsymbol{\theta}(i))\left\{\mathbf{F}_k\mathbf{q}(i)+\mathbf{V}_k o(i)\mathbf{q}(i) + \mathbf{W}_k\mathbf{d}(i)-\mathbf{q}(i+1)\right\} = \mathbf{0}. \tag{45}$$

## 4. State Feedback Control

This section addresses the problem of finding the state feedback control law (6), working on the parallel distributed compensation concept that stabilizes the system (1), (2) and achieves some level of attenuation. Moreover, since the LMI solvers mostly limit the maximal number of LMIs entering a solution, the number of LMIs in the following design conditions is minimized.

**Theorem 10.** *The state feedback control* (6) *to system* (1), (2) *exists if there exist a symmetric positive definite matrix* $\mathbf{X} \in \mathbb{R}^{n\times n}$, *matrices* $\mathbf{Y}_h \in \mathbb{R}^{r\times n}$, *and a positive scalar* $\gamma \in \mathbb{R}$ *such that*

$$\mathbf{X} = \mathbf{X}^T > 0, \quad \gamma > 0, \tag{46}$$

$$\begin{bmatrix} -\mathbf{X} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{F}_h\mathbf{X}-\mathbf{G}_h\mathbf{Y}_h & \mathbf{W}_h & -\mathbf{X} & * & * \\ \mathbf{V}_h\mathbf{X} & \mathbf{0} & \mathbf{0} & -\mathbf{X} & * \\ \mathbf{C}\mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0, \tag{47}$$

$$\begin{bmatrix} -\mathbf{X} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \dfrac{\mathbf{F}_h\mathbf{X}+\mathbf{F}_l\mathbf{X}-\mathbf{G}_h\mathbf{Y}_l-\mathbf{G}_l\mathbf{Y}_h}{2} & \dfrac{\mathbf{W}_h+\mathbf{W}_l}{2} & -\mathbf{X} & * & * \\ \dfrac{\mathbf{V}_h\mathbf{X}+\mathbf{V}_l\mathbf{X}}{2} & \mathbf{0} & \mathbf{0} & -\mathbf{X} & * \\ \mathbf{C}\mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{48}$$

*for all* $h \in \langle 1,2,\ldots,s\rangle$, $h < l \le s$, $h,l \in \langle 1,2,\ldots,s\rangle$, *and* $\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i)) \ne 0$, *respectively.*

*When the above conditions hold, the control law gain matrices are given as*

$$\mathbf{K}_h = \mathbf{Y}_h\mathbf{X}^{-1}. \tag{49}$$

*Proof.* Considering the control law (6), then (27) implies

$$E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i)) \times \left\{\mathbf{q}^T(i)\left(-\mathbf{P}+\mathbf{H}_{hl}^T\mathbf{P}\mathbf{H}_{hl}+\gamma^{-1}\mathbf{C}^T\mathbf{C}\right)\mathbf{q}(i) + \mathbf{q}^T(i)\mathbf{H}_{lh}^T\mathbf{P}\mathbf{W}_h\mathbf{d}(i)+\mathbf{d}^T(i)\mathbf{W}_h^T\mathbf{P}\mathbf{H}_{hl}\mathbf{q}(i) + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h+\mathbf{d}^T(i)\mathbf{W}_h^T\mathbf{P}\mathbf{W}_h\mathbf{d}(i) - \gamma\mathbf{d}^T(i)\mathbf{d}(i)\right\}, \tag{50}$$

where

$$\mathbf{H}_{hl} = \mathbf{F}_h - \mathbf{G}_h\mathbf{K}_l. \tag{51}$$

Permuting the subscripts $h$ and $l$ in (50) gives

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad \le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_l(\boldsymbol{\theta}(i)) \\ &\qquad \times \Big\{\mathbf{q}^T(i)\left(-\mathbf{P} + \mathbf{H}_{lh}^T\mathbf{P}\mathbf{H}_{lh} + \gamma^{-1}\mathbf{C}^T\mathbf{C}\right)\mathbf{q}(i) \\ &\qquad + \mathbf{q}^T(i)\,\mathbf{H}_{lh}^T\mathbf{P}\mathbf{W}_l\mathbf{d}(i) \\ &\qquad + \mathbf{d}^T(i)\,\mathbf{W}_l^T\mathbf{P}\mathbf{H}_{lh}\mathbf{q}(i) + \mathbf{V}_l^T\mathbf{P}\mathbf{V}_l \\ &\qquad + \mathbf{d}^T(i)\,\mathbf{W}_l^T\mathbf{P}\mathbf{W}_l\mathbf{d}(i) - \gamma\mathbf{d}^T(i)\,\mathbf{d}(i)\Big\} \end{aligned} \tag{52}$$

