Research Article

Multisensor Estimation Fusion of Nonlinear Cost Functions in Mixed Continuous-Discrete Stochastic Systems

Il Young Song, Vladimir Shin, Seokhyoung Lee, and Won Choi

1 Department of Sensor Systems, Hanwha Corporation R&D Center, 52-1 Oesam-dong, Yuseong-gu, Daejeon 305-106, Republic of Korea
2 Department of Information and Statistics, Research Institute of Natural Science, Gyeongsang National University, 501 Jinjudae-ro, Gyeongsangnam-do, Jinju 660-701, Republic of Korea
3 Department of Automation & Control Research, Hyundai Industrial Research Institute, 1000 Bangeojinsunhwan-doro, Dong-gu, Ulsan 682-792, Republic of Korea
4 Department of Mathematics, Incheon National University, 119 Academy-ro, Yeonsu-gu, Incheon 406-772, Republic of Korea

Correspondence should be addressed to Vladimir Shin; vishin@gnu.ac.kr and Won Choi; choiwon@incheon.ac.kr

Received 27 November 2013; Accepted 26 February 2014; Published 10 April 2014

Academic Editor: Zhengguang Wu

Copyright © 2014 Il Young Song et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We propose centralized and distributed fusion algorithms for estimation of nonlinear cost function (NCF) in multisensory mixed continuous-discrete stochastic systems. The NCF represents a nonlinear multivariate functional of state variables. For polynomial NCFs, we propose a closed-form estimation procedure based on recursive formulas for high-order moments for a multivariate normal distribution. In general case, the unscented transformation is used for calculation of nonlinear estimates of a cost functions. To fuse local state estimates, the mixed differential difference equations for error cross-covariance between local estimates are derived. The subsequent application of the proposed fusion estimators for a multisensory environment demonstrates their effectiveness.

1. Introduction

Multisensor data fusion is typically motivated by reducing the overall redundant information obtained from different sensors, increasing information gain by using multiple sensors, increasing the accuracy, and decreasing the uncertainty of the system. Further, multisensor data fusion can give benefits such as extended temporal and spatial coverage, reduced ambiguity, enhanced spatial resolution, and increased dimensionality of the measurement space. This process has attracted growing interest for potential applications in many fields including guidance, robotics, aerospace, target tracking, signal processing, and control [1–3]. In general, two basic fusion approaches are commonly used to process measured sensor data.

If a central processor receives the measurement data from all local sensors directly and processes them in real time, the correlative result is known as the centralized estimation process. One advantage of the centralized estimation is that it involves minimal information loss. However, the centralized estimation approach has several serious drawbacks, including poor survivability and reliability, as well as heavy communication and computational burdens.

In practice, especially when sensors are dispersed over a wide geographic area, there are limitations on the amount of communications allowed among sensors. Also, sensors are provided with processing capabilities. In this case, a certain amount of computation can be performed at the individual sensors and a compressed version of sensor data can be transmitted to a fusion center where the received information is appropriately combined to yield the global inference. The advantage of the distribution of filters is that the parallel structures would lead to increase of the input data rates and make easy fault detection and isolation. However, the accuracy of the distributed estimators is generally lower than that of the centralized estimator. Recently, various distributed and parallel versions of the standard continuous and discrete Kalman filters have been reported for linear dynamic systems.
within a multisensor environment [1, 2, 4–9]. For nonlinear
dynamic state-space models, different variants of suboptimal
nonlinear filters, such as the unscented Kalman filter, the
extended Kalman filter, and their extensions, are proposed in
order to enhance the performance of the nonlinear
estimation in multisensory environment [10–14].

However, some applications require the estimation fusion
of nonlinear functions of state variables, representing useful
information for system control, for example, a quadratic form
of a state vector, which can be interpreted as a current distance
between targets or as the energy of an object [3]. We refer to
the nonlinear function as the nonlinear cost function (NCF).
Aside from the aforementioned papers, most of the authors
have not focused on the estimation of the NCF, considering
instead only a state estimation. To the best of our knowledge,
there are no methods reported in the literature for estimation
fusion of NCFs in a multisensory environment.

Therefore, in this paper, the estimation fusion problem of
NCFs of state variables is considered for mixed continuous-
discrete linear systems under a multisensory environment.
The continuous-discrete approach allows system to avoid
discretization by propagating the estimate and error covari-
ance between measurements in continuous time using an
integration routine such as Runge-Kutta. This approach
yields the optimal or suboptimal estimate continuously at
times, including times between the data arrival instants.
The advantage of the continuous-discrete estimator over the
alternative approaches using system discretization is that, in
the former, it is not necessary for the sample times to be
equally spaced. This means that the cases of irregular and
intermittent measurements are easy to handle.

Therefore, the aim of this paper is to develop fusion esti-
mators for arbitrary NCFs under multisensory environment.
Centralized and decentralized estimation fusion algorithms
for NCFs are proposed and their accuracies are compared.

This paper is organized as follows. Section 2 presents
a statement of the estimation fusion problem for NCFs.
In Section 3, the globally optimal centralized estimator is
derived. In Section 4, we present the main result pertaining
to the distributed estimation of NCFs. Here, the key
equations for cross-covariance between the local continuous-
discrete estimators are derived. In Section 5, two computa-
tion procedures for calculation of estimates of NCFs and
cross-covariance are proposed. The procedures are based
on the unscented transformation and recursive formulas for
moments of multivariate normal distributions. In Section 6,
we study the comparative analysis of the proposed fusion
estimators via two theoretical examples. In Section 7, the
efficiency of the fusion estimators is studied for the case of an
unmanned marine probe system. Finally, we conclude our
results in Section 8.

2. Problem Statement

The general continuous-discrete Kalman multisensory
frame-work involves the estimation of the state of a
continuous-time linear dynamic system given discrete
measurements

\[
\dot{x}_t = F_t x_t + G_t v_t, \quad t \geq 0,
\]

\[
y_{t_k}^{(i)} = H_{t_k}^{(i)} x_{t_k} + w_{t_k}^{(i)}, \quad t_k > t_{k+1} > \cdots > t_0 = 0, \quad k = 1, 2, \ldots; \quad i = 1, \ldots, L,
\]

where \( x_t \in \mathbb{R}^n \) is a state vector, \( y_{t_k}^{(i)} \in \mathbb{R}^{m_i} \) is a measurement vector from \( i \)th sensor \( (i = 1, \ldots, L) \), \( v_t \in \mathbb{R}^q \) is a zero-
mean Gaussian white system noise with intensity \( Q_t \), that is,

\[
E(v_t v_t^T) = Q_t, \quad w_{t_k}^{(i)} \in \mathbb{R}^{m_i}, \quad i = 1, \ldots, L,
\]

represent white sequences (measurement errors) \( w_{t_k}^{(i)} \sim \mathcal{N}(0, R_{t_k}^{(i)}) \), and \( \delta_i \) is the
Dirac delta-function. We assume that the initial state \( x_0 \sim \mathcal{N}(\bar{x}_0, P_0) \) system and measurement noises \( v_t, w_{t_k}^{(1)}, \ldots, w_{t_k}^{(L)} \)
are mutually uncorrelated.

A problem associated with such systems (1) is that of
estimation of the nonlinear cost function of the state variables

\[
z_t = f(x_t) : \mathbb{R}^n \to \mathbb{R}
\]

from the overall noisy sensor measurements

\[
y_{t} = \begin{cases} y_{1, t}^{(1)} & \ldots & y_{1, t}^{(L)} \\ \vdots & \ddots & \vdots \\ y_{L, t}^{(1)} & \ldots & y_{L, t}^{(L)} \end{cases},
\]

\[
y_{t}^{(i)} = \begin{cases} y_{1, t}^{(i)} & \ldots & y_{1, t}^{(i)} \\ \vdots & \ddots & \vdots \\ y_{L, t}^{(i)} & \ldots & y_{L, t}^{(i)} \end{cases},
\]

where \( y_{t}^{(i)} \) is a measurement from \( i \)th sensor \( (i = 1, \ldots, L) \), and \( y_t \) is a vector from \( \mathbb{R}^m \).

