Research Article

Stability of Delayed Hopfield Neural Networks with Variable-Time Impulses

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In this paper the globally exponential stability criteria of delayed Hopfield neural networks with variable-time impulses are established. The proposed criteria can also be applied in Hopfield neural networks with fixed-time impulses. A numerical example is presented to illustrate the effectiveness of our theoretical results.

1. Introduction

Hopfield neural networks [1], which were referred by Hopfield in 1984, have attracted many attentions of researchers and have been applied in many fields such as pattern recognition, associative memory, and combinatorial optimization. Stability, a crucial dynamic feature of Hopfield neural networks, has been intensively investigated over the past decades. Some significant sufficient results can be referred in [2–6].

It is well known that time delay is unavoidable due to finite switching speeds of the amplifiers and it may cause oscillations or instability of dynamic systems. The effects of time delay on the dynamical behavior of neural networks are nonnegligible. Some stability criteria for delayed Hopfield neural networks have been proposed in [7–10]. Meanwhile, impulsive phenomena exist in a wide variety of evolutionary processes, such as financial systems and nanoscale electronic circuits in which many state variables change instantaneously, in the form of impulses. On the other hand, impulsive control is also applied widely in many fields of information science, electronics, automated control systems, computer networking, artificial intelligence, robotics and telecommunications, and so forth. Neural networks may jump instantaneously because of environmental changes (such as external noise and disturbance). We may also introduce impulses deliberately to stabilize the oscillating and chaotic neural networks. Many

researchers have investigated impulsive Hopfield neural networks and have obtained many interesting stability results [11–19].

However, up to now, the vast majority of stability results for impulsive Hopfield neural networks are focused on the case of fixed-time impulses. As we know, variable-time impulses arise naturally in biological and physiological systems. The primary difference between neural network with fixed-time impulses and neural network with variable-time impulses is the impulsive instant. In the neural network with fixed-time impulses, the impulsive instant is completely fixed and not about the state of system. But in neural network with variable-time impulses, the impulsive instant is not fixed and determined by state of system. In [20], we have focused on BAM neural networks with variable-time impulses and have obtained some crucial theoretical results. In [21], we have investigated the stabilizing effects of impulses for Hopfield neural networks and have shown that Hopfield neural networks with unstable continuous component may be still stable because of the stabilizing effects of impulses. In this paper, we focus on the destabilizing effects of Hopfield neural networks with variable-time impulses. It is shown that the impulsive Hopfield neural networks may preserve the global exponential stability of the impulse-free Hopfield neural networks even if the impulses have enlarging effects on the states of neurons. For this purpose, it is always assumed that the states of neurons enlarge at impulsive time.

This paper is organized as follows. In the coming section we introduce some notations, definition, and lemmas. In Section 3 we consider the stability of Hopfield neural networks with time delays and variable-time impulses and establish stability criteria. In Section 4, one example is given to illustrate the effectiveness of our theoretical results.

2. Preliminaries

In this paper, we consider the following Hopfield neural networks with variable-time impulses:

$$\begin{aligned}\dot{\tilde{x}}(t) &= -A\tilde{x}(t) + B\tilde{f}(\tilde{x}(t - \tau)) + \tilde{I}, \quad t \neq \tilde{t}_k(x), \\ \Delta\tilde{x}(t) &= \tilde{J}_k(\tilde{x}(t^-)), \quad t = \tilde{t}_k(x), \\ \tilde{x}(t_0 + s) &= \tilde{\phi}(s), \quad s \in [-\tau, 0],\end{aligned}\quad (1)$$

where $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ is the neuron state vector, $\tau > 0$ is the transmission delay, $A = \text{diag}(a_1, \dots, a_n)$ with $a_i > 0$, $i = 1, \dots, n$, $B = [b_{ij}]_{n \times n}$ is the delayed connection weight matrix, and $\tilde{f}(\cdot) = (\tilde{f}_1(\cdot), \dots, \tilde{f}_n(\cdot))^T$ is neuron activation function. $\tilde{I}$ is the external input, $0 \leq t_0 < \tilde{t}_1(\tilde{x}) < \dots < \tilde{t}_k(\tilde{x}) < \dots, \tilde{t}_k(\tilde{x}) \rightarrow \infty$ as $k \rightarrow \infty$, $\tilde{x}(t^+) = \lim_{s \rightarrow t^+} \tilde{x}(s)$, and $\tilde{x}(t^-) = \lim_{s \rightarrow t^-} \tilde{x}(s)$. In this paper we always assume that $\tilde{x}(t) = \tilde{x}(t^+)$ as $t = \tilde{t}_k(\tilde{x})$. We further assume that