and with (28) it yields

$$\begin{aligned} &2\left(E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\right) \\ &\quad \le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_l(\boldsymbol{\theta}(i))\,\mathbf{q}^{\circ T}(i)\left(\mathbf{P}_{hl}^{\circ} + \mathbf{P}_{lh}^{\circ}\right)\mathbf{q}^{\circ}(i) \\ &\quad < 0, \end{aligned} \tag{53}$$

where, in analogy with (30),

$$\begin{aligned} &\mathbf{P}_{hl}^{\circ} \\ &= \begin{bmatrix} -\mathbf{P} + \mathbf{H}_{hl}^T\mathbf{P}\mathbf{H}_{hl} + \mathbf{V}_h^T\mathbf{P}\mathbf{V}_h + \gamma^{-1}\mathbf{C}^T\mathbf{C} & \mathbf{H}_{hl}^T\mathbf{P}\mathbf{W}_h \\ \mathbf{W}_h^T\mathbf{P}\mathbf{H}_{hl} & -\gamma\mathbf{I}_{r_w} + \mathbf{W}_h^T\mathbf{P}\mathbf{W}_h \end{bmatrix} \\ &< 0. \end{aligned} \tag{54}$$

Rearranging the computation, (30) can be written as

$$\begin{aligned} &2\left(E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\right) \\ &\quad \le \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_h(\boldsymbol{\theta}(i))\,\mathbf{q}^{\circ T}(i)\left(\mathbf{P}_{hh}^{\circ} + \mathbf{P}_{hh}^{\circ}\right)\mathbf{q}^{\circ}(i) \\ &\qquad + 2\sum_{h=1}^{s-1}\sum_{l=h+1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_l(\boldsymbol{\theta}(i))\,\mathbf{q}^{\circ T}(i)\left(\mathbf{P}_{hl}^{\circ} + \mathbf{P}_{lh}^{\circ}\right)\mathbf{q}^{\circ}(i) \\ &\quad < 0, \end{aligned} \tag{55}$$

$$\begin{aligned} &\left(E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\right) \\ &\quad \le \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_h(\boldsymbol{\theta}(i))\,\mathbf{q}^{\circ T}(i)\,\mathbf{P}_{hh}^{\circ}\mathbf{q}^{\circ}(i) \\ &\qquad + 2\sum_{h=1}^{s-1}\sum_{l=h+1}^{s}\mu_h(\boldsymbol{\theta}(i))\,\mu_l(\boldsymbol{\theta}(i))\,\mathbf{q}^{\circ T}(i) \\ &\qquad\qquad \times\left(\frac{1}{2}\mathbf{P}_{hl}^{\circ} + \frac{1}{2}\mathbf{P}_{lh}^{\circ}\right)\mathbf{q}^{\circ}(i) \\ &\quad < 0, \end{aligned} \tag{56}$$

respectively.

Replacing $\mathbf{F}$ in (23) by the closed-loop system matrix (51) then, with respect to (55) and (56), it is

$$\mathbf{P}_{hh}^{\circ} = \begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{P}\mathbf{H}_{hh} & \mathbf{P}\mathbf{W}_h & -\mathbf{P} & * & * \\ \mathbf{P}\mathbf{V}_h & \mathbf{0} & \mathbf{0} & -\mathbf{P} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0, \tag{57}$$

$$\begin{aligned} &\frac{\mathbf{P}_{hl}^{\circ} + \mathbf{P}_{lh}^{\circ}}{2} = \begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{P}\dfrac{\mathbf{H}_{hl} + \mathbf{H}_{lh}}{2} & \mathbf{P}\dfrac{\mathbf{W}_h + \mathbf{W}_l}{2} & -\mathbf{P} & * & * \\ \mathbf{P}\dfrac{\mathbf{V}_h + \mathbf{V}_l}{2} & \mathbf{0} & \mathbf{0} & -\mathbf{P} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} \\ &\quad < 0, \end{aligned} \tag{58}$$

respectively. Defining the transform matrix

$$\mathbf{T}_A = \operatorname{diag}\left[\mathbf{P}^{-1} \quad \mathbf{I}_{r_w} \quad \mathbf{P}^{-1} \quad \mathbf{P}^{-1} \quad \mathbf{I}_m\right], \tag{59}$$

then premultiplying and postmultiplying of the left-hand side and the right-hand side of (58) and (59) by (51) give

$$\begin{bmatrix} -\mathbf{P}^{-1} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ (\mathbf{F}_h - \mathbf{G}_h\mathbf{K}_h)\,\mathbf{P}^{-1} & \mathbf{W}_h & -\mathbf{P}^{-1} & * & * \\ \mathbf{V}_h\mathbf{P}^{-1} & \mathbf{0} & \mathbf{0} & -\mathbf{P}^{-1} & * \\ \mathbf{C}\mathbf{P}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{60}$$

