Research Article

On the Octonionic Inclined Curves in the 8-Dimensional Euclidean Space

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We describe octonionic inclined curves and harmonic curvatures for the octonionic curves. We give characterizations for an octonionic curve to be an octonionic inclined curve. And finally, we obtain some characterisations for the octonionic inclined curves in terms of the harmonic curvatures.

1. Introduction

Hamilton [1] revealed quaternion in 1843 as an explanation of group construction and also performed to mechanics in three-dimensional space. For quaternions, same features are provided as complex numbers with the discrimination that the commutative rule is not effective in their case. The octonions [2, 3] form the widest normed algebra after the algebra of real numbers, complex numbers, and quaternions. The octonions are also known as Cayley Graves numbers and also have an algebraic structure defined on the eight-dimensional real vector space in such a way that two octonions can be added, multiplied, and divided with the fact that multiplication is neither commutative nor associative. Inclined curves in Euclidean n space were studied by Özdamar and Hacısalihoğlu [4]. The Serret-Frenet formulae for an octonionic curves in $\mathbb{R}^7$ and $\mathbb{R}^8$ are given by Bektaş and Yüce [5]. But, to our knowledge, there has been no study on the octonionic inclined curves in the eight-dimensional Euclidean space. Such a study is the object of this paper. Our main aim in the present work is to study the differential geometry of a smooth curve in the eight-dimensional Euclidean space.

2. Preliminaries

The octonions can be thought of as octal of real numbers. Octonion is a real linear combination of the unit octonions: $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ where e_0 is the scalar or real

element; it may be assimilated with the real number 1. That is, every real octonion (in this study we use octonion instead of real octonion, since two concepts are the same) A can be expressed in the manner $A = \sum_{i=0}^7 a_i e_i$. Hence an octonion can be decomposed in terms of its scalar (S_A) and vector (V_A) parts as $S_A = a_0$ and $V_A = \sum_{i=1}^7 a_i e_i$. Addition and extraction of octonions are made by adding and quarrying corresponding terms and thereby their factors, like quaternions. Multiplication is more complex. Multiplication is distributive over addition, so the product of two octonions can be calculated by summing the product of all the terms, again like quaternions. The product of each term can be given by multiplication of the coefficients and a multiplication table of the unit octonions, like this one [3, 6]:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Most of diagonal elements of the table are antisymmetric, making it almost a skew symmetric matrix except for the elements on the main diagonal, the row, and the column for

which $\mathbf{e}_0$ is an operand. The table can be epitomized by the relations: [7], $e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k$, where is a completely antisymmetric tensor with value +1 when $ijk = 123, 145, 176, 246, 257, 347, 365$, and $e_0 e_i = e_i e_0 = e_i$, with e_0 the scalar element, and $i, j, k = 1, \dots, 7$.

The overhead declaration though is not unique but is only one of 480 possible declarations for octonion multiplication. The others can be acquired by permuting the nonscalar elements, so can be noted to have different bases. Alternately they can be acquired by fixing the product rule for a few terms and deducing the rest from the other properties of the octonions. The 480 different algebras are isomorphic, so are in practice identical, and there is rarely a need to consider which particular multiplication rule is used [8, 9]. We denote the set of octonion

$$O = \left\{ A \mid A = \sum_{i=0}^7 a_i e_i, \quad e_0 e_i = e_i e_0 = e_i, \right. \\ \left. e_0 = +1, \quad e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k \right\}, \quad (2)$$

where $ijk = 123, 145, 176, 246, 257, 347, 365$. δ_{ij} is Kronecker delta. ε_{ijk} is completely antisymmetric tensor with value +1. O is spanned by +1 and the +1 imaginary units $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ each with square -1 , so that $O = \mathbb{R} \oplus \mathbb{R}^7$ [10]. Then octonions are isomorphic to $\mathbb{R}^8$ [11]. Just as the complex numbers and quaternions could be used to describe $\mathbb{R}^2$ and $\mathbb{R}^4$, the octonions may be used to describe points in $\mathbb{R}^8$ using the obvious identification [12]. Let $A = \sum_{i=0}^7 a_i e_i$ be an octonion. If $a_0 = 0$, then A is called a spatial (pure) octonion (or A is called a spatial (pure) octonion whenever $A + \bar{A} = 0$). Before we define the octonionic product, we give information about vector product in $\mathbb{R}^7$. Seven-dimensional Euclidean space and three-dimensional Euclidean space are the only Euclidean spaces to have a vector product. We know that we can express an octonion as the sum of a real part, S_A and a pure part, V_A in $\mathbb{R}^7$. So we get $A = S_A + V_A$. This guides to a vector product on $\mathbb{R}^7$ described by $V_A \wedge V_B = V_A V_B + \langle V_A, V_B \rangle$ [13]. Moreover this is given [13] by $V_A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $V_B = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$.