Typical examples of such NCFs may be an arbitrary quadratic
form \( f(x_t) = x_t^T \Omega x_t \), representing an energy-like function of
an object or square distance \( f(x_t) = d^2(x_t, x_0^0) \) between the
current \( x_t \) and nominal \( x_0^0 \) states, respectively.

We propose centralized and distributed estimation fusion
algorithms for NCFs in the subsequent sections.

3. Global Optimal Solution-Centralized
Estimator

In this section, the best global optimal (in mean–square error
sense) estimation algorithm for an NCF is derived. In the
centralized fusion set-up, a multisensory dynamic system (1)
can be reformulated into a composite form

\[
\dot{x}_t = F_t x_t + G_t v_t, \quad t \geq 0,
\]

\[
y_{t_k} = H_{t_k} x_{t_k} + w_{t_k}, \quad y_{t_k} \in \mathbb{R}^{m}, \quad m = m_1 + \cdots + m_L,
\]

where

\[
y_t = [ y_{t_k}^{(1)T} \ldots y_{t_k}^{(L)T} ]^T, \quad H_t = [ H_{t_k}^{(1)T} \ldots H_{t_k}^{(L)T} ]^T, \quad w_{t_k} = [ w_{t_k}^{(1)T} \ldots w_{t_k}^{(L)T} ]^T, \quad w_t \sim \mathcal{N}(0, R_t),
\]

\[
A_t = \text{diag} \{ R_{t_k}^{(1)}, \ldots, R_{t_k}^{(L)} \}.
\]
Then, the optimal mean-square estimate \( \hat{x}_t^{CF} = E(x_t | y_{[t,t_k]}) \) of the state \( x_t \) based on the overall sensor measurements (3) and error covariance \( P_t^{CF} = \text{cov}(e_t^{CF}, e_t^{CF}) \), are given by the centralized continuous-discrete Kalman filter equations [15, 16]. We refer to the filter as centralized filter (CF):

\[
\begin{align*}
\hat{x}_t^{CF} &= F_t \hat{x}_t^{CF}, \quad t_{k-1} \leq t \leq t_k, \quad \hat{x}_{t_k-1}^{CF} = \hat{x}_{t_k-1}^{CF}, \\
p_t^{CF} &= F_t p_t^{CF} F_t^T + G_t Q_t G_t^T, \quad p_{t_k-1}^{CF} = p_{t_k-1}^{CF}, \\
\end{align*}
\]

Measurement update at time \( t = t_k \):

\[
\begin{align*}
\hat{x}_{t_k}^{CF} &= \hat{x}_{t_k}^{CF} + K_{t_k} \left( y_{t_k} - H_{t_k} \hat{x}_{t_k}^{CF} \right), \quad \hat{x}_0^{CF} = \hat{x}_0^{CF}, \\
K_{t_k} &= P_{t_k} H_{t_k}^T \left( H_{t_k} P_{t_k} H_{t_k}^T + R_{t_k} \right)^{-1}, \\
P_{t_k} &= \left( I_n - K_{t_k} H_{t_k} \right) P_{t_k}^{CF}, \quad P_0^{CF} = P_0, \\
\end{align*}
\]

Next, using the time update equations (6a).

4. Distributed Estimation Fusion Algorithm for Nonlinear Cost Function

4.1. Local Kalman Estimates. From the local sensor \( y_{i,t}^{(i)} \), the corresponding local Kalman state estimate \( \hat{x}_{i,t}^{(i)} = E(x_t | y_{[t,t_k]}^{(i)}) \) can be calculated using the continuous-discrete Kalman filter equations [15, 16]. Thus, we have the following:

\[
\begin{align*}
\hat{x}_t^{(i)} &= F_t \hat{x}_t^{(i)}, \quad t_{k-1} \leq t \leq t_k, \quad \hat{x}_{t_k-1}^{(i)} = \hat{x}_{t_k-1}^{(i)}, \\
p_t^{(i)} &= F_t p_t^{(i)} F_t^T + G_t Q_t G_t^T, \quad p_{t_k-1}^{(i)} = p_{t_k-1}^{(i)}, \\
\end{align*}
\]

Measurement update at time \( t = t_k \):

\[
\begin{align*}
\hat{x}_{t_k}^{(i)} &= \hat{x}_{t_k}^{(i)} + K_{t_k}^{(i)} \left( y_{t_k}^{(i)} - H_{t_k}^{(i)} \hat{x}_{t_k}^{(i)} \right), \quad \hat{x}_0^{(i)} = \hat{x}_0^{(i)}, \\
K_{t_k}^{(i)} &= P_{t_k}^{(i)} H_{t_k}^{(i)^T} \left( H_{t_k}^{(i)} P_{t_k}^{(i)} H_{t_k}^{(i)^T} + R_{t_k}^{(i)} \right)^{-1}, \\
P_{t_k}^{(i)} &= \left( I_n - K_{t_k}^{(i)} H_{t_k}^{(i)} \right) P_{t_k}^{(i)}, \quad P_0^{(i)} = P_0. \\
\end{align*}
\]

Next, we propose suboptimal distributed estimation fusion algorithm based on the local Kalman estimates and error covariance \( \hat{x}_0^{CF}, p_0^{CF} \), corresponding to the fusion center as well as \( i = 1, \ldots, L \).

4.2. Distributed Fusion Estimator. The proposed distributed algorithm is comprised of two stages: first, the local Kalman estimates \( \hat{x}_{i,t}^{(i)} \) are transformed to local optimal (in a mean-square sense) nonlinear estimates of an NCF \( \hat{x}_{i,t}^{(i)} \), \( t = 1, \ldots, L \), and, at the second stage, the transformed estimates \( \hat{x}_{i,t}^{(i),j} \), \( j = 1, \ldots, L \) are linearly fused based on the fusion formula with scalar weights [5, 6, 8].

The optimal local mean-square estimate of NCF \( z_t = f(x_t) \) based on the local sensor measurements \( y_{[t,t_k]}^{(i)} \) represents a conditional mean; that is,

\[
\hat{z}_t^{(i)} = E(z_t | y_{[t,t_k]}^{(i)}) = \int f(x_t) \rho \left( x_t | y_{[t,t_k]}^{(i)} \right) dx_t, \\
\]

where \( \rho \left( x_t | y_{[t,t_k]}^{(i)} \right) = N(\hat{x}_t^{(i)}, p_t^{(i)}) \) is a conditionally Gaussian probability density function with conditional mean \( \hat{x}_t^{(i)} = E(x_t | y_{[t,t_k]}^{(i)}) \) and covariance \( p_t^{(i)} \) determined by CF equations (6a) and (6b) for composite linear models (4) and (5), including all sensor measurements. Thus, estimate (7) represents the optimal minimum mean-square error (MMSE) continuous-discrete estimator

\[
\hat{z}_t^{opt} = \int f(x_t) N(\hat{x}_t^{CF}, p_t^{CF}) dx_t, \\
\]

which depends on the centralized Kalman estimate \( \hat{x}_t^{CF} \) and its error covariance \( p_t^{CF} \).