$$\begin{bmatrix} -\mathbf{P}^{-1} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \boldsymbol{\Psi}(3,1) & \dfrac{\mathbf{W}_h + \mathbf{W}_l}{2} & -\mathbf{P}^{-1} & * & * \\ \dfrac{\mathbf{V}_h\mathbf{P}^{-1} + \mathbf{V}_l\mathbf{P}^{-1}}{2} & \mathbf{0} & \mathbf{0} & -\mathbf{P}^{-1} & * \\ \mathbf{C}\mathbf{P}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{61}$$

$$\boldsymbol{\Psi}(3,1) = \frac{(\mathbf{F}_h - \mathbf{G}_h\mathbf{K}_l)\,\mathbf{P}^{-1}}{2} + \frac{(\mathbf{F}_l - \mathbf{G}_l\mathbf{K}_h)\,\mathbf{P}^{-1}}{2}. \tag{62}$$

Thus, with the notations

$$\mathbf{X} = \mathbf{P}^{-1}, \qquad \mathbf{Y}_h = \mathbf{K}_h\mathbf{P}^{-1}, \qquad \mathbf{Y}_l = \mathbf{K}_l\mathbf{P}^{-1}. \tag{63}$$

Equations (60) and (61) imply (47) and (48), respectively. This concludes the proof. □

Applying this result directly to system (1), (2) would lead, in the case of a large number of rules, to a possibly conservative analysis, since a common symmetric positive definite matrix $\mathbf{P}$ is used to verify all Lyapunov inequalities. Considering the enhanced bounded real lemma form (36), (37), main aim of next theorem is to extend the affine TS model properties using slack matrix variables to decouple Lyapunov matrix and the system matrices in LMIs.

**Theorem 11.** *The state feedback control* (6) *to system* (1) *and* (2) *exists if there exist symmetric positive definite matrices* $\mathbf{X}, \mathbf{T} \in \mathbb{R}^{n\times n}$, *matrices* $\mathbf{Y}_h \in \mathbb{R}^{r\times n}$, *and a positive scalar* $\gamma \in \mathbb{R}$ *such that for all* $h$

$$\mathbf{X} = \mathbf{X}^T > 0, \quad \mathbf{T} = \mathbf{T}^T > 0, \quad \gamma > 0, \tag{64}$$

$$\begin{bmatrix} -\mathbf{T} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{F}_h\mathbf{X} - \mathbf{G}_h\mathbf{Y}_h & \mathbf{W}_h & \mathbf{T} - 2\mathbf{X} & * & * \\ \mathbf{V}_h\mathbf{X} & \mathbf{0} & \mathbf{0} & -\mathbf{X} & * \\ \mathbf{C}\mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{65}$$

$$\begin{bmatrix} -\mathbf{T} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{\Phi}(3,1) & \dfrac{\mathbf{W}_h + \mathbf{W}_l}{2} & \mathbf{T} - 2\mathbf{X} & * & * \\ \dfrac{\mathbf{V}_h + \mathbf{V}_l}{2}\mathbf{X} & \mathbf{0} & \mathbf{0} & -0.5\mathbf{X} & * \\ \mathbf{C}\mathbf{X} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{66}$$

$$\mathbf{\Phi}(3,1) = \frac{\mathbf{F}_h\mathbf{X} + \mathbf{F}_l\mathbf{X} - \mathbf{G}_h\mathbf{Y}_l - \mathbf{G}_l\mathbf{Y}_h}{2}. \tag{67}$$

*When the above conditions hold, the set of control law gain matrices is given as* (49).

*Proof.* With respect to the control law form (6), then (41), (43), and (44) give

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad\le E_o\left\{\mathbf{q}^T(i+1)\right\}\mathbf{P}E_o\{\mathbf{q}(i+1)\} \\ &\qquad + \gamma^{-1}\mathbf{q}^T(i)\mathbf{C}^T\mathbf{C}\mathbf{q}(i) - \gamma\mathbf{d}^T(i)\mathbf{d}(i) - \mathbf{q}^T(i)\mathbf{P}\mathbf{q}(i) \\ &\qquad + E_o\left\{\mathbf{q}^T(i+1)\right\}\mathbf{Q}\sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i)) \\ &\qquad\qquad \times\left(\mathbf{H}_{hl}\mathbf{q}(i) + \mathbf{W}_h\mathbf{d}(i) - E_o\{\mathbf{q}(i+1)\}\right) \\ &\qquad + \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i)) \\ &\qquad\qquad \times\left(\mathbf{H}_{hl}\mathbf{q}(i) + \mathbf{W}_h\mathbf{d}(i) - E_o\{\mathbf{q}(i+1)\}\right)^T \\ &\qquad\qquad \times\mathbf{Q}E_o\{\mathbf{q}(i+1)\} \\ &\qquad + 2\mathbf{q}^T(i)\sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mathbf{V}_h^T\mathbf{Q}\mathbf{V}_h\mathbf{q}(i) < 0, \end{aligned} \tag{68}$$