Let $\zeta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \wedge (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$, then,

$$\begin{aligned} \zeta = & (a_2 b_3 - a_3 b_2 + a_4 b_5 - a_5 b_4 + a_7 b_6 - a_6 b_7, \\ & a_3 b_1 - a_1 b_3 + a_4 b_6 - a_6 b_4 + a_5 b_7 - a_7 b_5, \\ & a_1 b_2 - a_2 b_1 + a_4 b_7 - a_7 b_4 + a_6 b_5 - a_5 b_6, \\ & a_5 b_1 - a_1 b_5 + a_6 b_2 - a_2 b_6 + a_7 b_3 - a_3 b_7, \\ & a_1 b_4 - a_4 b_1 + a_3 b_6 - a_6 b_3 + a_7 b_2 - a_2 b_7, \\ & a_1 b_7 - a_7 b_1 + a_2 b_4 - a_4 b_2 + a_5 b_3 - a_3 b_5, \\ & a_2 b_5 - a_5 b_2 + a_3 b_4 - a_4 b_3 + a_6 b_1 - a_1 b_6). \end{aligned} \quad (3)$$

The vector product in $\mathbb{R}^7$ has all the properties expect Jacobi identity. So generally for pure octonions $V_A \wedge (V_B \wedge V_C) + V_B \wedge (V_C \wedge V_A) + V_C \wedge (V_A \wedge V_B) \neq 0$ or equivalently

the triple vector product identity fails $V_A \wedge (V_B \wedge V_C) \neq \langle V_A, V_C \rangle V_B - \langle V_A, V_B \rangle V_C$. The identities which it does satisfy are as follows.

Theorem 1. Let V_A, V_B , and V_C be spatial octonions and let θ be the angle between V_A and V_B . If c is an random real number then the succeeding identicalnesses run for vector products in $\mathbb{R}^7$ [13]. Consider the following:

- (i) $V_A \wedge (V_B + V_C) = V_A \wedge V_B + V_A \wedge V_C$;
- (ii) $V_A \wedge V_A = 0$;
- (iii) $V_A \wedge V_B = -V_B \wedge V_A$;
- (iv) $\langle V_A, V_A \wedge V_B \rangle = 0$;
- (v) $\|V_A \wedge V_B\| = \|V_A\| \|V_B\| \sin \theta$;
- (vi) $\langle V_A \wedge V_B, V_C \rangle = \langle V_B \wedge V_C, V_A \rangle = \langle V_C \wedge V_A, V_B \rangle$;
- (vii) $V_A \wedge (V_A \wedge V_B) = \langle V_A, V_B \rangle V_A - \langle V_A, V_A \rangle V_B$.

If we take widely information about cross product in $\mathbb{R}^7$, we can read the references [14]. Now we can describe octonion product. The octonionic product of two octonions is served as follows [15]:

$$A \times B = S_A S_B - \langle V_A, V_B \rangle + S_A V_B + S_B V_A + V_A \wedge V_B, \quad \forall A, B \in O, \quad (4)$$

where we have used the dot and cross products in $\mathbb{R}^7$. $\bar{A}$ is called conjugate of A and described as noted below

$$\bar{A} = S_A - V_A = a_0 - \sum_{i=1}^7 a_i e_i, \quad (5)$$

where we have used the conjugates of basis elements as $\bar{e}_0 = e_0$ and $\bar{e}_j = -e_j$ ($j = 1, \dots, 7$). The inner product of octonions qualifies as follows:

$$\langle A, B \rangle : O \times O \longrightarrow \mathbb{R}, \quad (A, B) \longrightarrow \langle A, B \rangle = \frac{1}{2} (A \times \bar{B} + B \times \bar{A}) = \sum_{i=0}^7 a_i b_i. \quad (6)$$

Hence it is called the octonionic inner product. The norm of an octonion A is defined by

$$\|A\| = \sqrt{A \times \bar{A}} = \sqrt{\sum_{i=0}^7 a_i^2}. \quad (7)$$

If $\|A\| = 1$, then A is called a unit octonion. The only octonion with norm 0 is 0, and every nonzero octonion has a unique inverse; namely [16],

$$A^{-1} = \frac{\bar{A}}{\|A\|^2}. \quad (8)$$

For all the normed division algebras, the norm provides the identicalness [16]

$$\|A \times B\| = \|A\| \|B\|. \quad (9)$$

If we take $n = 7, 8$ in the study named “A characterization of inclined curves in Euclidean n space,” we can obtain the following definitions.