In distributed fusion, the fusion center tries to get the best estimate of an NCF with the processed data received from each local sensor \( y_{[t,t_k]}^{(i)} \), \( i = 1, \ldots, L \). In Sections 4 and 5, we propose the distributed estimation fusion algorithm based on the \( L \) local Kalman estimates \( \hat{x}_{i,t}^{(i)} = E(x_t | y_{[t,t_k]}^{(i)}) \), \( i = 1, \ldots, L \), which are available at the fusion center.
Mathematical Problems in Engineering

where scalar weights \( a_i^{(l)} \) in \( \mathbb{R} \) are defined as

\[
a_i = [a_i^{(1)} \ldots a_i^{(L)}] \in \mathbb{R}^L, \\
a_i = \frac{1}{L^2} \sum_{j=1}^L P_{j,i}^{-1}, \quad 1_L = [1 \ldots 1]^T \in \mathbb{R}^L,
\]

where

\[
P_{z,\tau} = \left[ \text{tr} \left( P_{z,\tau}^{(ij)} \right) \right]_{i,j=1}^L \in \mathbb{R}^{LxL}, \\
P_{z,\tau}^{(ij)} = \text{cov} \left( e_{z,\tau}^{(i)}, e_{z,\tau}^{(j)} \right),
\]

\[
e_{z,\tau}^{(i)} = z_t - \bar{z}_t = f(x_t) - \bar{z}_t,
\]

where \( i, j = 1, \ldots, L \).

Since the local NCF estimates \( \bar{z}_i^{(l)} \) in (10) represent a nonlinear transformation of the local state estimates and their error covariance, \( \bar{z}_i^{(l)} = \bar{z}_t^{(l)}(x_i^{(l)}, P_t^{(i)}) \), the cross-covariance \( P_{z,\tau}^{(ij)} \) in (12) depends on the local covariance \( P_t^{(i)} = \text{cov}(e_t^{(i)}, e_t^{(j)}), i = 1, \ldots, L \), determined by the Kalman equations (9a) and (9b), and the local cross-covariance \( P_{t}^{(ij)} = \text{cov}(e_t^{(i)}, e_t^{(j)}, i \neq j, \) which can be derived by the equations

\[
\begin{align*}
\text{Time update between measurements:} & \\
P_{t}^{(ij)} & = F_t^{(i)} P_{t}^{(ij)} + P_{t}^{(ij)} F_t^{(j)}^T + G_t Q_t G_t^T, \quad t_{k-1} \leq \tau \leq t_k \quad (13a) \\
P_{t}^{(ij)} & = P_{t}^{(i)}, \quad t_{k-1} = t_k, \\
\text{Measurement update at time } \tau = t_k: & \\
P_{t}^{(ij)} & = \left( I_t - K_t^{(i)} H_t^{(i)} \right) P_{t}^{(ij)} \left( I_t - K_t^{(i)} H_t^{(j)} \right)^T, \quad i, j = 1, \ldots, L, \quad i \neq j, \\
P_{t}^{(ij)} & = P_{t} \quad (13b)
\end{align*}
\]

where the filter gains \( K_t^{(i)}, i = 1, \ldots, L \), are determined by (9a) and (9b).

The derivation of (13a) and (13b) is given in the appendix.

4.3 Discussion

(1) The local error cross-covariances \( P_{z,\tau}^{(ij)} \) and weights \( a_i^{(j)} \) can be precomputed, because they do not depend on the sensor measurements \( y_t^{(i)}, i = 1, \ldots, L \), but only on the noise statistics \( Q_t, R_t^{(i)} \), the system matrices \( F_t, G_t, H_t^{(i)} \), the initial conditions \( \bar{z}_0, P_0 \), and the NCF \( z_t = f(x_t) \), which are the part of system models (1) and (2). Thus, once the measurement schedule has been settled, the real-time implementation of the distributed estimator requires only the computational of the local estimates \( \bar{z}_i^{(l)}, \bar{z}_t \) and the final fusion estimate \( \bar{z}_t^{\text{opt}} \) of an NCF.

(2) The implementation of the distributed estimator consists of two stages: off-line and on-line. The off-line stage is more complex than the off-line stage. This is because it requires the computation of the local cross-covariance and weights. However, it is not essential because this stage can be precomputed. The on-line stage (real-time implementation) requires the computation of only the local and fusion estimates. Therefore, the complexity of the on-line stage is not critical for the distributed estimator. However, to compute \( \bar{z}_t^{\text{opt}} \), the centralized estimator requires all sensor measurements together at each time instant \( k = 1, 2, \ldots \), whereas the distributed estimator computes \( \bar{z}_i^{(l)} \) and \( \bar{z}_t \) sequentially.

In the following, we discuss two computational algorithms for evaluation of fusion estimate (10) depending on the type of NCF.

5. Numerical Calculation of Estimates of Nonlinear Cost Function

5.1 Multivariate Polynomial Cost Function Recursive Procedure. Let a special NCF (2) represent an arbitrary multivariate polynomial function of the form

\[
z = f(x) = \sum_{0 \leq t_1 + \ldots + t_n \leq n} D_{t_1 t_2 \ldots t_n} x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}, \quad (14)
\]

\[
\ell_1, \ldots, \ell_n \geq 0.
\]

Then, the local estimate \( \bar{z}_i^{(l)} = \mathbf{E}(f(x_t) | y) \) has a closed-form solution because conditional expectation \( \mathbf{E}(f(x_t) | y) \) and cross-covariance \( P_{z,\tau}^{(ij)} \) depend on high-order moments \( \mathbf{E}(x_t^{\ell_1} x_t^{\ell_2} \ldots x_t^{\ell_n} | y) \) or \( \mathbf{E}(x_t^{\ell_1} x_t^{\ell_2} \ldots x_t^{\ell_n}) \) of a multivariate Gaussian distribution, which can be calculated explicitly in terms of first- and second-order moments \( \mathbf{E}(x_t) = \mathbf{E}(x_t^{\ell_1}) \) and \( P_{x,\tau}^{(ij)} \), \( i, j = 1, \ldots, n \), using recursive formulas [17–19]. For example,

\[
m_{t_1 t_2 \ldots t_n} = \sum_{i=2}^n P_{t_1}^{(i)} m_{t_1-1 t_2 \ldots t_n} + (\ell_1 - 1) P_{t_1}^{(i)} m_{t_1-2 t_2 \ldots t_n}
\]

with the first term vanishing when \( \ell_1 = 1 \) [19].

The following example illustrates the closed-form computational procedure.

Consider an arbitrary quadratic cost function

\[
z_t = f(x_t) = x_t^T \Omega_t x_t, \quad \Omega_t = \Omega_t, \quad \Omega_t > 0. \quad (16)
\]

Show that the optimal local estimate \( \bar{z}_i^{(l)} \) can be calculated explicitly in terms of a local state estimate and its error covariance. Using formula \( \mathbf{E}(x_t^T \Omega x_t) = \text{tr}(\Omega_t (P_t + m_m T)) \), \( m = \mathbf{E}(x), P = \text{cov}(x, x) \) [17], we obtain an optimal local estimate for the quadratic cost function

\[
\bar{z}_t^{(l)} = \mathbf{E}(x_t^T \Omega_t x_t | y) = \text{tr} \left\{ \Omega_t \left( P_t^{(i)} + \bar{x}_t \bar{x}_t^T \right) \right\}. \quad (17)
\]
Mathematical Problems in Engineering

where the local Kalman estimate and error covariance \((\hat{x}_i^{(0)}, P_i^{(0)})\) satisfy (9a) and (9b).