where $\mathbf{H}_{hl}$ is given in (51). Thus, considering (42) and (43), it is

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad\le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i)\mathbf{P}_{hl}^{\bullet}\mathbf{q}^{\bullet}(i) < 0, \end{aligned} \tag{69}$$

where

$$\mathbf{P}_{hl}^{\bullet} = \begin{bmatrix} -\mathbf{P} + \mathbf{C}^T\mathbf{C} + \mathbf{V}_h^T\mathbf{Q}\mathbf{V}_h & \mathbf{0} & \mathbf{H}_{hl}^T\mathbf{Q} \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & \mathbf{W}_h^T\mathbf{Q} \\ \mathbf{Q}\mathbf{H}_{hl} & \mathbf{Q}\mathbf{W}_h & \mathbf{P} - 2\mathbf{Q} \end{bmatrix} < 0. \tag{70}$$

Permuting the subscripts $h$ and $l$ in (69) leads to

$$\begin{aligned} &E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i)) \\ &\quad\le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i)\mathbf{P}_{lh}^{\bullet}\mathbf{q}^{\bullet}(i) < 0 \end{aligned} \tag{71}$$

and so

$$\begin{aligned} &2\left(E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\right) \\ &\quad\le \sum_{h=1}^{s}\sum_{l=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i)\left(\mathbf{P}_{hl}^{\bullet} + \mathbf{P}_{lh}^{\bullet}\right)\mathbf{q}^{\bullet}(i) < 0, \end{aligned} \tag{72}$$

$$\begin{aligned} &\left(E_o\{v(\mathbf{q}(i+1))\} - v(\mathbf{q}(i))\right) \\ &\quad\le \sum_{h=1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_h(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i)\mathbf{P}_{hh}^{\bullet}\mathbf{q}^{\bullet}(i) \\ &\qquad + 2\sum_{h=1}^{s-1}\sum_{l=h+1}^{s}\mu_h(\boldsymbol{\theta}(i))\mu_l(\boldsymbol{\theta}(i))\mathbf{q}^{\bullet T}(i) \\ &\qquad\qquad \times\left(\frac{1}{2}\mathbf{P}_{hl}^{\bullet} + \frac{1}{2}\mathbf{P}_{lh}^{\bullet}\right)\mathbf{q}^{\bullet}(i) < 0, \end{aligned} \tag{73}$$

respectively. Subsequently, exploiting the Schur complement property, it can be obtained that

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{Q}\mathbf{H}_{hh} & \mathbf{Q}\mathbf{W}_h & \mathbf{P} - 2\mathbf{Q} & * & * \\ \mathbf{Q}\mathbf{V}_h & \mathbf{0} & \mathbf{0} & -0.5\mathbf{Q} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0, \tag{74}$$

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{Q}\dfrac{\mathbf{H}_{hl} + \mathbf{H}_{lh}}{2} & \mathbf{Q}\dfrac{\mathbf{W}_h + \mathbf{W}_l}{2} & \mathbf{P} - 2\mathbf{Q} & * & * \\ \mathbf{Q}\dfrac{\mathbf{V}_h + \mathbf{V}_{hl}}{2} & \mathbf{0} & \mathbf{0} & -0.5\mathbf{Q} & * \\ \mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_m \end{bmatrix} < 0. \tag{75}$$

Since it was supposed that $\mathbf{Q}$ is positive definite, performing a congruence transformation by

$$\mathbf{T}_B = \operatorname{diag}\left[\mathbf{Q}^{-1} \quad \mathbf{I}_{r_w} \quad \mathbf{Q}^{-1} \quad \mathbf{Q}^{-1} \quad \mathbf{I}_m\right], \tag{76}$$

that is, premultiplying and postmultiplying the left-hand side and the right-hand side of (74) and (75) by (76), then

$$\begin{bmatrix} -\mathbf{Q}^{-1}\mathbf{P}\,\mathbf{Q}^{-1} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \mathbf{H}_h\mathbf{Q}^{-1} & \mathbf{W}_h & \mathbf{\Pi}(3,3) & * & * \\ \mathbf{V}_h\mathbf{Q}^{-1} & \mathbf{0} & \mathbf{0} & -0.5\mathbf{Q}^{-1} & * \\ \mathbf{C}\mathbf{Q}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{77}$$