$$(A1) \text{ there is } l_i > 0, i = 1, \dots, n, \text{ such that } |\tilde{f}_i(\alpha) - \tilde{f}_i(\beta)| \leq l_i |\alpha - \beta|, \text{ for any } \alpha, \beta \in R.$$

Throughout this paper, it is always assumed that there is at least one equilibrium point of (1). As usual, we shift an equilibrium point $x^* = (x_1^*, \dots, x_n^*)^T$ to the origin by transformation $x = \tilde{x} - x^*$. Then system (1) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + Bf(x(t - \tau)), \quad t \neq t_k(x), \\ \Delta x(t) &= J_k(x(t^-)), \quad t = t_k(x), \\ x(t_0 + s) &= \phi(s), \quad s \in [-\tau, 0],\end{aligned}\quad (2)$$

where $f(x(t)) = \tilde{f}(x(t) + x^*) - \tilde{f}(x^*)$, $J_k(x(t^-)) = \tilde{J}_k(x(t^-) + x^*)$, $t_k(x) = \tilde{t}_k(x + x^*)$, and $\phi(s) = \tilde{\phi}(s) - x^*$.

In the sequel, we introduce some notations, basic definition, and lemmas:

$$\begin{aligned}R_+ &= [0, +\infty); \quad S_k = \{(t, x) \in R_+ \times R^n : t = t_k(x)\}; \\ N_k &= \{t \in R_+ : \text{there is } x \in R^n \text{ such that } (t, x) \in S_k\}; \\ d_k &= d(N_{k-1}, N_k) = \inf_{\substack{\tilde{t} \in N_k \\ t \in N_{k-1}}} |\tilde{t} - t|; \\ d_k^* &= d^*(N_{k-1}, N_k) = \sup_{\substack{\tilde{t} \in N_k \\ t \in N_{k-1}}} |\tilde{t} - t|; \\ L &= \text{diag}(l_1, \dots, l_n);\end{aligned}\quad (3)$$

$\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ denote the maximum and the minimum eigenvalues of the corresponding matrix, respectively.

$\|\cdot\|$ denotes the Euclidean norm of a vector or a square matrix.

$A > 0$ denotes that A is positive definite matrix.

$PC = PC(I, R^n) = \{\psi : I \rightarrow R^n \mid \psi(t^+) = \psi(t) \text{ for } t \in I, \psi(t^-) \text{ exists for } t \in I, \psi(t^-) = \psi(t) \text{ for all } t \in I \text{ but at most finite points}\}$, where I is an interval:

$$\|y\|_\tau = \sup_{\tau \leq s \leq 0} \|y(s)\| \quad \text{for } y \in PC([-\tau, 0], R^n). \quad (4)$$

Definition 1. The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of (1) is said to be globally exponentially stable if, for any solution $\tilde{x}(t, t_0, \tilde{\phi})$ with the initial condition $\tilde{\phi} \in PC$, there are constant $\varepsilon > 0$ and $M > 1$ such that $\|\tilde{x}(t, t_0, \tilde{\phi}) - x^*\| \leq M \|\tilde{\phi} - x^*\|_\tau e^{-\varepsilon(t-t_0)}$.

Lemma 2 (Berman and Plemmons [22]). Let $A \in R^{n \times n}$; then

$$\lambda_{\min}(A) x^T x \leq x^T A x \leq \lambda_{\max}(A) x^T x, \quad (5)$$

for any $x \in R^n$, if A is a symmetric matrix.