Definition 2. Let $\gamma : I \rightarrow \mathbb{R}^7$ be a curve in $\mathbb{R}^7$ with the arc length parameter s and let u be a unit constant vector of $\mathbb{R}^7$. For all $s \in I$, if

$$\langle \gamma'(s), u \rangle = \cos \varphi = \text{constant}, \quad \varphi \neq \frac{\pi}{2} \quad (10)$$

then the curve is called an inclined curve in $\mathbb{R}^7$, where $\gamma'(s)$ is the unit tangent vector to the curve γ at its point $\gamma(s)$, and φ is a constant angle between the vectors γ' and u [4].

We can give same definition in $\mathbb{R}^8$.

Definition 3. Let $\gamma : I \rightarrow \mathbb{R}^7$ be a curve in $\mathbb{R}^7$ with an arc length parameter s and let u be a unit constant vector. Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6\}$, $3 \leq r \leq 7$ be the Frenet 7-frame of γ at its point $\gamma(s)$. If the angle, between $\gamma'(s)$ and u , is $\varphi = \varphi(s)$ we define the function

$$H_i : I \rightarrow \mathbb{R}, \quad 3 \leq i \leq r-2 \quad (11)$$

by

$$\langle n_{i+1}(s), u \rangle = H_i(s) \cos \varphi \quad (12)$$

as the harmonic curvature, with order i , of the curve γ at its point $\gamma(s)$. We define also $H_0 = 0$ [4].

We can give same definition in $\mathbb{R}^8$.

Now we are going to give some definitions and theorems about octonionic curves in $\mathbb{R}^7$ and $\mathbb{R}^8$.

Definition 4. The seven-dimensional Euclidean space $\mathbb{R}^7$ is consubstantiated by the space of spatial real octonions $O_P = \{\gamma \in O \mid \gamma + \bar{\gamma} = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in $\mathbb{R}$ and let $s \in I$ be the parameter along the smooth curve

$$\begin{aligned} \gamma : I \subset \mathbb{R} &\rightarrow O_P \\ s &\rightarrow \gamma(s) = \sum_{i=1}^7 \gamma_i(s) e_i. \end{aligned} \quad (13)$$

Then the curve is called spatial octonionic curve or octonionic curve in $\mathbb{R}^7$ [5].

Theorem 5. The seven-dimensional Euclidean space $\mathbb{R}^7$ is consubstantiated by the space of spatial real octonions $O_P = \{\gamma \in O \mid \gamma + \bar{\gamma} = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in $\mathbb{R}$ and let $s \in I$ be the parameter along the smooth curve

$$\begin{aligned} \gamma : I \subset \mathbb{R} &\rightarrow O_P \\ s &\rightarrow \gamma(s) = \sum_{i=1}^7 \gamma_i(s) e_i. \end{aligned} \quad (14)$$

Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6\}$ be the Frenet trihedron of the differentiable Euclidean space curve in the Euclidean space $\mathbb{R}^7$. Then Frenet equations are

$$\begin{aligned} \mathbf{t}'(s) &= k_1(s) \mathbf{n}_1(s), \\ \mathbf{n}_1'(s) &= -k_1(s) \mathbf{t}(s) + k_2(s) \mathbf{n}_2(s), \\ \mathbf{n}_2'(s) &= -k_2(s) \mathbf{n}_1(s) + k_3(s) \mathbf{n}_3(s), \\ \mathbf{n}_3'(s) &= -k_3(s) \mathbf{n}_2(s) + k_4(s) \mathbf{n}_4(s), \\ \mathbf{n}_4'(s) &= -k_4(s) \mathbf{n}_3(s) + k_5(s) \mathbf{n}_5(s), \\ \mathbf{n}_5'(s) &= -k_5(s) \mathbf{n}_4(s) + k_6(s) \mathbf{n}_6(s), \\ \mathbf{n}_6'(s) &= -k_6(s) \mathbf{n}_5(s), \end{aligned} \quad (15)$$

where k_i , $1 \leq i \leq 6$ curvature functions.

We may state Frenet formulae of the Frenet apparatus in the matrix form:

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_1' \\ \mathbf{n}_2' \\ \mathbf{n}_3' \\ \mathbf{n}_4' \\ \mathbf{n}_5' \\ \mathbf{n}_6' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & -k_4 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & -k_5 & 0 & k_6 \\ 0 & 0 & 0 & 0 & 0 & -k_6 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \\ \mathbf{n}_6 \end{bmatrix}. \quad (16)$$

This is the Serret-Frenet formulae for the spatial octonionic curve γ in $\mathbb{R}^7$ [5].

$\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6, k_1, k_2, k_3, k_4, k_5, k_6\}$ is the Frenet apparatus for spatial octonionic curve γ in $\mathbb{R}^7$.