$$\begin{bmatrix} -\mathbf{Q}^{-1}\mathbf{P}\mathbf{Q}^{-1} & * & * & * & * \\ \mathbf{0} & -\gamma\mathbf{I}_{r_w} & * & * & * \\ \dfrac{\mathbf{H}_{hl}+\mathbf{H}_{lh}}{2}\mathbf{Q}^{-1} & \dfrac{\mathbf{W}_h+\mathbf{W}_l}{2} & \mathbf{\Pi}(3,3) & * & * \\ \dfrac{\mathbf{V}_h+\mathbf{V}_l}{2}\mathbf{Q}^{-1} & \mathbf{0} & \mathbf{0} & -0.5\mathbf{Q}^{-1} & * \\ \mathbf{C}\mathbf{Q}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} < 0, \tag{78}$$

$$\mathbf{\Pi}(3,3) = \mathbf{Q}^{-1}\mathbf{P}\mathbf{Q}^{-1} - 2\mathbf{Q}^{-1}. \tag{79}$$

Thus, with the notations

$$\begin{aligned} \mathbf{T} &= \mathbf{Q}^{-1}\mathbf{P}\mathbf{Q}^{-1}, \qquad \mathbf{X} = \mathbf{Q}^{-1}, \\ \mathbf{Y}_h &= \mathbf{K}_h\mathbf{Q}^{-1}, \qquad \mathbf{Y}_l = \mathbf{K}_l\mathbf{Q}^{-1}. \end{aligned} \tag{80}$$

Equations (77) and (78) imply (65) and (66), respectively. This concludes the proof. □

Note, the LMIs in Theorem 11 are linear and the equivalent Lyapunov matrix $\mathbf{T}$, used to verify all inequalities, is a symmetric positive definite matrix.

## 5. Illustrative Example

To demonstrate the control properties, the trailer-like mobile robot kinematics is considered [33], described by the following set of differential equations:

$$\begin{aligned} \frac{d\left(q_1(t)+q_2(t)\right)}{dt} &= v\frac{1}{l}\tan\left(u(t)\right), \\ \frac{dq_2(t)}{dw(t)} &= \frac{1}{L}\tan\left(q_1(t)\right), \\ \frac{dq_3(t)}{dw(t)} = \sin\left(q_2(t)\right), &\qquad \frac{dq_4(t)}{dw(t)} = \cos\left(q_2(t)\right), \\ \frac{dw(t)}{dt} &= \frac{1}{v\cos\left(q_1(t)\right)}, \end{aligned} \tag{81}$$

where $u(t)$ is the steering angle (rad), $q_1(t)$ is the angle difference between truck and trailer (rad), $q_2(t)$ is the angle of trailer (rad), $q_3(t)$ is $x$ position of rear end of trailer (m), $q_4(t)$ is $y$ position of rear end of trailer (m), $v$ is the constant speed of the backward movement (m/sek), $l$ is the length of truck (m), and $L$ is the length of trailer (m).

The control purpose is to realize the backward movement of the trailer-like mobile robot along the $q_3(t) = 0$, that is, to regulate $q_1(t) \sim q_3(t)$ by manipulating the steering angle $u(t)$, where, without forward movements,

$$\lim_{t\to\infty} q_1(t) = 0, \qquad \lim_{t\to\infty} q_2(t) = 0, \qquad \lim_{t\to\infty} q_3(t) = 0. \tag{82}$$

Consequently, the variable $q_4(t)$ is not necessary to take into account.

If $q_1(t)$ and $u(t)$ are always small values, the above model can be simplified as

$$\begin{aligned} \frac{dq_1(t)}{dt} &= -\frac{v}{L}q_1(t) + \frac{v}{l}\tan\left(u(t)\right) \doteq -\frac{v}{L}q_1(t) + \frac{v}{l}u(t), \\ \frac{dq_2(t)}{dt} &= \frac{v}{L}\sin\left(q_1(t)\right) \doteq \frac{v}{L}q_1(t), \\ \frac{dq_3(t)}{dt} &= v\cos\left(q_1(t)\right)\sin\left(q_2(t)\right) \doteq v\sin\left(q_2(t)\right). \end{aligned} \tag{83}$$

Applying Euler discretization, the next difference equation forms are obtained (compare, e.g., [34]):

$$\begin{aligned} q_1(i+1) &\doteq \left(1 - \frac{v\Delta t}{L}\right)q_1(i) + \frac{v\Delta t}{l}u(i), \\ q_2(i+1) &\doteq q_2(i) + \frac{v\Delta t}{L}q_1(i), \\ q_3(i+1) &\doteq q_3(i) + v\Delta t\sin\left(\frac{q_2(i+1)+q_2(i)}{2}\right) \\ &= q_3(i) + v\Delta t\sin\left(q_2(i) + \frac{v\Delta t}{2L}q_1(i)\right), \end{aligned} \tag{84}$$

where $\Delta t$ (s) is the sampling period.