Lemma 3. Consider the following differential inequality:

$$\begin{aligned}D^+V(t) &\leq -\alpha V(t) + \beta V(t - \tau), \quad t \neq t_k, \\ V(t) &\leq c_k V(t^-), \quad t = t_k,\end{aligned}\quad (6)$$

where $V(t) \geq 0$, $\alpha > \beta > 0$, and $c_k > 1$, $k = 1, 2, \dots$. Denote $\delta^* = \sup_k \{t_{k+1} - t_k\}$, $\delta = \inf_k \{t_{k+1} - t_k\}$. For $d^* \geq \delta^*$, $0 < d \leq \delta$, and $\theta > 0$ such that $-\alpha + \beta e^{-\theta\tau} = -\theta$, where $U = \max_k \{c_k^{-1} \dots c_{k+N}^{-1}\}$ for $\underline{N} = \lfloor \tau/d^* \rfloor > 0$ and $U = 1$ for $\underline{N} = 0$, one has

$$V(t) \leq M \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\theta(t-t_0)}, \quad t \in [t_{k-1}, t_k], \quad (7)$$

where $c_0 = 1$, $\bar{V}(t_0) = \sup_{s \in [-\tau, 0]} V(t_0 + s)$, and $M \geq 1$ is a constant.

Proof. See the Appendix section. $\square$

Remark 4. In Lemma 3, if $\tau < d^*$, we have $U = 1$. Based on the proof, we know that $\theta = \varepsilon$ (please see the Appendix section for meaning of ε). If $\tau \geq d^*$, we obtain $U < 1$. It is obvious to know that $\theta > \varepsilon$. Generally speaking, (7) is more valuable because the convergence rate in (7) is larger than that in (A.1) when $\tau \geq d^*$. Because $c_k > 1$, we can use Lemma 3 to investigate the stability of impulsive differential systems in which the impulses are with destabilizing effects.

The solutions of system (1) may hit the same switching surface S_k finite or infinite times causing “beating phenomenon” or “pulse phenomenon.” In a similar way in [17], we can get the following lemma easily which guarantees that beating phenomenon does not exist.

Lemma 5. Suppose that

- (i) for any $k = 1, 2, \dots, \tilde{t}_k(x)$ is bounded;
- (ii) for any $(t_0, \tilde{\phi})$ there is a solution of continuous subsystem of system (1) in $[t_0, +\infty)$;
- (iii) $(\partial \tilde{t}_k(\tilde{x}) / \partial \tilde{x})(-A\tilde{x} + B\tilde{f}(\tilde{x}(t - \tau)) + I) < 1$;
- (iv) $\tilde{t}_k(\tilde{x}) \geq \tilde{t}_k(\tilde{x} + \tilde{f}_k(\tilde{x}))$, and $\tilde{t}_{k+1}(\tilde{x} + \tilde{f}_k(\tilde{x})) > \tilde{t}_k(\tilde{x})$.

Then there is solution of system (1) in $[t_0, +\infty)$, and it hits each switching surface $S_k : t = \tilde{t}_k(\tilde{x})$ exactly once in turn.

From Theorem 1.1.4 in [23], we know that there exists a unique solution of system (1) without impulses in our paper on $[t_0, +\infty)$, if $\|f(x) - f(y)\| \leq K\|x - y\|$, which yields that there exists a unique solution for any $\sum_k = \{(t, x) : t_{k-1}(x) \leq t < t_k(x)\}$ and any initial condition. Therefore, by mathematical induction, we know the global uniqueness and existence of solution. From now on we always assume that there exists a unique solution of system (1) $\tilde{x}(t) = \tilde{x}(t, t_0, \tilde{\phi})$ satisfying the conditions of Lemma 5; namely, it hits each switching surface $S_i, i \geq k$, only once [11]. In addition, we also always assume that $\{t_i\}_{i=1}^{\infty}$ are the moments that integral curve $(t, \tilde{x}(t, t_0, \tilde{\phi}))$ hits each switching surface $\{S_{k_i}\}_{i=1}^{\infty}$ in turn; namely, $t_i = \tilde{t}_{k_i}(x(t_i))$ and $t_i < t_{i+1}$.

3. Main Results

In this section, we establish some sufficient criteria for the exponential stability of system (1).