Remark 6. What has been achieved in this theorem is reputable in local differential geometry. We have done this for two especial goals:

- (1) to designate the demonstration for the Serret-Frenet formulae and Frenet apparatus of the curve γ in $\mathbb{R}^7$. We will roll the outcomes of this theorem comprehensively in the next theorem;
- (2) to indicate how octonions are to be used in designating curvature numbers of curves in general.

Definition 7. The eight-dimensional Euclidean space $\mathbb{R}^8$ is assimilated into the space of real octonion. Let $I = [0, 1]$ be an interval in $\mathbb{R}$ and let $s \in I$ be the parameter along the smooth curve

$$\begin{aligned} \beta : I \subset \mathbb{R} &\rightarrow O \\ s &\rightarrow \beta(s) = \sum_{i=0}^7 \gamma_i(s) e_i. \end{aligned} \quad (17)$$

Then the curve is called octonionic curve [5].

Theorem 8. The eight-dimensional Euclidean space $\mathbb{R}^8$ is assimilated into the space of real octonion. Let

$$\beta : I \subset \mathbb{R} \longrightarrow O$$

$$s \longrightarrow \beta(s) = \sum_{i=0}^7 \gamma_i(s) e_i \quad (18)$$

be a smooth curve in $\mathbb{R}^8$ described over I . Let the parameter s be selected that $\mathbf{T} = \beta'(s) = \sum_{i=0}^7 \gamma'_i(s) e_i$ has unit magnitude. Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7\}$ be the Frenet elements of β . Then the Frenet equations are

$$\mathbf{T}'(s) = K(s) \mathbf{N}_1(s),$$

$$\mathbf{N}_1'(s) = -K(s) \mathbf{T}(s) + k_1(s) \mathbf{N}_2(s),$$

$$\begin{aligned} \mathbf{N}_2'(s) &= -k_1(s) \mathbf{N}_1(s) + (k_2 - K)(s) \mathbf{N}_3(s), \\ \mathbf{N}_3'(s) &= -(k_2 - K)(s) \mathbf{N}_2(s) + k_3(s) \mathbf{N}_4(s), \\ \mathbf{N}_4'(s) &= -k_3(s) \mathbf{N}_3(s) + (k_4 - K)(s) \mathbf{N}_5(s), \\ \mathbf{N}_5'(s) &= -(k_4 - K)(s) \mathbf{N}_4(s) + k_5(s) \mathbf{N}_6(s), \\ \mathbf{N}_6'(s) &= -k_5(s) \mathbf{N}_5(s) + (k_6 + K)(s) \mathbf{N}_7(s), \\ \mathbf{N}_7'(s) &= -(k_6 + K)(s) \mathbf{N}_6(s), \end{aligned} \quad (19)$$

where $\mathbf{N}_1 = \mathbf{t} \times \mathbf{T}$, $\mathbf{N}_2 = \mathbf{n}_1 \times \mathbf{T}$, $\mathbf{N}_3 = \mathbf{n}_2 \times \mathbf{T}$, $\mathbf{N}_4 = \mathbf{n}_3 \times \mathbf{T}$, $\mathbf{N}_5 = \mathbf{n}_4 \times \mathbf{T}$, $\mathbf{N}_6 = \mathbf{n}_5 \times \mathbf{T}$ and $\mathbf{N}_7 = \mathbf{n}_6 \times \mathbf{T}$. $K = \|\mathbf{T}'(s)\|$.

We may express Frenet formulae of the Frenet apparatus in the matrix form:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \\ \mathbf{N}_3' \\ \mathbf{N}_4' \\ \mathbf{N}_5' \\ \mathbf{N}_6' \\ \mathbf{N}_7' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 & 0 & 0 & 0 & 0 \\ -K & 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_1 & 0 & (k_2 - K) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(k_2 - K) & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 & (k_4 - K) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(k_4 - K) & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & (k_6 + K) \\ 0 & 0 & 0 & 0 & 0 & 0 & -(k_6 + K) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \\ \mathbf{N}_6 \\ \mathbf{N}_7 \end{bmatrix}. \quad (20)$$

This is the Serret-Frenet formulae for octonionic curve β in $\mathbb{R}^8$ [5].

3. Octonionic Inclined Curves and Harmonic Curvatures

Definition 9. Let $\gamma(I)$ be spatial octonionic curve with an arc length parameter s and let U be an unit and constant spatial octonion. For all $s \in I$, let $\langle \gamma'(s), U \rangle$ be a constant defined by

$$\langle \gamma'(s), U \rangle = \cos \varphi = \text{constant}, \quad \varphi \neq \frac{\pi}{2}. \quad (21)$$

Then $\gamma(I)$ is called spatial octonionic inclined curve.