Defining the premise variable and the sector values as

$$\theta(i) = q_2(i) + \frac{v\Delta t}{2L}q_1(i), \qquad d_1 = p\theta(i), \qquad d_2 = \theta(i), \tag{85}$$

then (12) implies

$$\begin{aligned} \mu_1(\theta(i)) &= \frac{\theta(i) - \sin(\theta(i))}{(1-p)\theta(i)}, \\ \mu_2(\theta(i)) &= \frac{\sin(\theta(i)) - p\theta(i)}{(1-p)\theta(i)}, \end{aligned} \tag{86}$$

and the solitary nonlinear difference equation can be approximated as

$$\begin{aligned} q_3(i+1) = q_3(i) &+ \mu_2(\theta(i))\,v\Delta t\left(q_2(i) + \frac{v\Delta t}{2L}q_1(i)\right) \\ &+ \mu_1(\theta(i))\,vp\Delta t\left(q_2(i) + \frac{v\Delta t}{2L}q_1(i)\right). \end{aligned} \tag{87}$$

Considering the external disturbance $d(i)$ and the multiplicative noise $o(i)$ only in the indirect measurement of $x_1(i)$, it is given that [35]

$$q_1(i+1) \doteq \left(1 - \frac{v\Delta t}{L}\right) q_1(i) + \frac{v\Delta t}{l} u(i) + ro(i) q_1(i) + wd(i), \tag{88}$$

where $r$ is the constant gain of multiplicative noise and $w$ is the constant gain of disturbance. Subsequently, the output equation is defined as

$$y(i) = Lq_2(i) + q_3(i). \tag{89}$$

Now, introducing the state vector

$$\mathbf{q}^T(i) = [q_1(i) \quad q_2(i) \quad q_3(i)], \tag{90}$$

the discrete-time TS fuzzy model takes the form

$$\begin{aligned} &\mathbf{q}(i+1) \\ &\quad = \sum_{h=1}^{2} \mu_h(\boldsymbol{\theta}(i)) \left((\mathbf{F}_h + \mathbf{V}o(i))\mathbf{q}(i) + \mathbf{G}u(i) + \mathbf{W}d(i)\right), \\ &\quad y(i) = \mathbf{C}\mathbf{q}(i), \end{aligned} \tag{91}$$

where

$$\mathbf{F}_1 = \begin{bmatrix} 1 - \frac{v\Delta t}{L} & 0 & 0 \\ \frac{v\Delta t}{L} & 1 & 0 \\ \frac{p(v\Delta t)^2}{2L} & pv\Delta t & 1 \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} 1 - \frac{v\Delta t}{L} & 0 & 0 \\ \frac{v\Delta t}{L} & 1 & 0 \\ \frac{(v\Delta t)^2}{2L} & v\Delta t & 1 \end{bmatrix},$$
$$\mathbf{V} = \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \frac{v\Delta t}{l} \\ 0 \\ 0 \end{bmatrix},$$
$$\mathbf{W} = \begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [0 \quad L \quad 1]. \tag{92}$$

Note, the main reason of using the above approximation for $q_3(i+1)$ is to obtain a controllable pair $(\mathbf{F}_1, \mathbf{G})$. Be aware that for very small value of $p$, the stability of the system may be close to the limit.

To demonstrate the control properties, the next system parameters were considered:

$$l = 0.087, \quad L = 0.130, \quad v = -0.1, \quad \Delta t = 0.5,$$
$$r = 0.09, \quad w = 0.8, \quad p = \frac{10^{-2}}{\pi}. \tag{93}$$

Solving (64)–(66) with respect to the LMI variables $\mathbf{X}$, $\mathbf{T}$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\gamma$ using Self-Dual-Minimization (SeDuMi) package of MATLAB [36], the feedback gain matrix design problem was feasible with the results

$$\mathbf{X} = \begin{bmatrix} 1.2922 & 0.3811 & 0.0000 \\ 0.3811 & 0.2454 & 0.0051 \\ 0.0000 & 0.0051 & 0.0006 \end{bmatrix},$$
$$\gamma = 2.0757,$$
$$\mathbf{T} = \begin{bmatrix} 0.7413 & 0.2150 & 0.0000 \\ 0.2150 & 0.2280 & 0.0051 \\ 0.0000 & 0.0051 & 0.0006 \end{bmatrix},$$
$$\mathbf{Y}_1 = \mathbf{Y}_2 = [-3.4175 \quad -0.4977 \quad 0.0001], \tag{94}$$

giving the feedback gain matrices

$$\mathbf{K}_1 = \mathbf{K}_2 = [-4.3084 \quad 5.6390 \quad -47.2534]. \tag{95}$$

In consequence, the linear state feedback control law was obtained for such system.