Theorem 6. Assume that, in addition to condition (A1), the following conditions are satisfied:

- (A2) $\tilde{f}_k(\tilde{x}(t)) = S_k(\tilde{x}(t) - x^*)$ with $S_k = \text{diag}(s_{1k}, s_{2k}, \dots, s_{nk})$;
- (A3) $d = \inf_k \{d_k\} > 0, d^* = \sup_k \{d_k^*\} < \infty$;
- (A4) there are a symmetric positive definite matrix P , constants $\mu > 0, q > 0, c_k > 0, \varepsilon > 0$, and $\alpha > \beta$ such that

$$\begin{aligned} -A^T P - PA + \mu^{-1} PBB^T P &\leq -\alpha P, \\ \mu L^T L &\leq \beta P, \\ P + S_k^T P + PS_k + S_k^T PS_k &\leq c_k P, \end{aligned} \quad (8)$$

$c_k < e^{\varepsilon d}$, and $-\alpha + \beta e^{\varepsilon \tau} U = -\varepsilon$, where $U = \max_k \{c_k^{-1} \dots c_{k+N}^{-1}\}$ for $\underline{N} = \lfloor \tau/d^* \rfloor > 0$ and $U = 1$ for $\underline{N} = 0$.

Then the equilibrium point x^* of system (1) is globally exponentially stable.

Proof. Based on (A2), we know that $J_k(x(t^-)) = S_k x(t^-)$. We choose the Lyapunov function of system (2) as follows:

$$V(t, x(t)) = x^T(t) Px(t). \quad (9)$$

Let $V(t) = V(t, x(t))$ briefly. When $t \neq t_i$, we have

$$\begin{aligned} D^+ V(t) &= \dot{x}^T(t) Px(t) + x^T(t) P \dot{x}(t) \\ &= (-Ax(t) + Bf(x(t - \tau)))^T Px(t) \\ &\quad + x^T(t) P (-Ax(t) + Bf(x(t - \tau))) \\ &\leq x^T(t) (-A^T P - PA) x(t) \\ &\quad + \mu^{-1} x^T(t) PBB^T Px(t) + \mu f^T(x(t)) f(x(t)) \\ &\leq x^T(t) (-A^T P - PA + \mu^{-1} PBB^T) x(t) \\ &\quad + \mu^{-1} x^T(t - \tau) L^T L x(t - \tau) \\ &\leq -\alpha V(t) + \beta V(t - \tau). \end{aligned} \quad (10)$$

When $t = t_i$, we have

$$\begin{aligned} V(t_i) &= x^T(t_i) Px(t_i) \\ &= (x(t_i^-) + S_{k_i} x(t_i^-))^T P (x(t_i^-) + S_{k_i} x(t_i^-)) \\ &= x^T(t_i^-) (P + S_{k_i}^T P + PS_{k_i} + S_{k_i}^T PS_{k_i}) x(t_i^-) \\ &\leq c_{k_i} V(t_i^-). \end{aligned} \quad (11)$$

Therefore, we have

$$\begin{aligned} D^+ V(t) &\leq -\alpha V(t) + \beta V(t - \tau), \quad t \neq t_i, \\ V(t_i) &\leq c_{k_i} V(t_i^-), \quad i = 1, 2, \dots \end{aligned} \quad (12)$$

On the basis of (7) and Lemma 3, we have

$$V(t) \leq M \prod_{l=1}^i c_{k_l} \bar{V}(t_0) e^{-\varepsilon(t-t_0)}, \quad t \in [t_i, t_{i+1}). \quad (13)$$

From condition (A4), we know that there is $0 < \zeta < \varepsilon$ such that $c_l \leq e^{(\varepsilon-\zeta)d} \leq e^{(\varepsilon-\zeta)(t_{l+1}-t_l)}$, $l = 1, 2, \dots$. Therefore, we have

$$\begin{aligned} V(t) &\leq M \prod_{l=1}^i e^{(\varepsilon-\zeta)(t_{l+1}-t_l)} e^{-\varepsilon(t-t_0)} \\ &= M e^{-\zeta(t-t_0)} e^{(\varepsilon-\zeta)(t_{i+1}-t-t_0-t_1)} \\ &\leq M e^{-\zeta(t-t_0)} e^{(\varepsilon-\zeta)d^*}. \end{aligned} \quad (14)$$

Denote $\bar{M} = e^{(\varepsilon+\zeta)d^*}$; then we have

$$V(t) \leq M \bar{M} \bar{V}(t_0) e^{-\zeta(t-t_0)}. \quad (15)$$

Based on (9) and Lemma 2, we obtain that

$$\|x(t)\| \leq \sqrt{\frac{M \bar{M} \lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\bar{\phi}\|_{\tau} e^{-(\zeta/2)(t-t_0)}. \quad (16)$$