5.2. General Cost Function and Unscented Transformation. During the last decade, the unscented transformation (UT) has become a powerful approach for designing computationally effective algorithms for nonlinear models [10–12, 14, 20]. Following this, the procedure to calculate the best local estimate of an NCF (conditional mean)

\[
\hat{x}_i^{(0)} = \mathbb{E} \left[ f \left( x_i \right) \mid y_{\lfloor t, z_i \rfloor}^{(2)} \right]
\]

(18)

using the UT can be summarized as follows.

Generate the sigma points \(\{X_{i,j}\}_{j=0}^{2n}\) with corresponding weights \(\{W_{i,j}\}_{j=0}^{2n}\):

\[
X_{i,0}^{(0)} = x_i^{(0)}, \quad W_0 = \frac{\ell}{n + \ell},
\]

\[
X_{i,s}^{(0)} = x_i^{(0)} + \sqrt{(n + \ell) P_i^{(0)}}, \quad W_s = \frac{1}{2(n + \ell)}, \quad s = 1, \ldots, n,
\]

\[
X_{i,s+n}^{(0)} = x_i^{(0)} - \sqrt{(n + \ell) P_i^{(0)}}, \quad W_{s+n} = \frac{1}{2(n + \ell)},
\]

(19)

where \(\sqrt{P_i^{(0)}}\) is the \(s\)th column of the matrix square root of \(P_i^{(0)}\) and \(\ell\) is the scaling parameter influencing the spread of points in the space and thus the accuracy of the approximation [20]. Propagate each of these sigma points through the nonlinear function as

\[
\xi_{i,j}^{(s)} = f \left( X_{i,j}^{(s)} \right), \quad s = 0, 1, \ldots, 2n
\]

(20)

and the resulting best local estimate of the NCF is given as

\[
\hat{x}_i^{(0)} = \sum_{s=0}^{2n} W_{i,j}^{(s)} \xi_{i,j}^{(s)}, \quad i = 1, \ldots, L.
\]

(21)

Similar to (19)–(21), the local cross-covariance \(P_{i,j}^{(0)}\) in (12) can be calculated based on the UT. But, in a special case of a polynomial NCF (14), they are calculated for a multivariate Gaussian distribution of a composite random vector \(U^T_t = [x_t^T, \hat{x}_t^T, \hat{x}_t^{(0)T}]\) via the recursive formulas (15).

The best way to gain some insight into the proposed centralized and distributed estimators is to look at some theoretical examples. The comparison analysis of the proposed estimators will be demonstrated in the next section.

6. Theoretical Comparison of Estimators

6.1. Example 1: Estimation of Power of a Constant Scalar Unknown. Consider a simple example of an application of the obtained results. We estimate the quadratic cost function \(z = \theta^2\) of a random constant \(\theta \sim \mathcal{N}(0, \sigma_\theta^2)\), given two multiple discrete sensor measurements \(y_{i_k}^{(1)}\) and \(y_{i_k}^{(2)}\) of \(\theta\) corrupted by uncorrelated Gaussian white noises. The mixed continuous-discrete model describing this situation is

\[
\text{System: } x_t = 0, \quad t \geq 0, \quad x_0 \equiv \theta \sim \mathcal{N}(0, \sigma_\theta^2),
\]

\[
\text{Sensor 1: } y_{i_k}^{(1)} = x_t + w_{i_k}^{(1)}, \quad w_{i_k}^{(1)} \sim \mathcal{N}(0, r_1),
\]

(22)

\[
\text{Sensor 2: } y_{i_k}^{(2)} = x_t + w_{i_k}^{(2)}, \quad w_{i_k}^{(2)} \sim \mathcal{N}(0, r_2).
\]

Here, we derive precise equations for the MSEs for the proposed fusion estimators and demonstrate a comparative analysis.

6.1.1. Centralized Optimal Estimate of Quadratic Cost Function, \(\hat{z}_t^{\text{opt}}\). Using (17) at \(\Omega_k = 1\), the global optimal estimate of the quadratic cost function takes the form

\[
\hat{z}_t^{\text{opt}} = \mathbb{E} \left[ \theta^2 \mid y_{\lfloor t, z_t \rfloor}^{(1)} \right] = \int \theta^2 \mathcal{N} \left( \hat{\theta}_t^{\text{CF}}, P_t^{\text{CF}} \right) d\theta = P_t^{\text{CF}} \left( \hat{\theta}_t^{\text{CF}} \right)^2,
\]

(23)

where \(P_t^{\text{CF}} \equiv \hat{x}_t^{\text{CF}} = \mathbb{E}(x_t \mid y_{\lfloor t, z_t \rfloor}^{(1)})\) is the best global MMSE estimate of an unknown state \(x_t = \theta\) based on the overall sensor measurements \(y_{\lfloor t, z_t \rfloor}^{(1)}\) and \(y_{\lfloor t, z_t \rfloor}^{(2)}\) and \(P_t^{\text{CF}} = \mathbb{E}[(\theta - \hat{\theta}_t^{\text{CF}})^2]\) is its error variance. Using the continuous-discrete Kalman filter equations (6a) and (6b), we get

Time update between measurements:

\[
\hat{x}_i^{(0)} = 0, \quad t_{k-1} \leq \tau < t_k, \quad \hat{\theta}_i^{\text{CF}} = \hat{\theta}_{i-1}^{\text{CF}},
\]

(24a)

\[
\hat{p}_i^{\text{CF}} = 0, \quad P_{i+k-1}^{\text{CF}} = P_{i+k-1}^{\text{CF}},
\]

Measurement update at time \(\tau = t_k:\)

\[
\hat{\theta}_i^{\text{CF}} = \hat{\theta}_i^{\text{CF}} + K_i^{(1)} \left( y_{i_k}^{(1)} - \hat{\theta}_i^{\text{CF}} \right) + K_i^{(2)} \left( y_{i_k}^{(2)} - \hat{\theta}_i^{\text{CF}} \right),
\]

\[
\hat{\theta}_0^{\text{CF}} = 0,
\]

(24b)

\[
K_i^{(1)} = \frac{r_2 P_{i+k}^{\text{CF}}}{r_1 r_2 + (r_1 + r_2) P_{i+k}^{\text{CF}}},
\]

\[
K_i^{(2)} = \frac{r_1 P_{i+k}^{\text{CF}}}{r_1 r_2 + (r_1 + r_2) P_{i+k}^{\text{CF}}},
\]

\[
p_{i+k}^{\text{CF}} = \left( 1 - K_i^{(1)} - K_i^{(2)} \right) P_{i+k}^{\text{CF}}, \quad P_{i+k}^{\text{CF}} = \sigma_\theta^2.
\]

Using induction, we obtain the exact formula for the MSE

\[
p_{i+k}^{\text{CF}} = \mathbb{E} \left[ (\theta - \hat{\theta}_i^{\text{CF}})^2 \right] = \begin{cases} P_{i+k}^{\text{CF}}, & t_{k-1} \leq \tau < t_k, \\ P_{i+k}^{\text{CF}} & \tau = t_k, \end{cases}
\]

where

\[
p_{i+k}^{\text{CF}} = \frac{r_2 \sigma_\theta^2}{r + k \sigma_\theta^2}, \quad r = \frac{r_1 r_2}{r_1 + r_2}, \quad k = 0, 1, 2, \ldots.
\]

(25)
The estimation accuracy between the unknown power \( z = \theta^2 \) and its global fusion estimate

\[
\hat{z}_\tau^{opt} = \begin{cases} 
   p_{t_k}^{CF} + (\hat{\theta}_{t_{k-1}}^{CF})^2, & t_{k-1} \leq \tau < t_k, \\
   p_{t_k}^{CF} + (\hat{\theta}_{t_k}^{CF})^2, & \tau = t_k, 
\end{cases}
\]  

(26)

also can be measured in terms of the MSE \( P_\tau^{opt} = E[(\hat{\theta}^2 - \hat{z}_\tau^{opt})^2] \). We have

\[
P_\tau^{opt} = E \left[ (\hat{\theta}^2 - p_{t_k}^{CF} - (\hat{\theta}_{t_k}^{CF})^2)^2 \right] \\
= E(\hat{\theta}^4) + (p_{t_k}^{CF})^2 + E \left[ (\hat{\theta}_{t_k}^{CF})^4 \right] - 2p_{t_k}^{CF} E(\hat{\theta}^2) \\
- 2E \left[ \hat{\theta}^2 (\hat{\theta}_{t_k}^{CF})^2 \right] + 2p_{t_k}^{CF} E \left[ (\hat{\theta}_{t_k}^{CF})^4 \right],
\]

(27)

t_{k-1} \leq \tau \leq t_k.