Definition 10. $\gamma(I)$ octonionic curve is given by arc length parameter s . Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6\}$ be the Frenet trihedron in the point $\gamma(s)$ of the curve γ and let U be unit and constant spatial octonion such that angle $\varphi(s)$ is between $\gamma'(s)$ and U ,

$$H_i : I \longrightarrow \mathbb{R}, \quad 1 \leq i \leq 5 \quad (22)$$

be a function defined by

$$\langle n_{i+1}(s), U \rangle = H_i \cos \varphi = \text{constant}, \quad \varphi \neq \frac{\pi}{2}. \quad (23)$$

Then functions H_i are called i th Harmonic curvature in the point $\gamma(s)$ of the γ spatial octonionic curve with respect to u .

Definition 11. $\beta : I \subset \mathbb{R} \rightarrow O$ octonionic curve is given by arc length parameter s such that U is a unit and constant spatial octonion for every $s \in I$,

$$\langle \beta'(s), U \rangle = \cos \varphi = \text{constant}, \quad \varphi \neq \frac{\pi}{2}. \quad (24)$$

Then curve β is called octonionic inclined curve in O .

Definition 12. $\beta : I \subset \mathbb{R} \rightarrow O$ octonionic curve is given by arc length parameter s . Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7\}$ be the Frenet apparatus and let U be unit and constant such that angle $\varphi(s)$ is between $\mathbf{T}'(s)$ and U . Let

$$H_i : I \longrightarrow \mathbb{R}, \quad 1 \leq i \leq 6 \quad (25)$$

be a function defined by

$$\langle N_{i+1}(s), U \rangle = H_i \cos \varphi = \text{constant}, \quad \varphi \neq \frac{\pi}{2}. \quad (26)$$

Then functions H_i are called i th Harmonic curvature in the point $\beta(s)$ of the β octonionic curve with respect to U .

Theorem 13. Let $\gamma : I \rightarrow \mathbb{R}^7$ be spatial octonionic inclined curve given by arc length parameter s . Curvatures in the point

$\gamma(s)$ of curve γ are $k_i(s)$, $\zeta_i = 1/k_i(s)$ and $H_i(s)$, $1 \leq i \leq 6$ are harmonic curvatures; they are

$$\begin{aligned} H_1 &= \frac{k_1}{k_2}, & H_2 &= \frac{H'_1}{k_3}, \\ H_j &= (H'_{j-1} + H'_{j-2}k_i)\zeta_{j+1}, \\ &2 \leq j \leq 5. \end{aligned} \quad (27)$$

Proof. Let φ be an angle between the unit and constant spatial octonion U and $\mathbf{t}(s)$. Such that $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6\}$ Frenet apparatus in the point $\gamma(s)$ we obtain that

$$\langle \mathbf{t}(s), U \rangle = \cos \varphi. \quad (28)$$

Here, differentiating with respect to s , we find that $\langle \mathbf{t}'(s), U \rangle = 0$. By the aid of (15), we obtain that $\langle \mathbf{n}_1(s), U \rangle = 0$. If derivative of this function with respect to s is taken, we find that $\langle \mathbf{n}'_1(s), U \rangle = 0$. Here, using (15),

$$\langle -k_1(s)\mathbf{t}(s) + k_2(s)\mathbf{n}_2(s), U \rangle = 0 \quad (29)$$

is obtained. Thus, if (21) and (23) are used,

$$(-k_1(s) + k_2(s)H_1)\cos \varphi = 0, \quad \cos \varphi \neq \frac{\pi}{2} \quad (30)$$

is found. Thus,

$$H_1 = \frac{k_1(s)}{k_2(s)}. \quad (31)$$

By the aid of (15), we obtain that $\langle \mathbf{n}_2(s), U \rangle = H_1 \cos \varphi$. If derivative of this function with respect to s is taken, and (15), (21), and (23) are used, we find that $H_2 = H'_1/k_3$. For the higher harmonic curvatures let us differentiate (23), with respect to s for j ; then $\langle \mathbf{n}'_{j+1}(s), U \rangle = H'_{j-1} \cos \varphi$. By the aid of (15), $\langle -k_j(s)\mathbf{n}_1(s) + k_{j+1}(s)\mathbf{n}_{i+2}(s), U \rangle = H'_{j-1} \cos \varphi$ we get $H_j = (H'_{j-1} + H'_{j-2}k_i)\zeta_{j+1}$, $2 \leq j \leq 5$. $\square$

Theorem 14. Let $\gamma : I \rightarrow \mathbb{R}^7$ be a spatial octonionic inclined curve. Such that $\gamma(s) = \sum_{i=1}^7 \gamma_i(s)e_i$,

$$\beta(s) = \sum_{i=0}^7 \gamma_i(s)e_i, \quad (32)$$

obtained from γ , octonionic curve, is an octonionic inclined curve.