By virtue of $x(t) = \tilde{x}(t) - x^*$, we have

$$\|\tilde{x}(t) - x^*\| \leq \sqrt{\frac{M\bar{M}\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\tilde{\phi} - x^*\|_{\tau} e^{-(\zeta/2)(t-t_0)}, \quad (17)$$

which yields that the equilibrium point x^* of system (1) is globally exponentially stable. $\square$

Remark 7. Because $\alpha > \beta$, we know that impulse-free neural network is stable. The impulses may be of destabilizing effects due to $c_k > 1$. It is shown that impulsive Hopfield neural networks will preserve the global exponential stability of the impulse-free Hopfield neural networks even if the impulses have enlarging effects on the states of neurons.

As mentioned in [17], the impulsive differential systems in which impulses occur in fixed time can be viewed as particular impulsive differential systems with variable-time impulses. Therefore, based on Theorem 6, we can obtain the stability criterion for the following format of Hopfield neural networks:

$$\begin{aligned} \dot{\tilde{x}}(t) &= -A\tilde{x}(t) + B\tilde{f}(\tilde{x}(t - \tau)) + \tilde{I}, \quad t \neq \tilde{t}_k, \\ \Delta\tilde{x}(t) &= \tilde{J}_k(\tilde{x}(t^-)), \quad t = \tilde{t}_k, \\ \tilde{x}(t_0 + s) &= \tilde{\phi}(s), \quad s \in [-\tau, 0]. \end{aligned} \quad (18)$$

Theorem 8. Assume that (A1), (A2) hold, and

$$(A5) \quad \delta = \inf_k \{\delta_k\} > 0, \delta^* = \sup_k \{\delta_k^*\} < \infty;$$

(A6) there are a symmetric positive definite matrix P , constants $\mu > 0$, $q > 0$, $c_k > 0$, $\varepsilon > 0$, and $\alpha > \beta$ such that

$$\begin{aligned} -A^T P - PA + \mu^{-1} P B B^T P &\leq -\alpha P, \\ \mu L^T L &\leq \beta P, \\ P + S_k^T P + P S_k + S_k^T P S_k &\leq c_k P, \end{aligned} \quad (19)$$

$c_k < e^{\varepsilon d}$, and $-\alpha + \beta e^{\varepsilon \tau} U = -\varepsilon$, where $U = \max_k \{c_k^{-1} \cdots c_{k+N}^{-1}\}$ for $N = \lfloor \tau/\delta^* \rfloor > 0$ and $U = 1$ for $N = 0$.

Then the equilibrium point x^* of system (18) is globally exponentially stable.

4. Numeric Example

In this section, we consider one example to illustrate the effectiveness of theoretical results.

Example 1. Consider the following system:

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= -\tilde{x}_1(t) + 0.1 \sin(\tilde{x}_1(t - 0.8)) \\ &\quad + 0.4 \sin(\tilde{x}_2(t - 0.8)) + \pi, \\ \dot{\tilde{x}}_2(t) &= -\tilde{x}_2(t) + 0.4 \sin(\tilde{x}_1(t - 0.8)) \\ &\quad + 0.1 \sin(\tilde{x}_2(t - 0.8)) + \pi, \quad t \neq \tilde{t}_k(\tilde{x}), \\ \Delta\tilde{x}_1(t) &= -2.2(\tilde{x}_1(t) - \pi), \\ \Delta\tilde{x}_2(t) &= -2.2(\tilde{x}_2(t) - \pi), \quad t = \tilde{t}_k(\tilde{x}), \end{aligned} \quad (20)$$

where $\tilde{t}_k(\tilde{x}) = -(2/5\pi^2) \arctan^2(\tilde{x}_1 + \tilde{x}_2 - 2\pi) + 0.7k$. It is easy to obtain that $A = \text{diag}(1, 1)$, $B = \begin{bmatrix} 0.1 & 0.4 \\ 0.4 & 0.1 \end{bmatrix}$, $S_k = \text{diag}(-2.2, -2.2)$, $\tau = 0.8$, $d = 0.6$, $d^* = 0.8$, $L = E$, and $x^* = (\pi, \pi)^T$.

Now we verify that there is no beating phenomenon in system (20).

