Research Article

Boundary Stabilization of the Wave Equation with Time-Varying and Nonlinear Feedback

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We study the stabilization of the wave equation with variable coefficients in a bounded domain and a time-varying and nonlinear term. By the Riemannian geometry methods and a suitable assumption of nonlinearity and the time-varying term, we obtain the uniform decay of the energy of the system.

1. Introduction

There are many results concerning the boundary stabilization of classical wave equations. See [1–6] for linear cases and [7–12] for nonlinear ones. The stability of the wave equation with variable coefficients has attracted much attention. See [13–23], and many others. In [20], by the methods in [11, 24], the authors study the stability of the wave equation with nonlinear term and time-varying term. However, under the condition the nonlinear term has upper bound and the time-varying term has lower bound, the stability of the wave equation was not studied in [20]. In this paper, our purpose is to study the stability of the wave equation under the condition the nonlinear term has upper bound and the time-varying term has lower bound.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ . It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. Define

$$\mathcal{A}u = -\operatorname{div} A(x) \nabla u \quad \text{for } u \in H^1(\Omega), \quad (1)$$

where div is the divergence operator of the standard metric of $\mathbb{R}^n$; $A(x) = (a_{ij}(x))$ is symmetric, positively definite matrices for each $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$.

We consider the stabilization of the wave equations with variable coefficients and time-varying delay in the dissipative boundary feedback:

$$\begin{aligned} u_{tt} + \mathcal{A}u &= 0 \quad (x, t) \in \Omega \times (0, +\infty), \\ u(x, t)|_{\Gamma_2} &= 0 \quad t \in (0, +\infty), \\ \frac{\partial u(x, t)}{\partial \nu_{\mathcal{A}}} + \phi(t) g_1 u_t(x, t) &= 0 \quad (x, t) \in \Gamma_1 \times (0, +\infty), \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega, \end{aligned} \quad (2)$$

$g_1 \in C(\mathbb{R})$ and there exists a positive constant c_1 such that

$$\begin{aligned} g_1(0) &= 0, \quad sg_1(s) \leq |s|^2 \quad \text{for } s \in \mathbb{R}, \\ |g_1(s)| &\geq c_1 |s| \quad \text{for } |s| > 1, \end{aligned} \quad (3)$$

and $\phi(t) \in C([0, +\infty))$ satisfies

$$\phi(t) \geq \phi_0 \quad \forall t \geq 0, \quad (4)$$

$$\lim_{t \rightarrow +\infty} \frac{F(t)}{t} = 0, \quad (5)$$

where ϕ_0 is a positive constant and $F(t) = \max_{0 \leq \rho \leq t} \phi(\rho)$.

$\partial u / \partial \nu_{\mathcal{A}}$ is the conormal derivative

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \langle A(x) u, \nu \rangle, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard metric of the Euclidean space $\mathbb{R}^n$ and $\nu(x)$ is the outside unit normal vector for each $x \in \Gamma$. Moreover, the initial data (u_0, u_1) belongs to a suitable space.

Define the energy of the system (2) by

$$E(t) = \frac{1}{2} \int_{\Omega} \left(u_t^2 + \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \right) dx. \quad (7)$$

We define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \quad (8)$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold with an inner product:

$$\langle X, Y \rangle_g = \langle A^{-1}(x) X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g \quad X, Y \in \mathbb{R}_x^n. \quad (9)$$

Let D_g denote the Levi-Civita connection of the metric g . For the variable coefficients, the main assumptions are as follows.

Assumption A. There exists a vector field H on $\overline{\Omega}$ and a constant $\rho_0 > 0$ such that

$$D_g H(X, X) \geq \rho_0 |X|_g^2 \quad \text{for } X \in \mathbb{R}_x^n, \quad x \in \overline{\Omega}. \quad (10)$$

Moreover we assume that

$$\sup_{x \in \overline{\Omega}} \operatorname{div} H < \inf_{x \in \overline{\Omega}} \operatorname{div} H + 2\rho_0, \quad (11)$$

$$\begin{aligned} H \cdot \nu &\leq 0 \quad x \in \Gamma_2, \\ H \cdot \nu &\geq \delta \quad x \in \Gamma_1, \end{aligned} \quad (12)$$

where δ is a positive constant.

Assumption (10) was introduced by [13] as a checkable assumption for the exact controllability of the wave equation with variable coefficients. For examples on the condition, see [13, 14].

Based on Assumption (10), Assumption A was given by [19] to study the stabilization of the wave equation with variable coefficients and boundary condition of memory type.

Define

$$H_{\Gamma_2}^1(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_2} = 0\}. \quad (13)$$

To obtain the stabilization of the system (2), we assume the system (2) is well-posed such that

$$u \in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H_{\Gamma_2}^1(\Omega)). \quad (14)$$

The main result of this paper is stated as follows.