Proof. Let $\beta : I \rightarrow O$ be an octonionic curve given by arc length parameter s .

Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7\}$ be the Frenet apparatus and let U be unit and constant spatial octonion. If we use Definition 11, we get the following statement:

$$\begin{aligned} \langle \beta'(s), u \rangle &= \langle \mathbf{T}(s), U \rangle, \\ \mathbf{T}(s) &= S_{\mathbf{T}(s)} + \vec{V}_{\mathbf{T}(s)}, \quad U = S_U + \vec{V}_U, \end{aligned} \quad (33)$$

where

$$\mathbf{T}(s) = S_{\mathbf{T}(s)} + \vec{V}_{\mathbf{T}(s)}. \quad (34)$$

We notice that

$$\begin{aligned} \langle \mathbf{T}(s), U \rangle &= \frac{1}{2} (\mathbf{T}(s) \times \bar{U} + U \times \overline{\mathbf{T}(s)}) \\ &= \frac{1}{2} \{ [(S_{\mathbf{T}(s)} + \vec{V}_{\mathbf{T}(s)}) \times \bar{U}] + [U \times (S_{\mathbf{T}(s)} - \vec{V}_{\mathbf{T}(s)})] \}. \end{aligned} \quad (35)$$

Since U is spatial octonion, then $U = \vec{V}_U$, $\bar{U} = -U$. Here, we can account for the product of octonion

$$\begin{aligned} \langle \mathbf{T}(s), U \rangle &= \frac{1}{2} \{ [S_{\mathbf{T}(s)} \cdot 0 - \langle \vec{V}_{\mathbf{T}(s)}, -U \rangle + S_{\mathbf{T}(s)} \cdot (-U) + 0 \cdot \vec{V}_{\mathbf{T}(s)} + \vec{V}_{\mathbf{T}(s)} \wedge (-U)] \\ &\quad + [0 \cdot S_{\mathbf{T}(s)} - \langle U, \vec{V}_{\mathbf{T}(s)} \rangle + S_{\mathbf{T}(s)} \cdot U + 0 \cdot \vec{V}_{\mathbf{T}(s)} + U \wedge (-\vec{V}_{\mathbf{T}(s)})] \} \\ &= \frac{1}{2} \{ \langle \vec{V}_{\mathbf{T}(s)}, U \rangle - S_{\mathbf{T}(s)}U - \vec{V}_{\mathbf{T}(s)} \wedge U \\ &\quad + \langle U, \vec{V}_{\mathbf{T}(s)} \rangle + S_{\mathbf{T}(s)}U - U \wedge \vec{V}_{\mathbf{T}(s)} \} \\ &= \langle \vec{V}_{\mathbf{T}(s)}, U \rangle \end{aligned} \quad (36)$$

and so $\langle \beta'(s), U \rangle = \cos \varphi$ is obtained. Then, β curve is octonionic inclined curve. $\square$

Theorem 15. Let $\beta : I \rightarrow O$ be an octonionic inclined curve given by arc length parameter s . Such that $K_i(s)$ are curvatures in the point $\beta(s)$, $\delta_i(s) = 1/K_i(s)$, $1 \leq i \leq 7$, are curvature radii and $H_i(s)$, $1 \leq i \leq 6$, are harmonic curvatures, they are

$$\begin{aligned} H_1 &= \frac{1.\text{curvature}}{2.\text{curvature}}, & H_2 &= \frac{H'_1}{(k_2 - K)(s)}, \\ H_3 &= \frac{H'_2 + (k_2 - K)H_1}{k_3}, \\ H_4 &= \frac{H'_3 + k_3H_2}{(k_4 - K)(s)}, & H_5 &= \frac{H'_4 + (k_4 - K)H_3}{k_5}, \\ H_6 &= \frac{H'_5 + k_5H_4}{(k_6 + K)(s)}, \end{aligned} \quad (37)$$

where $K_1(s) = K$, $K_2(s) = k_1$, $K_3(s) = k_2 - K$, $K_4(s) = k_3$, $K_5(s) = k_4 - K$, $K_6(s) = k_5$, $K_7(s) = k_6 + K$.