Using the orthogonality property of the unbiased estimate \( \hat{\theta}_{t_k}^{CF} \) and the formulas for the fourth-order moments of a bivariate Gaussian random vector \( \left[ \theta \ \hat{\theta}_{t_k}^{CF} \right]^T \),

\[
E(\hat{\theta}^4) = 3(\sigma_0^2)^2, \quad E \left[ (\hat{\theta}_{t_k}^{CF})^4 \right] = 3 \text{Var}(\hat{\theta}_{t_k}^{CF})^2, \\
E \left[ \hat{\theta}^2 (\hat{\theta}_{t_k}^{CF})^2 \right] = \sigma_0^2 \text{Var}(\hat{\theta}_{t_k}^{CF}) + 2 \text{cov}(\theta, \hat{\theta}_{t_k}^{CF})^2,
\]

where

\[
\text{Var}(\hat{\theta}_{t_k}^{CF}) = \text{cov}(\theta, \hat{\theta}_{t_k}^{CF}) = \sigma_0^2 - p_{t_k}^{CF},
\]

(28)

we obtain

\[
P_\tau^{opt} = 2p_{t_k}^{CF} (2\sigma_0^2 - p_{t_k}^{CF}), \quad t_{k-1} \leq \tau \leq t_k.
\]

(29)

Taking into account (25), we get the exact MMSE for the centralized estimator; that is,

\[
P_\tau^{opt} = E \left[ (\hat{\theta}^2 - \hat{z}_\tau^{opt})^2 \right] = \begin{cases} 
   p_{t_{k-1}}^{opt}, & t_{k-1} \leq \tau < t_k, \\
   p_{t_k}^{opt}, & \tau = t_k, 
\end{cases}
\]

where

\[
P_{t_k}^{opt} = 2p_{t_k}^{CF} (2\sigma_0^2 - p_{t_k}^{CF}) = \frac{2r_i^2 \sigma_0^4 (r + 2k\sigma_0^2)}{(r + k\sigma_0^2)}, \quad r = \frac{r_1 r_2}{r_1 + r_2}, \quad k = 0, 1, 2, \ldots
\]

(30)

Together with the centralized estimator (26), we apply the distributed estimator developed in Section 4.

### 6.1.2. Distributed Fusion Estimate, \( \hat{z}^{(k)} \)

Using (9a) and (9b) and (13a) and (13b), the local estimates \( \hat{\theta}^{(0)}, \hat{\theta}^{(1)} \) and the formulas for the fourth-order moments of a bivariate Gaussian random vector \( \left[ \theta \ \hat{\theta}^{(k)} \right]^T \), error variances \( P_{\tau}^{(i)} = E(\epsilon^{(i)}_\tau)^2 \), and cross-covariance \( P_{\tau}^{(12)} = E(\epsilon^{(1)}_\tau \epsilon^{(2)}_\tau) \), \( i = 1, 2 \), are described by the following equations:

**Time update between measurements:**

\[
\begin{align*}
\hat{\theta}_{t_k}^{(0)} &= 0, \quad t_{k-1} \leq \tau \leq t_k, \\
\hat{\theta}_{t_k}^{(1)} &= \hat{\theta}_{t_k}^{(0)} - \sigma^{(0)}_0, \quad \hat{\theta}_{t_k}^{(2)} = \hat{\theta}_{t_k}^{(0)}, \quad t_{k-1} \leq \tau \leq t_k,
\end{align*}
\]

(31a)

**Measurement update at time \( \tau = t_k \):**

\[
\hat{\theta}_{t_k}^{(0)} = \hat{\theta}_{t_k}^{(0)} + K_{t_k}^{(0)} (\hat{\theta}_{t_k}^{(0)} - \hat{\theta}_{t_k}^{(0)}), \quad \hat{\theta}_{t_k}^{(0)} = 0,
\]

(31b)

The solution of (31a) and (31b) is given by

\[
P_{\tau}^{(ii)} = \begin{cases} 
   p_{t_{k-1}}^{(ii)}, & t_{k-1} \leq \tau < t_k, \\
   p_{t_k}^{(ii)}, & \tau = t_k,
\end{cases}
\]

(32)

\[
P_{\tau}^{(12)} = \begin{cases} 
   p_{t_{k-1}}^{(12)}, & t_{k-1} \leq \tau < t_k, \\
   p_{t_k}^{(12)}, & \tau = t_k,
\end{cases}
\]

Next, using formula (10), one can obtain two local estimates for the quadratic cost as \( \hat{z}^{(i)} = P_{t_{k-1}}^{(ii)} + (\hat{\theta}^{(i)}_{t_k})^2, i = 1, 2 \), where \( \hat{\theta}^{(1)}_{t_k} \) and \( \hat{\theta}^{(2)}_{t_k} \) are calculated by (31a) and (31b). In the second stage,
using fusion formulas (11) and (12), we obtain the distributed fusion estimate
\[ \hat{z}_{\text{fus}}^\tau = a_1^\tau \hat{x}_{\text{fus}}^\tau + a_2^\tau \hat{z}_\tau, \quad a_1^\tau + a_2^\tau = 1, \]
where
\[ a_1^\tau = \frac{p_{z,\tau}^{(11)} - p_{z,\tau}^{(12)}}{p_{z,\tau}^{(11)} - 2 p_{z,\tau}^{(12)} + p_{z,\tau}^{(22)}}, \]
\[ a_2^\tau = \frac{p_{z,\tau}^{(11)} - p_{z,\tau}^{(12)}}{p_{z,\tau}^{(11)} - 2 p_{z,\tau}^{(12)} + p_{z,\tau}^{(22)}}, \]
\[ p_{z,x,\tau}^{(ij)} = \text{cov}(\tilde{x}_{\tau}^{(i)}, \tilde{x}_{\tau}^{(j)}), \quad \tilde{x}_{\tau}^{(i)}, \tilde{x}_{\tau}^{(j)} = \theta^2 - \hat{z}_{\tau}, \quad i, j = 1, 2. \]

Calculating the cross-covariance \( P_{z,x,\tau}^{(ij)} \) based on the formulas for high-order moments of a Gaussian distribution (28), we get
\[ P_{z,x,\tau}^{(ij)} = E[(\theta^2 - \hat{z}_{\tau}^{(i)})(\theta^2 - \hat{z}_{\tau}^{(j)})] = 2P_{\tau}^{(12)} + 2P_{\tau}^{(22)} + 2a_1^\tau a_2^\tau P_{\tau}^{(11)}, \]
where
\[ P_{\tau}^{(ij)} = E[(\theta^2 - \hat{z}_{\tau}^{(i)})(\theta^2 - \hat{z}_{\tau}^{(j)})] = 2P_{\tau}^{(12)} + 2P_{\tau}^{(22)} + 2a_1^\tau a_2^\tau P_{\tau}^{(11)}, \]
\[ P_{z,x,\tau}^{(ij)} = \text{cov}(\tilde{x}_{\tau}^{(i)}, \tilde{x}_{\tau}^{(j)}), \quad \tilde{x}_{\tau}^{(i)}, \tilde{x}_{\tau}^{(j)} = \theta^2 - \hat{z}_{\tau}, \quad i, j = 1, 2. \]