The obtained control law insures the stable control in the mean square sense. The closed-loop subsystem matrix eigenvalue spectra are as follows:

$$\begin{aligned} \rho(\mathbf{F}_1 - \mathbf{G}\mathbf{K}_1) &= \{0.9987 \quad -0.0451 \pm 0.3903i\}, \\ \rho(\mathbf{F}_2 - \mathbf{G}\mathbf{K}_2) &= \{-0.1960 \quad 0.5523 \pm 0.4860i\}, \end{aligned} \tag{96}$$

which implies also the whole closed-loop system stability.

To compare, using the same system parameters, (46)–(48) give

$$\mathbf{X} = \begin{bmatrix} 1.3880 & 0.4283 & 0.0000 \\ 0.4283 & 0.2350 & 0.0050 \\ 0.0000 & 0.0050 & 0.0006 \end{bmatrix}, \quad \gamma = 2.3289,$$
$$\mathbf{Y}_1 = \mathbf{Y}_2 = [-3.4941 \quad -0.7661 \quad 0.0001] \tag{97}$$

and, by Theorem 10, the design parameters were computed as follows:

$$\begin{aligned} \mathbf{K}_1 = \mathbf{K}_2 &= [-4.0909 \quad 5.0975 \quad -42.2792], \\ \rho(\mathbf{F}_1 - \mathbf{G}\mathbf{K}_1) &= \{0.9987 \quad 0.0174 \pm 0.3993i\}, \\ \rho(\mathbf{F}_2 - \mathbf{G}\mathbf{K}_2) &= \{-0.1275 \quad 0.5805 \pm 0.4883i\}, \end{aligned} \tag{98}$$

which give less relative damping of closed-loop system solutions than there were obtained in the first case.

Since the conditions of Theorems 10 and 11 are all satisfied and the design tasks were feasible, it is possible to conclude that the TS fuzzy control of given stochastic system with state-multiplicative noise is asymptotically stable in the mean with the quadratic performance by applying the linear control law.

The simulation presents the closed-loop system properties, where the multiplicative noise parameters were chosen as in (5), the disturbance was modeled as noise with covariance $\sigma_w^2 = 0.2$, and the initial system state vector was (in degrees)

$$\mathbf{q}_T(0) = [0.5 \quad 0.5 \quad 0.15]. \tag{99}$$

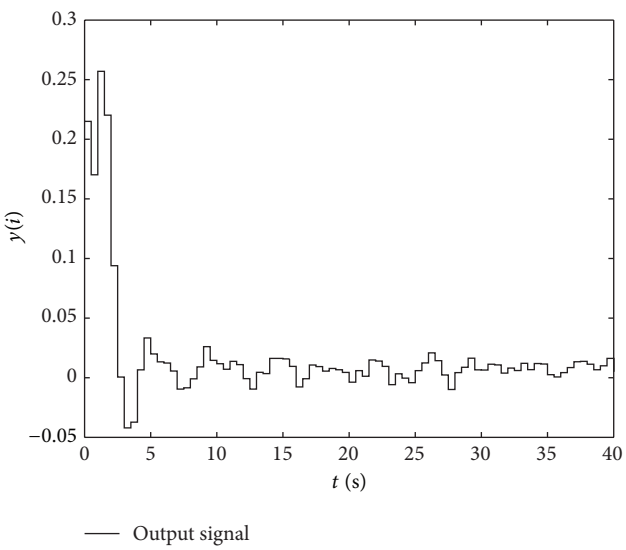


Figure 1: The response of $y(i)$ for the system with TS fuzzy controller designed according to the conditions of Theorem 11.