Proof. $\beta : I \rightarrow O$ curve is given by regular octonionic. U is an unit and a constant spatial octonion, and such that $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6, \mathbf{N}_7\}$ is Frenet apparatus in the point $\beta(s)$,

$$\langle \mathbf{T}(s), U \rangle = \cos \varphi = \text{constant} \quad (38)$$

is written. If derivative with respect to s of this equation is taken, we obtain that $\langle \mathbf{T}'(s), U \rangle = 0$. Here, using (19), $\langle K(s)\mathbf{N}_1(s), U \rangle = 0$ is found. Because of $K(s)\mathbf{N}_1(s) \neq 0$, we write as $\langle \mathbf{N}_1(s), U \rangle = 0$. Thus, $\langle \mathbf{N}_1'(s), U \rangle = 0$ is obtained. Here, using (19),

$$-K(s) \langle \mathbf{T}(s), U \rangle + k_1(s) \langle \mathbf{N}_2(s), U \rangle = 0 \quad (39)$$

is found. In addition, from (26) for $i = 1$ we obtain that

$$\langle \mathbf{N}_2(s), U \rangle = H_1(s) \cos \varphi. \quad (40)$$

By taking (24) and (40) into consideration,

$$(-K(s) + k_1(s) H_1(s)) \cos \varphi = 0, \quad \cos \varphi \neq 0 \quad (41)$$

$$H_1(s) = \frac{K(s)}{k_1(s)} = \frac{1.\text{curvature}}{2.\text{curvature}}$$

is found. On the other hand, if derivative of (40) with respect to s is taken,

$$\langle \mathbf{N}_2'(s), U \rangle = H_1'(s) \cos \varphi \quad (42)$$

is found. Here, using (19),

$$-k_1(s) \langle \mathbf{N}_1(s), U \rangle + (k_2 - K)(s) \langle \mathbf{N}_3(s), U \rangle = H_1'(s) \cos \varphi \quad (43)$$

is found. In addition, from (26) for $i = 2$ we obtain that

$$\langle \mathbf{N}_3(s), U \rangle = H_2(s) \cos \varphi. \quad (44)$$

By taking (24) and (44) into consideration,

$$(k_2 - K)(s) H_2(s) \cos \varphi = H_1'(s) \cos \varphi \quad (45)$$

is obtained. Thus,

$$H_1(s) = \frac{H_1'(s)}{(k_2 - K)(s)}. \quad (46)$$

If derivative of (44) with respect to s is taken,

$$\langle \mathbf{N}_3'(s), U \rangle = H_2'(s) \cos \varphi \quad (47)$$

is found. Here, using (19),

$$-(k_2 - K)(s) \langle \mathbf{N}_2(s), U \rangle + k_3(s) \langle \mathbf{N}_4(s), U \rangle = H_2'(s) \cos \varphi \quad (48)$$

is found. In addition, from (26) for $i = 3$ we obtain that

$$\langle \mathbf{N}_4(s), U \rangle = H_3(s) \cos \varphi. \quad (49)$$

By taking (40) and (49) into consideration,

$$-(k_2 - K)(s) H_1(s) \cos \varphi + k_3(s) H_3(s) \cos \varphi = H_2'(s) \cos \varphi \quad (50)$$

is obtained. Thus,

$$H_3(s) = \frac{H_2'(s) + (k_2 - K)(s) H_1(s)}{k_3(s)}. \quad (51)$$

Similarly, If derivative of (49) and following equations with respect to s is taken,

$$\begin{aligned} \langle \mathbf{N}_5(s), U \rangle &= H_4(s) \cos \varphi, \\ \langle \mathbf{N}_6(s), U \rangle &= H_5(s) \cos \varphi, \end{aligned} \quad (52)$$

we get

$$\begin{aligned} H_4(s) &= \frac{H_3'(s) + k_3(s) H_2(s)}{(k_4 - K)(s)}, \\ H_5(s) &= \frac{H_4'(s) + (k_4 - K)(s) H_3(s)}{k_5(s)}, \\ H_6(s) &= \frac{H_5'(s) + k_5(s) H_4(s)}{(k_6 + K)(s)}. \end{aligned} \quad (53)$$

□

Theorem 16. γ is a spatial octonionic curve given by arc length parameter s . And let H_i , $1 \leq i \leq 5$ be harmonic curvatures in the point $\gamma(s)$. γ is octonionic inclined curve if and only if $\sum_{i=1}^5 H_i^2$ is constant.

Proof. ($\Rightarrow$) Let γ be a spatial octonionic curve given by arc length parameter s . Then, there is a U unit and constant spatial octonion. Therefore,

$$\langle \gamma'(s), U \rangle = \cos \varphi \quad (54)$$

is constant for γ spatial octonionic inclined curve with respect to arc length parameter s such that $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4, \mathbf{n}_5, \mathbf{n}_6\}$ is basis of spatial octonion in the point $\gamma(s)$; spatial octonion U

$$U = \langle \mathbf{t}(s), U \rangle \mathbf{t}(s) + \sum_{i=1}^6 \langle \mathbf{n}_i(s), U \rangle \mathbf{n}_i(s) \quad (55)$$