Finally, the overall MSE \( P_{\tau}^{\text{fus}} \) can be evaluated as
\[ P_{\tau}^{\text{fus}} = E[(\theta^2 - \hat{z}_{\tau}^{\text{fus}})^2] = \begin{cases} P_{\tau}^{\text{fus}}, & t_{k-1} \leq \tau < t_k, \\ P_{\tau}^{\text{fus}}, & \tau = t_k, \end{cases} \]
where
\[ P_{\tau}^{\text{fus}} = (a_1^\tau)^2 P_{z,x,\tau}^{(11)} + (a_1^\tau a_2^\tau)^2 P_{z,x,\tau}^{(12)} + 2a_1^\tau a_2^\tau P_{\tau}^{(11)} P_{\tau}^{(12)}, \]
\[ k = 0, 1, 2, \ldots. \]

Here, the scalar weights \( a_1^\tau \) and \( a_2^\tau \) and cross-covariance \( P_{z,x,\tau}^{(ij)} \), \( i, j = 1, 2 \), are determined by (32)–(34) at \( \tau = t_k \), \( k = 1, 2, \ldots. \)

6.1.3. Comparative Analysis of Centralized and Distributed Estimators. The MSE is an important value that can be used to reflect the accuracy of NCF estimation. The exact MSEs \( P_{\tau}^{\text{opt}} \) and \( P_{\tau}^{\text{fus}} \) are illustrated in Figure 1 for \( \sigma_0^2 = 1, r_1 = 2, r_2 = 3 \). Not surprisingly, Figure 1 illustrates that the centralized estimator exhibits a performance that is completely superior to the distributed estimator; that is, \( P_{\tau}^{\text{opt}} < P_{\tau}^{\text{fus}} \). From Figure 1, we also observe that the difference between two fusion estimators is negligible for steady-state regimes \( k \gg 1 \). Thus, for the example, application of the distributed estimator can produce good results in real-time processing requirements.

\[ \hat{z}_{\tau}^{\text{opt}} = E(x_{\tau}^2 | y_{[t_1,t_k]}), \]
\[ = \int x^2 N(x_{\tau}^{\text{CF}}, P_{\tau}^{\text{CF}}) dx = P_{\tau}^{\text{CF}} + (x_{\tau}^{\text{CF}})^2, \]
\[ t_{k-1} \leq \tau \leq t_k, \]

where the estimate \( x_{\tau}^{\text{CF}} \) and its error variance \( P_{\tau}^{\text{CF}} \) are described by the continuous-discrete Kalman filter equations (6a) and (6b)

\[ \hat{x}_{\tau}^{\text{CF}} = a\hat{x}_{\tau}^{\text{CF}}, \quad t_{k-1} \leq \tau \leq t_k, \quad \hat{x}_{\tau}^{\text{CF}} = \hat{x}_{\tau}^{\text{CF}}, \]
\[ \hat{P}_{\tau}^{\text{CF}} = 2a\hat{P}_{\tau}^{\text{CF}} + q, \quad \hat{P}_{\tau}^{\text{CF}} = \hat{P}_{\tau}^{\text{CF}}. \]

Figure 1: MSEs of fusion estimators for quadratic cost function \( z = \theta^2 \).

6.2. Example 2: Estimation of Power of a Scalar Signal. Let the scalar signal \( x_t \) with two sensors be described by
\[ \dot{x}_t = ax_t + v_t, \quad a < 0, \ t \in [0, T_k], \]
\[ \dot{x}_t = x_t^{(i)} + w_t^{(i)}, \quad i = 1, 2, \]

where \( v_t \) is zero-mean white Gaussian noise with intensity \( q \) and \( w_t^{(i)} \sim \mathcal{N}(0, r_i) \) are uncorrelated white Gaussian sequences. Let \( x_0 \sim \mathcal{N}(0, \sigma_0^2) \), and an NCF represents power of the signal; that is, \( z_t = f(x_t) = x_t^2 \).

In a similar way as in Example 1, we can derive equations for MSEs for the proposed estimators.

6.2.1. Centralized Optimal Estimate of Quadratic Cost Function, \( \hat{z}_{\tau}^{\text{opt}} \). The global MMSE fusion estimate of the power of signal takes the form
\[ \hat{z}_{\tau}^{\text{opt}} = E(x_{\tau}^2 | y_{[t_1,t_k]}), \]
\[ = \int x^2 N(x_{\tau}^{\text{CF}}, P_{\tau}^{\text{CF}}) dx = P_{\tau}^{\text{CF}} + (x_{\tau}^{\text{CF}})^2, \]
\[ t_{k-1} \leq \tau \leq t_k, \]
Measurement update at time $\tau = t_k$:

$$\hat{x}_{t_k}^{CF} = \hat{x}_{t_k}^{CF-} + K_{t_k}^{(1)} (y_{t_k}^{(1)} - \hat{x}_{t_k}^{CF-}) + K_{t_k}^{(2)} (y_{t_k}^{(2)} - \hat{x}_{t_k}^{CF-}),$$

$$\hat{x}_{t_k}^{CF-} = \hat{x}_0,$$

$$K_{t_k}^{(1)} = \frac{r_2 p_{t_k}^{CF-}}{r_1 r_2 + (r_1 + r_2) p_{t_k}^{CF-}},$$

$$K_{t_k}^{(2)} = \frac{r_1 p_{t_k}^{CF-}}{r_1 r_2 + (r_1 + r_2) p_{t_k}^{CF-}},$$

$$p_{t_k}^{CF} = (1 - K_{t_k}^{(1)} - K_{t_k}^{(2)}) p_{t_k}^{CF-}, \quad p_{t_k}^{CF-} = \sigma_0^2.$$(38b)

Solving (38a) and (38b) for the error variance, we get

$$p_{t_k}^{CF} = \left( p_{t_k}^{CF-} + \frac{q}{2a} \right) e^{2a(r-t_{k-1})} - \frac{q}{2a}, \quad t_{k-1} \leq \tau \leq t_k,$$

$$p_{t_k}^{CF} = \frac{r_1 r_2 p_{t_k}^{CF-}}{r_1 r_2 + (r_1 + r_2) p_{t_k}^{CF-}}, \quad k = 0, 1, \ldots; \quad p_{t_k}^{CF-} = \sigma_0^2.$$(39)

To find the overall MSE $p_{t_k}^{opt} = E[(x_n^2 - z_{opt}^2)^2]$, we use the same way as in the derivation of formula (29). We obtain

$$p_{t_k}^{opt} = 2p_{t_k}^{CF} (2\alpha_{2,\tau} - p_{t_k}^{CF}), \quad \alpha_{2,\tau} = E(\hat{x}_0^2),$$

$$t_{k-1} \leq \tau \leq t_k,$$(40)

where the second-order moment of the signal $\alpha_{2,\tau}$ satisfies the Lyapunov equation

$$\alpha_{2,\tau} = 2a\alpha_{2,\tau} + q, \quad \tau \geq 0, \quad \alpha_{2,0} = E(\hat{x}_0^2) = \sigma_0^2 + \hat{x}_0^2.$$(41)

Finally, using relation (40) between $p_{t_k}^{CF}$ and $p_{t_k}^{opt}$, we get

$$p_{t_k}^{opt} = \begin{cases} 2p_{t_k}^{CF} (2\alpha_{2,\tau} - p_{t_k}^{CF}), & t_{k-1} \leq \tau < t_k, \\ 2p_{t_k}^{CF} (2\alpha_{2,t_k} - p_{t_k}^{CF}), & \tau = t_k, \end{cases}$$

where

$$\alpha_{2,\tau} = \left( \alpha_{2,0} + \frac{q}{2a} \right) e^{2a\tau} - \frac{q}{2a}, \quad \tau \geq 0.$$(42)

Together with centralized estimator (37), we apply the distributed estimator.