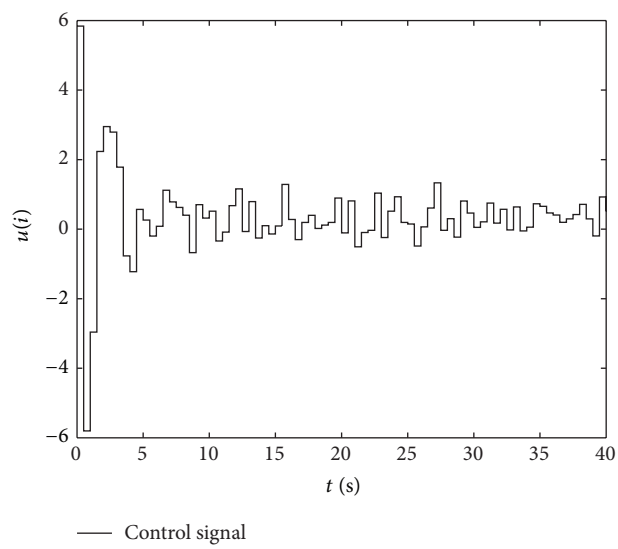


Figure 2: Input signal $u(i)$ for the system with TS fuzzy controller designed according to the conditions of Theorem 11.

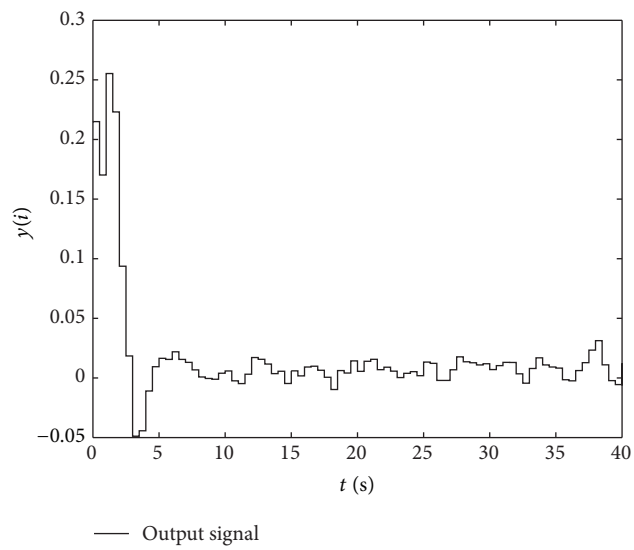


Figure 3: The response of $y(i)$ for the system with TS fuzzy controller designed according to the conditions of Theorem 10.

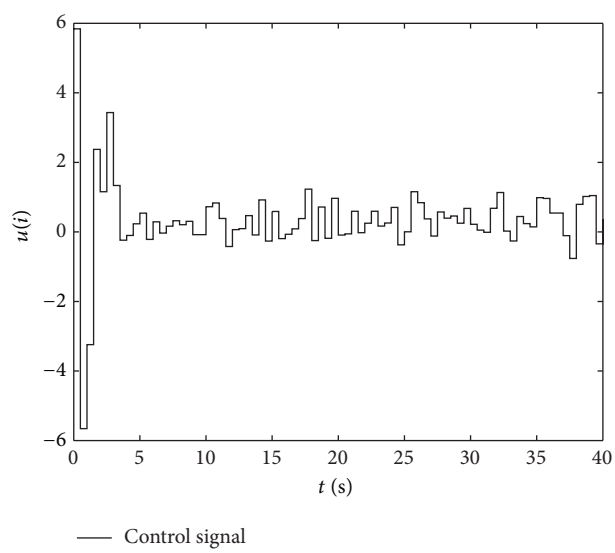


Figure 4: Input signal $u(i)$ for the system with TS fuzzy controller designed according to the conditions of Theorem 10.

Figure 1 shows the output response $y(i)$ of the controlled nonlinear system with the TS fuzzy controller (6) designed according to the conditions of Theorem 11, where the associated input signal $u(i)$ is presented in Figure 2. For comparison, Figures 3 and 4 show the responses of $y(i)$ and the control input $u(i)$ for the closed-loop system with the fuzzy controller designed according to the conditions of Theorem 10, respectively.

## 6. Concluding Remarks

The paper presents the control design principle for TS fuzzy discrete-time stochastic multivariable dynamic systems with state-multiplicative noise. The stability of the control scheme is established in the mean square sense and with the quadratic performance using an enhanced representation of bounded real lemma for such TS fuzzy stochastic systems. As a result, the Lyapunov matrix and the system parameter matrices are decoupled in the resulting LMIs. This provides a suitable way for determination of state control by solving these naturally affine LMI problems. Compared with the previous results, the number of assumptions is reduced since the solutions obtained according to the conditions of both presented theorems tend to receive a linear state controller.

Presented applications can be considered as a task concerning the class of $H_\infty$ stabilization control problems where the design conditions were newly formulated. This formulation poses the problem as a stabilization problem with

a fuzzy state feedback controller in the parallel distributed form whose gain matrices take no special structures and allows finding a solution to the control law without restrictive assumptions and additional specifications on the design parameters. The procedures used here for discrete-time TS fuzzy stochastic systems with state-multiplicative noise can be similarly extended for the continuous-time case.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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