is obtained. Since U is a unit,

$$\|U\|^2 = U \times \overline{U} = 1. \quad (56)$$

Here, using (55),

$$\begin{aligned} \|U\|^2 &= \left(\langle \mathbf{t}(s), U \rangle \mathbf{t}(s) + \sum_{i=1}^6 \langle \mathbf{n}_i(s), U \rangle \mathbf{n}_i(s) \right) \\ &\quad \times \langle \mathbf{t}(s), U \rangle \mathbf{t}(s) + \sum_{i=1}^6 \langle \mathbf{n}_i(s), U \rangle \mathbf{n}_i(s), \end{aligned} \quad (57)$$

if we use Definition 10 in the last equation we can write

$$\begin{aligned} 1 &= \left(\cos \varphi t + \sum_{i=0}^5 H_i(s) \mathbf{n}_{i+1}(s) \cos \varphi \right) \\ &\quad \times \cos \varphi t + \sum_{i=0}^5 H_i(s) \mathbf{n}_{i+1}(s) \cos \varphi. \end{aligned} \quad (58)$$

From octonionic product, we have

$$1 = \cos^2 \varphi + \sum_{i=1}^5 H_i^2(s) \cos^2 \varphi, \quad (59)$$

where

$$\sum_{i=1}^5 H_i^2(s) = \tan^2 \varphi = \text{constant}. \quad (60)$$

($\Leftarrow$) In contrast, suppose that $\sum_{i=1}^5 H_i^2(s)$ is constant for γ spatial octonionic curve. It is study to show that $\langle \gamma'(s), U \rangle = \cos \varphi$. Therefore, there is φ angle so that $\tan^2 \varphi = a$. Thus, we define U spatial octonion, where

$$U = \cos \varphi t + \sum_{i=2}^6 H_{i-1}(s) \mathbf{n}_i(s) \cos \varphi. \quad (61)$$

Here, we demonstrate that u is a constant. Thus, if derivative of (61) with respect to s is taken,

$$\begin{aligned} \frac{1}{\cos \varphi} \frac{dU}{ds} &= t' + \sum_{i=2}^6 H_{i-1}'(s) \mathbf{n}_i(s) + \sum_{i=2}^6 H_{i-1}(s) \mathbf{n}_i'(s), \\ \frac{1}{\cos \varphi} \frac{dU}{ds} &= t' + H_1' \mathbf{n}_2 + H_2' \mathbf{n}_3 + H_3' \mathbf{n}_4 + H_4' \mathbf{n}_5 + H_5' \mathbf{n}_6 \\ &\quad + H_1 \mathbf{n}_2' + H_2 \mathbf{n}_3' + H_3 \mathbf{n}_4' + H_4 \mathbf{n}_5' + H_5 \mathbf{n}_6' \end{aligned} \quad (62)$$

is found. On the other hand,

$$\langle \mathbf{n}_3(s), U \rangle = H_2 \cos \varphi \implies \langle \mathbf{n}_3'(s), U \rangle = H_2' \cos \varphi \quad (63)$$

is obtained. Here, using (15),

$$H_2' = -k_3 H_1 + k_4 H_3 \quad (64)$$

is obtained. Similarly,

$$\begin{aligned} H_3' &= -k_4 H_2 + k_5 H_4, \\ H_4' &= -k_5 H_3 + k_6 H_5, \\ H_5' &= -k_6 H_4. \end{aligned} \quad (65)$$

Finally, we get

$$\frac{1}{\cos \varphi} \frac{dU}{ds} = 0. \quad (66)$$

Thus, u is a constant. On the other hand,

$$\|U\|^2 = U \times \bar{U}, \quad (67)$$

$$\begin{aligned} \|U\|^2 &= \left(\cos \varphi t + \sum_{i=2}^6 H_{i-1}(s) \mathbf{n}_i(s) \cos \varphi \right) \\ &\quad \times \cos \varphi t + \sum_{i=2}^6 H_{i-1}(s) \mathbf{n}_i(s) \cos \varphi \\ &= \cos^2 \varphi + \cos^2 \varphi \left(\sum_{i=1}^5 H_i^2(s) \right) \\ &= 1 \end{aligned} \quad (68)$$

is obtained. Thus,

$$\begin{aligned} \langle t(s), U \rangle &= \frac{1}{2} (t \times \bar{U} + U \times \bar{t}) \\ &= \cos \varphi \end{aligned} \quad (69)$$

is found. Therefore, γ is an inclined curve. $\square$

Theorem 17. β is an octonionic curve given by arc length parameter s . And let $H_i, 1 \leq i \leq 6$ be harmonic curvatures in the point $\beta(s)$. β is an octonionic inclined curve if and only if $\sum_{i=1}^6 H_i^2$ is constant.

Proof. The result is straightforward. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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