### 6.2.2. Distributed Fusion Estimate, $\hat{z}_\tau^{fus}$. The distributed fusion equations for the example follow the same basic pattern as in Section 6.1.2. The local estimates $\hat{x}_\tau^{(i)} = \text{E}(x_\tau | y_{t_i}^{(i)})$, corresponding error variances $p_{t_i}^{(i)} = \text{E}(e_i^{(i)})$, and cross-covariance $p_{t_i}^{(i)(j)} = \text{E}(e_i^{(i)}e_j^{(j)})$ are described by the following:

**Time update between measurements:**

$$\hat{x}_{\tau}^{(i)} = a \hat{x}_{\tau-1}^{(i)}, \quad t_{k-1} \leq \tau < t_k, \quad \hat{x}_{\tau-1}^{(i)} = \hat{x}_{\tau-1}^{(i)} - a \hat{z}_{\tau-1}^{(i)} = \hat{x}_{\tau-1}^{(i)}, \quad p_{t_i}^{(i)(i)} = 2a p_{t_i-1}^{(i)(i)} + q, \quad i = 1, 2,$$

$$p_{t_i}^{(i)(1)} = 2a p_{t_i-1}^{(i)(1)} + q, \quad t_{k-1} \leq \tau < t_k, \quad p_{t_i}^{(i)(2)} = p_{t_i-1}^{(i)(2)}, \quad i = 1, 2.$$ (44a)

**Measurement update at time $\tau = t_k$:**

$$\hat{x}_{t_k}^{(i)} = \hat{x}_{t_k}^{(i)} - K_{t_k}^{(i)} \left( y_{t_k}^{(i)} - \hat{z}_{t_k}^{(i)} \right), \quad \hat{x}_0 = \hat{x}_0,$$

$$K_{t_k}^{(i)} = \frac{p_{t_k}^{(i)}}{r_i + p_{t_k}^{(i)}},$$

$$p_{t_k}^{(i)} = (1 - K_{t_k}^{(i)}) p_{t_k}^{(i)} - \sigma_0^2,$$

$$p_{t_k}^{(i)} = \left( 1 - K_{t_k}^{(i)} \right) \left( 1 - K_{t_k}^{(i)} \right) p_{t_k}^{(i)} + p_{t_k}^{(i)} - \sigma_0^2.$$ (44b)

The solution of (44a) and (44b) is given by

$$p_{t_k}^{(i)} = \begin{cases} p_{t_{k-1}}^{(i)} + \frac{q}{2a} e^{2a(r-t_{k-1})} - \frac{q}{2a}, & \frac{r_i}{r_i + k \sigma_0^2}, \\ p_{t_{k}}^{(i)}, & \tau = t_k, \end{cases}$$

where

$$p_{t_k}^{(i)} = \begin{cases} p_{t_{k-1}}^{(i)} + \frac{q}{2a} e^{2a(r-t_{k-1})} - \frac{q}{2a}, & \frac{r_i}{r_i + k \sigma_0^2}, \\ p_{t_{k+1}}^{(i)}, & \tau = t_k, \end{cases}$$

$$p_{t_k}^{(i)} = \frac{r_i}{r_i + k \sigma_0^2} e^{2a(r-t_{k-1})} - \frac{q}{2a}, \quad i = 1, 2,$$

$$p_{t_k}^{(i)(1)} = \frac{r_i}{r_i + k \sigma_0^2} e^{2a(r-t_{k-1})} - \frac{q}{2a}, \quad \hat{x}_{\tau}^{(i)} = \hat{x}_{\tau}^{(i)} - a \hat{z}_{\tau}^{(i)}, \quad a \hat{z}_{\tau}^{(i)}, \quad \hat{x}_{\tau}^{(i)} = \hat{x}_{\tau}^{(i)} - a \hat{z}_{\tau}^{(i)}.$$(45)

Next, two local estimates for the power of signal $z_i = x_i^2$ take the form $\hat{z}_{\tau}^{(i)} = p_{t_k}^{(i)(i)} (\hat{x}_{\tau}^{(i)} + \hat{z}_{\tau}^{(i)}), i = 1, 2$. Combining $\hat{z}_{\tau}^{(1)}$ and $\hat{z}_{\tau}^{(2)}$ based on (II), we obtain the distributed fusion estimate

$$\hat{z}_{\tau}^{fus} = e^{(i)} \hat{z}_{\tau}^{(1)} + e^{(2)} \hat{z}_{\tau}^{(2)}, \quad e^{(1)} + e^{(2)} = 1,$$

$$t_{k-1} \leq \tau \leq t_k.\]
where

\[ a_t^{(1)} = \frac{p_{z,z}^{(22)} - p_{z,z}^{(12)}}{p_{z,z}^{(11)} - 2p_{z,z}^{(12)} + p_{z,z}^{(22)}}, \]

\[ a_t^{(2)} = \frac{p_{z,z}^{(22)} - p_{z,z}^{(12)}}{p_{z,z}^{(11)} - 2p_{z,z}^{(12)} + p_{z,z}^{(22)}}, \]

with the covariance \( P_{z,z}^{ij} \) which is calculated as

\[ P_{z,z}^{(i)} = E \left[ (x_t^2 - z_{t}^{(i)}) (x_t^2 - z_{t}^{(i)})^T \right] = 4S_i P_{fus}^{(i)} + 4m_i^2 P_{fus}^{(i)}, \]

\[ P_{z,z}^{(12)} = E \left[ (x_t^2 - z_{t}^{(1)}) (x_t^2 - z_{t}^{(2)}) \right] = 4m_t^2 P_{z,z}^{(12)} + 4P_{z,z}^{(11)} + 4P_{z,z}^{(22)} \]

\[ + 4P_{z,z}^{(12)} (S_t - P_{z,z}^{(1)}) - P_{z,z}^{(22)}), \]

\[ \tau_{k-1} \leq \tau \leq \tau_k, \]

where

\[ m_t = E(x_t) = m_{t_k} e^{\alpha(t-t_{k-1})}, \quad k = 1, 2, \ldots; \quad m_0 = x_0, \]

\[ S_t = \text{Var}(x_t) = \left( S_{t_{k-1}} + \frac{q}{2a} \right) e^{2\alpha(t-t_{k-1})} - \frac{q}{2a}, \]

\[ S_0 = \sigma_0^2. \]

Finally, the overall MSE \( P_{fus}^{\tau} \) of the fusion estimate \( z_{fus}^{\tau} \) is evaluated as

\[ P_{fus}^{\tau} = E \left[ (x_t^2 - z_{fus}^{\tau})^2 \right] = \left\{ \begin{array}{ll}
P_{fus}^{\tau}, & \tau_{k-1} \leq \tau \leq \tau_k, \\
P_{fus}^{\tau_k}, & \tau = \tau_k,
\end{array} \right. \]

where

\[ P_{fus}^{\tau} = (a_t^{(1)})^2 P_{z,z}^{(1)} + (a_t^{(2)})^2 P_{z,z}^{(2)} + 2a_t^{(1)} a_t^{(2)} P_{z,z}^{(12)}, \]

Here, the weights \( a_t^{(i)} \) and cross-covariance \( P_{z,z}^{ij} \) are determined by (46) and (47), respectively.