Theorem 1. *Let Assumption A holds true. Then there exist positive constants C, C_2 , such that*

$$E(t) \leq C_1 h \left(\frac{C_2 E(0)}{t} \right) + \frac{C_1 F(t)}{t} E(0), \quad t > 0. \quad (15)$$

2. Basic Inequality of the System

In this section we work on Ω with two metrics at the same time, the standard dot metric $\langle \cdot, \cdot \rangle$ and the Riemannian metric $g = \langle \cdot, \cdot \rangle_g$ given by (8).

If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g , via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X \rangle_g, \quad (16)$$

where X is any vector field on $(\mathbb{R}^n, g)$. The following lemma provides further relations between the two metrics; see [13] in Lemma 2.1.

Lemma 2. *Let $x = (x_1, \dots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$. Let f, h be functions and let $\mathcal{H}, X$ be vector fields. Then*

$$\begin{aligned} (a) \quad & \langle H(x), A(x) X(x) \rangle_g = \langle H(x), X(x) \rangle, \quad x \in \mathbb{R}^n; \\ (b) \quad & \end{aligned} \quad (17)$$

$$\nabla_g f = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) f_{x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla f, \quad x \in \mathbb{R}^n, \quad (18)$$

where ∇f is the gradient of f in the standard metric;

$$\begin{aligned} (c) \quad & \nabla_g f(h) = \langle \nabla_g f, \nabla_g h \rangle_g = \langle \nabla f, A(x) \nabla h \rangle, \quad x \in \mathbb{R}^n, \\ & \text{where the matrix } A(x) \text{ is given in formula (1).} \end{aligned} \quad (19)$$

To prove Theorem 1, we still need several lemmas further. Define

$$E_0(t) = \frac{1}{2} \int_{\Omega} \left(u_t^2 + |\nabla_g u|_g^2 \right) dx. \quad (20)$$

Then, we have

$$E(t) = E_0(t) + \xi \int_{t-\tau(t)}^t \int_{\Gamma_1} u_t^2(x, \rho) d\Gamma d\rho. \quad (21)$$

Lemma 3. *Let (u) be the solution of system (2). Then there exists a constant C_1 such that*

$$E(0) - E(T) = C_1 \int_0^T \int_{\Gamma_1} \phi(t) u_t(x, t) g_1(u_t(x, t)) d\Gamma dt, \quad (22)$$

where $T \geq 0$. The assertion (22) implies that $E(t)$ is decreasing.

Proof. Differentiating (7), we obtain

$$\begin{aligned} E'(t) &= \int_{\Omega} \left(u_t u_{tt} + \nabla_g u \cdot \nabla_g u_t \right) dx \\ &= \int_{\Gamma_1} \phi(t) u_t(x, t) g_1(u_t(x, t)) d\Gamma. \end{aligned} \quad (23)$$

Then the inequality (22) holds true. $\square$

3. Proofs of Theorem 1

From Proposition 2.1 in [13], we have the following identities.

Lemma 4. Suppose that $u(x, t)$ solves equation $u_{tt} + \mathcal{A}u = 0$, $(x, t) \in \Omega \times (0, +\infty)$ and that $\mathcal{H}$ is a vector field defined on $\overline{\Omega}$. Then, for $T \geq 0$,

$$\begin{aligned} & \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} \mathcal{H}(u) d\Gamma dt \\ & + \frac{1}{2} \int_0^T \int_{\Gamma} \left(u_t^2 - |\nabla_g u|_g^2 \right) \mathcal{H} \cdot \nu d\Gamma dt \\ & = (u_t, \mathcal{H}(u))|_0^T + \int_0^T \int_{\Omega} D_g \mathcal{H}(\nabla_g u, \nabla_g u) dx dt \\ & + \frac{1}{2} \int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 \right) \operatorname{div} \mathcal{H} dx dt. \end{aligned} \quad (24)$$

Moreover, assume that $P \in C^1(\overline{\Omega})$. Then

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 \right) P dx dt \\ & = (u_t, uP)|_0^T + \frac{1}{2} \int_0^T \int_{\Omega} \nabla_g P(u^2) dx dt \\ & - \int_0^T \int_{\Gamma} Pu \frac{\partial u}{\partial \nu_{\mathcal{A}}} d\Gamma dt. \end{aligned} \quad (25)$$

Lemma 5. Suppose that all assumptions in Theorem 1 hold true. Let u be the solution of the system (2). Then there exist positive constants C, T_0 for which

$$E(T) \leq \frac{C}{T} \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt, \quad (26)$$

where $T \geq T_0$.