- (1) It is obvious that, for $k = 1, 2, \dots$, $\tilde{t}_k(x) = -(2/5\pi^2) \arctan^2(\tilde{x}_1 + \tilde{x}_2 - 2\pi) + 0.7k$ is bounded.
- (2) Based on [24], it is easy to predicate the existence of solutions for system (20).
- (3) Let $y = \tilde{x}_1 + \tilde{x}_2 - 2\pi$, $F_1 = \sin(\tilde{x}_1(t - 0.8))$, and $F_2 = \sin(\tilde{x}_2(t - 0.8))$. We have

$$\begin{aligned} &\frac{\partial \tilde{t}_k(\tilde{x})}{\partial \tilde{x}} (-A\tilde{x} + B\tilde{f}(\tilde{x}(t - \tau)) + I) \\ &= -\frac{4}{5\pi^2} \frac{\arctan(y)}{1 + y^2} (-y + 0.5F_1 + 0.5F_2) \\ &\leq \frac{4}{5\pi^2} \frac{|y|}{1 + y^2} (|y| + 0.5|F_1| + 0.5|F_2|) \\ &\leq \frac{4}{5\pi^2} \left(\frac{y^2}{1 + y^2} + 0.5 \left(\frac{|F_1| + |F_2|}{1 + y^2} \right) \right) \leq \frac{8}{5\pi^2} < 1. \end{aligned} \quad (21)$$

- (4) It is easy to see that $\tilde{t}_k(\tilde{x}) \geq t_k(\tilde{x} + \tilde{J}_k(\tilde{x}))$ and $\tilde{t}_{k+1}(\tilde{x} + \tilde{J}_k(\tilde{x})) > \tilde{t}_k(\tilde{x})$.

Therefore, all the conditions of Lemma 5 are satisfied; that is to say, there is no beating phenomenon in system (20).

For convenience, choose $P = E$, $\mu = 1$, $\alpha = 1.7$, $\beta = 1$, and $c_k = 1.44$. It is easy to verify that $\varepsilon = 0.792$ and all the conditions of Theorem 6 are satisfied. Therefore, system (20) is globally exponentially stable, although the impulses are of destabilizing effects, as shown in Figure 1.

Appendix

Proof of Lemma 3

Let ε be the largest positive solution satisfying the inequality $-\alpha + \beta e^{\varepsilon c} \leq -\varepsilon$. We claim that

$$V(t) \leq \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\varepsilon(t-t_0)}, \quad (A.1)$$

where $t \in [t_{k-1}, t_k)$, $k = 1, 2, \dots$

FIGURE 1: The time response curves of system (20).

First, for $t \in [t_0 - \tau, t_0]$, we have $V(t) \leq \bar{V}(t_0) \leq \bar{V}(t_0)e^{-\varepsilon(t-t_0)}$. Particularly, $V(t_0) \leq \bar{V}(t_0)e^{-\varepsilon(t_0-t_0)} = c_0 \bar{V}(t_0)e^{-\varepsilon(t_0-t_0)}$. Now we show that (A.1) holds for $k = 1$; namely,

$$V(t) \leq c_0 \bar{V}(t_0) e^{-\varepsilon(t-t_0)}, \quad t \in [t_0, t_1]. \quad (\text{A.2})$$

If (A.2) does not hold, there is $t^* \in [t_0, t_1)$ such that

$$\begin{aligned} V(t^*) &= c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-t_0)}, \\ D^+ V(t^*) &> -\varepsilon c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-t_0)}. \end{aligned} \quad (\text{A.3})$$

However, based on (6), we have

$$\begin{aligned} D^+ V(t^*) &\leq -\alpha V(t^*) + \beta V(t^* - \tau) \\ &\leq -\alpha c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-t_0)} + \beta c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-\tau-t_0)} \\ &= (-\alpha + \beta e^{\varepsilon\tau}) c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-t_0)} \\ &\leq -\varepsilon c_0 \bar{V}(t_0) e^{-\varepsilon(t^*-t_0)}, \end{aligned} \quad (\text{A.4})$$

which contradicts (A.3).