6.2.3. Comparative Analysis of Centralized and Distributed Estimators. The model parameters are subjected to \( a = -2, \)

\( q = 10, r_1 = 0.2, r_2 = 0.3, x_{0} \sim \mathcal{N}(1, 1), t_k - t_{k-1} = 0.1, \)

\( k = 1, 2, \ldots, 20. \) Figure 2 illustrates the MSEs of the power of signals \( P_t^{\text{opt}} \) and \( P_t^{\text{fus}}. \) As we can see in Figure 2, the centralized estimator \( z_{t}^{\text{opt}} \) is better than the distributed one \( z_{t}^{\text{fus}}; \) that is, \( P_t^{\text{opt}} < P_t^{\text{fus}}. \) However, the difference between \( P_t^{\text{opt}} \) and \( P_t^{\text{fus}} \) is negligible. The relative error \( \Delta_t = [(P_t^{\text{fus}} - P_t^{\text{opt}})/P_t^{\text{opt}}] \) [100% within the observation period \( t_k \in [0; 2] \) is about 6%. For this reason, the distributed estimator for NCFs is suitable for real implementation in multisensory systems.

7. Application of Fusion Algorithms

A comparative experimental analysis of the proposed estimators is considered for the motion of unmanned marine prober (UMP). In a marine inspection environment, UMP systems are often considered because they offer the benefits of convenience and human safety.

Assume a scenario in which the UMP detected an oil tanker accident, from which oil has spread out on a surface of the water without the influence of wind. As an initial action, the UMP estimates the length of a contour of the oil spread (Figure 3).

To control the size of a surface, the UMP needs to compute the distance from the oil tanker \( d_t \) at every time instance representing an NCF

\[ d_t = f(x_t) = \sqrt{x_{1,t}^2 + x_{2,t}^2}, \quad x_t = [x_{1,t} \ x_{2,t}]^T. \]

where \( x_{1,t} \) and \( x_{2,t} \) are coordinates of UMP.

Here, we verify the proposed fusion estimators using a linearized model of UMP [3]:

\[ x_{1,t+1} = x_{1,t} - 2x_{2,t} + v_{1,t}, \]

\[ x_{2,t+1} = x_{1,t} - x_{2,t} + v_{2,t}, \]

where \( v_{1,t} \) and \( v_{2,t} \) are uncorrelated zero-mean white Gaussian noises with intensities \( q_1 = q_2 = 0.1, t \in [0; 3], x_{1,0} \sim \mathcal{N}(20; 0.2), \) and \( x_{2,0} \sim \mathcal{N}(0; 0.2). \)

Next, with the help of systemic sensors such as ultrasonic sensors, sonar, radar, or GPS, the UMP measures the relative coordinates \( x_{1,t} \) and \( x_{2,t} \) from the oil tanker, respectively. Then, the measurement model for the UMP is given by

\[ y_{1,t} = x_{1,t} + w_{1,t}, \quad y_{2,t} = x_{2,t} + w_{2,t}, \]

where \( w_{1,t} \) and \( w_{2,t} \) are uncorrelated zero-mean white Gaussian sequences with intensities \( r_1 = r_2 = 0.1. \)

Since the NCF is nonlinear, we apply the UT to calculate the local estimates \( z_{t}^{(i)} \) and fusion estimates \( z_{t}^{\text{opt}} \) and \( z_{t}^{\text{fus}}. \) The time update differential equations were solved by the
Runge-Kutta scheme of the fourth order with the integration step \( \Delta t = 0.01 \). To compare the MSEs \( P_{t}^{\text{opt}} \) and \( P_{t}^{\text{fus}} \), the Monte-Carlo method with 1000 runs was performed. Figure 4 illustrates the time histories of the MSEs for the both estimators.

As in Figure 4, the centralized estimate \( \hat{z}_{t}^{\text{opt}} \) has the best performance due to the lowest value of the MSE \( P_{t}^{\text{opt}} < P_{t}^{\text{fus}} \). As a result, we can confirm that we have verified that the decentralized estimator is more suitable for distributed processing in a multisensory environment.

8. Conclusion

In this paper, we derive a new centralized and decentralized estimator for nonlinear cost functions in mixed multisensor continuous-discrete stochastic systems. Computational approaches to their designing in practice are offered. Particular emphasis is given to a closed-form recursive procedure for a polynomial cost functions. The estimation accuracies of the proposed estimators are studied. In general, the centralized fusion estimator is considered as the most accurate, but, by the results of simulations with theoretical and real examples, the decentralized estimator demonstrates a reasonable accuracy. Furthermore, due to inherent drawbacks of centralized processing, the decentralized estimator may be more preferable in multisensory environment.

During the last decades, there has been extensive interest in the study of a class of physical systems modeled by hybrid system dynamics known as Markovian jump systems [21–23]. As a generalization of the obtained results for mixed continuous-discrete stochastic systems, we would like to point out that it is possible to extend the main results to Markovian jump systems.

Appendix

The derivation of the equation for cross-covariance (13a) and cross-covariance (13b) is given as follows.

The Kalman equations (1) and (9a) and (9b) yield the linear differential difference equations for the local error \( e^{(i)}_{\tau} = x^{(i)}_{\tau} - \tilde{x}^{(i)}_{\tau} \)

\[
\begin{align*}
  e^{(i)}_{\tau} &= \dot{x}^{(i)}_{\tau} - \dot{\tilde{x}}^{(i)}_{\tau} = F_{\tau} e^{(i)}_{\tau} + G_{\tau} \nu_{\tau}, \quad t_{k-1} \leq \tau \leq t_{k}, \\
  e^{(i)}_{x_{k}} &= x^{(i)}_{x_{k}} - \tilde{x}^{(i)}_{x_{k}} \\
  &= x^{(i)}_{x_{k}} - \tilde{x}^{(i)}_{x_{k}} - K_{k}^{(i)} \left[ H_{x_{k}}^{(i)} x^{(i)}_{x_{k}} + w^{(i)}_{x_{k}} - H_{x_{k}}^{(i)} \tilde{x}^{(i)}_{x_{k}} \right] \quad (A.1) \\
  &= \left( I_{n} - K_{k}^{(i)} H_{x_{k}}^{(i)} \right) e^{(i)}_{x_{k}} - K_{k}^{(i)} w^{(i)}_{x_{k}}, \\
  e^{(i)}_{w_{k}} &= x^{(i)}_{w_{k}} - \tilde{x}^{(i)}_{w_{k}}.
\end{align*}
\]
Consequently,

\[
\frac{d}{dt} \mathbf{P}_{ij} = F_{ij} \mathbf{P}_{ij} F_{ij}^T + G_t Q_t G_t^T + \mathbf{P}_{ij} F_{ij}^T + G_t Q_t G_t^T
\]

(A.2)

Taking into account that \(e_{ik}^{(j)}\) and \(e_{kj}^{(j)}\) do not depend on measurements \(y_{ik}^{(j)}\) and \(y_{kj}^{(j)}\), respectively, and white noises \(w_{ik}^{(j)}\) and \(w_{kj}^{(j)}\) are uncorrelated at \(i \neq j\), (A.3) yields linear recursive (13a) and (13b) for \(P_{ij}^{(j)}\).

This completes the derivation of (13a) and (13b).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Incheon National University Research Grant in 2014-2015 and in part by the National Research Foundation of Korea of the Ministry of Education, Science and Technology under Grant no. NRF-2012R1A1A2000679.

References