Proof. We let θ be a positive constant satisfying

$$\frac{1}{2} \sup_{x \in \overline{\Omega}} \operatorname{div} H < \theta < \frac{1}{2} \inf_{x \in \overline{\Omega}} \operatorname{div} H + \rho_0. \quad (27)$$

Set

$$\mathcal{H} = H, \quad P = \theta - \rho_0. \quad (28)$$

Substituting the identity (25) into the identity (24), we obtain

$$\begin{aligned} \Pi_{\Gamma} &= (u_t, H(u) + Pu)|_0^T \\ &+ \int_0^T \int_{\Omega} \left(D_g H(\nabla_g u, \nabla_g u) - \rho_0 |\nabla_g u|_g^2 \right) dx dt \\ &+ \int_0^T \int_{\Omega} \left(\left(\frac{1}{2} \operatorname{div} H + \rho_0 - \theta \right) u_t^2 \right. \\ &\quad \left. + \left(\theta - \frac{1}{2} \operatorname{div} H \right) |\nabla_g u|_g^2 \right) dx dt, \end{aligned} \quad (29)$$

where

$$\begin{aligned} \Pi_{\Gamma} &= \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma} \left(u_t^2 - |\nabla_g u|_g^2 \right) H \cdot \nu d\Gamma dt. \end{aligned} \quad (30)$$

Decompose Π_{Γ} as

$$\Pi_{\Gamma} = \Pi_{\Gamma_1} + \Pi_{\Gamma_2}. \quad (31)$$

Since $u|_{\Gamma_2} = 0$, we obtain $\nabla_{\Gamma} u|_{\Gamma_2} = 0$; that is,

$$\nabla_g u = \frac{\partial u}{\partial \nu_{\mathcal{A}}} \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g^2} \quad \text{for } x \in \Gamma_2. \quad (32)$$

Similarly, we have

$$H(u) = \left\langle H, \nabla_g u \right\rangle_g = \frac{\partial u}{\partial \nu_{\mathcal{A}}} \frac{H \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} \quad \text{for } x \in \Gamma_2. \quad (33)$$

Using the formulas (32) and (33) in the formula (30) on the portion Γ_2 , with (12), we obtain

$$\Pi_{\Gamma_2} = \frac{1}{2} \int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \frac{H \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Gamma dt \leq 0. \quad (34)$$

From (12), we have

$$\begin{aligned} \Pi_{\Gamma_1} &= \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu_{\mathcal{A}}} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma_1} \left(u_t^2 - |\nabla_g u|_g^2 \right) H \cdot \nu d\Gamma dt \\ &\leq C_{\varepsilon} \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 d\Gamma dt \\ &+ \varepsilon \int_0^T \int_{\Gamma_1} \left(u^2 + |\nabla_g u|_g^2 \right) d\Gamma dt \\ &+ \int_0^T \int_{\Gamma_1} \left(Cu_t^2 - \delta |\nabla_g u|_g^2 \right) d\Gamma dt \\ &\leq C \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 d\Gamma dt + \varepsilon E(t) \\ &+ C \int_0^T \int_{\Gamma_1} u_t^2 d\Gamma dt. \end{aligned} \quad (35)$$

Substituting the formulas (34) and (35) into the formula (29), with (27), we obtain

$$\begin{aligned} & \int_0^T E(t) dt \\ & \leq C(E(0) + E(T)) + C \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt. \end{aligned} \quad (36)$$

It follows from (22) that

$$\int_0^T E(t) dt \geq TE(T). \quad (37)$$

Substituting the formulas (22) and (37) into the formula (36), the inequality (26) holds. $\square$

Proof of Theorem 1. Since $E(t)$ is decreasing, with (4) and (26), for sufficiently large T , we have

$$\begin{aligned} E(T) &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} (\phi^2(t) g^2(u_t) + u_t^2) d\Gamma dt \\ &\leq \frac{C}{T} \left\{ \int_0^T \int_{\Gamma_1} (u_t^2 + g^2(u_t)) d\Gamma dt \right. \\ &\quad \left. + F(T) \int_0^T \int_{\Gamma_1} u_t \phi(t) g(u_t) d\Gamma dt \right\} \\ &\leq \frac{C}{T} \left\{ \int_0^T \int_{\{x \in \Gamma_1, |u_t| \leq 1\}} h(u_t g(u_t)) d\Gamma dt + CF(T) E(0) \right\} \\ &\leq \frac{C}{T} \int_0^T \int_{\Gamma_1} h(u_t g(u_t)) d\Gamma dt + \frac{CF(T)}{T} E(0) \\ &\leq C \text{meas}(\Gamma_1) h \left(\frac{\int_0^T \int_{\Gamma_1} u_t g(u_t) d\Gamma dt}{T \cdot \text{meas}(\Gamma_1)} \right) + \frac{CF(T)}{T} E(0) \\ &\leq C_1 h \left(\frac{C_2 E(0)}{T} \right) + \frac{C_1 F(T)}{T} E(0). \end{aligned} \quad (38)$$

Note that $E(t)$ is decreasing; the estimate (15) holds. $\square$

4. Application of the System (2)

Nonlinear feedback describes a property of a physical system; that is, the response by the physical system to an applied force is nonlinear in its effect. One of the applications of the system (2) is in sound waves, where the system (2) describes the reflection of sound in heterogeneous materials at surfaces of some materials with nonlinearity of interest in engineering practice. Theorem 1 indicates that the energy of the sound waves with the reflection of sound at surfaces in heterogeneous materials at surfaces of some materials with nonlinearity is uniform decay under a suitable assumption of the nonlinearity.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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