Now suppose that, for $k = \bar{k}$, $\bar{k} \geq 1$, (A.1) holds. Namely, $V(t) \leq \prod_{i=0}^{\bar{k}-1} c_i \bar{V}(t_0) e^{-\varepsilon(t-t_0)}$, $t \in [t_{\bar{k}-1}^-, t_{\bar{k}}^-]$. We prove that (A.1) is also satisfied for $k = \bar{k} + 1$. From (6) we can obtain $V(t_{\bar{k}}^-) = \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t_{\bar{k}}^- - t_0)}$. If this claim is not true, there is $t^{**} \in [t_{\bar{k}}^-, t_{\bar{k}+1}^-)$ such that

$$V(t^{**}) = \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)}, \quad (\text{A.5})$$

$$D^+ V(t^{**}) > -\varepsilon \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)}.$$

There is a nonnegative integer h ($0 \leq h \leq \bar{k} + 1$) such that $t^{**} - \tau \in [t_{h-1}, t_h)$ (if $h = 0$, $t_{h-1} = t_0 - \tau$). Then we have

$$V(t^{**} - \tau) \leq \prod_{i=0}^{h-1} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-\tau-t_0)}. \quad (\text{A.6})$$

By virtue of (6), we obtain

$$\begin{aligned} D^+ V(t^{**}) &\leq -\alpha V(t^{**}) + \beta V(t^{**} - \tau) \\ &\leq -\alpha \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)} \\ &\quad + \beta \prod_{i=0}^{h-1} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-\tau-t_0)} \\ &= \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)} \left(-\alpha + \beta \prod_{i=h}^{\bar{k}} c_i^{-1} e^{\varepsilon\tau} \right) \\ &\leq \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)} (-\alpha + \beta e^{\varepsilon\tau}) \\ &\leq -\varepsilon \prod_{i=0}^{\bar{k}} c_i \bar{V}(t_0) e^{-\varepsilon(t^{**}-t_0)}, \end{aligned} \quad (\text{A.7})$$

which contradicts (A.5), which yields that (A.1) holds for $k = \bar{k} + 1$.

We can find $k^* \geq 1$ such that $t_{k^*} - \tau \geq t_0$. For any $t \in [t_{k-1}, t_k)$, $k \leq k^*$, from (A.1) we have

$$\begin{aligned} V(t) &\leq \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\varepsilon(t-t_0)} \\ &\leq \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\varepsilon(t-t_0)} e^{(\theta-\varepsilon)(t-t_0)} \\ &\leq \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\theta(t-t_0)} e^{(\theta-\varepsilon)(k^* d^* + t_1 - t_0)} \\ &= M \prod_{i=0}^{k-1} c_i \bar{V}(t_0) e^{-\theta(t-t_0)}, \end{aligned} \quad (\text{A.8})$$

where $M = e^{(\theta-\varepsilon)(k^* d^* + t_1 - t_0)}$. That is to say, (7) holds for any $k \leq k^*$.

Now we show that (7) holds for $k > k^*$. For $k = k^* + 1$, if (7) does not hold, there is $t^{***} \in [t_{k^*}, t_{k^*+1})$ such that

$$V(t^{***}) = M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)}, \quad (\text{A.9})$$

$$D^+ V(t^{***}) > -\theta M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)}.$$

Note that there is a positive integer h^* ($1 \leq h^* \leq k^* + 1$) such that $t^{***} - \tau \in [t_{h^*-1}, t_{h^*})$. Then we have

$$\begin{aligned}
 & D^+V(t^{***}) \\
 & \leq -\alpha V(t^{***}) + \beta V(t^{***} - \tau) \\
 & \leq -\alpha M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)} \\
 & \quad + \beta M \prod_{i=0}^{h^*-1} c_i \bar{V}(t_0) e^{-\theta(t^{***}-\tau-t_0)} \\
 & \leq M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)} \left(-\alpha + \beta \prod_{i=h^*}^{k^*} c_i^{-1} e^{\theta\tau} \right) \\
 & \leq M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)} (-\alpha + \beta e^{\theta\tau} U) \\
 & \leq -\theta M \prod_{i=0}^{k^*} c_i \bar{V}(t_0) e^{-\theta(t^{***}-t_0)},
 \end{aligned} \tag{A.10}$$

which contradicts (A.9). Therefore (7) holds for $k = k^* + 1$.

By mathematical induction, it is easy to illustrate that (7) holds for $k > k^* + 1$. The proof is complete.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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