Research Article

A Geometric Modeling Method Based on TH-Type Uniform B-Splines

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A geometric modeling method based on TH-type uniform B-splines which are composed of trigonometric and hyperbolic polynomial with parameters is introduced in this paper. The new splines possess many important properties of quadratic and cubic B-splines. Taking different values of the parameters, one can not only locally adjust the shape of the curves, but also change the type of some segments of a curve between trigonometric and hyperbolic functions as well. The given curves can also interpolate directly control polygon locally by selecting special parameters. Moreover, the introduced splines can represent some quadratic curves and transcendental curves with selecting proper control points and parameters.

1. Introduction

B-splines are used as an important geometric modeling tool in computer aided geometric design (CAGD). However, there are still several limitations on B-splines in practical applications [1]. Firstly, for the fixed control points and the knot sequences, the shape of the curves and surfaces represented by B-splines is fixed. Secondly, the B-splines cannot represent conics (except parabolas) and some known curves such as the cycloid and the helix exactly. Although NURBS can overcome the shortcomings of B-splines, as the complexity of its rational basis functions and its derivatives and integrals are hard to compute, it is not convenient to the user. So, in order to avoid their inconveniences, recently, several new splines defined in different space from the usual polynomial space have been proposed for geometric modeling in CAGD [2–11]. T-type splines were introduced [2–7], which can exactly represent the ellipse, the cycloid, and the helix. Pottmann and Wagner [8] and Koch and Lyche [9] presented a kind of exponential splines in tension in that space $\{1, t, \cosh t, \sinh t\}$. Lü et al. [10] gave the explicit expressions for uniform splines. Li and Wang [11] generalized the curves and surfaces of exponential forms to algebraic hyperbolic spline forms of any degree, which can represent exactly some remarkable curves such as

the hyperbola and the catenary. However, H-type uniform B-splines in tension are not applicable to freeform polynomial curves of high orders, which severely restrict their applications in CAGD.

By comparing T-type uniform B-splines and H-type uniform B-splines, we found that T-type uniform B-splines are located on one side of the B-spline, and H-type uniform B-splines are located on the other side of the B-spline. Therefore, one thinks if the two different curves can be unified to produce new blending splines, then the new curve will have more plentiful modeling power. In order to construct more flexible curves for curves and surface modeling, Zhang et al. [12, 13] proposed a curve family, named FB-spline, that uses a unified basis $\{1, t, \cos t, \sin t\}$ and basis $\{1, t, \cosh t, \sinh t\}$. FB-splines inherited nearly all the properties that the T-type B-splines and the H-type B-splines have. However, the formulas for the FB-splines were rather complicated. Wang and Fang [14] unified and extended three types of splines by a new kind of spline (UE-spline for short) defined over the space $\{\cos \omega_i t, \sin \omega_i t, 1, t, \dots, t^l, \dots\}$, where the type of a curve can be switched by a frequency sequence $\{\omega_i\}$. However, the geometric meaning of the sequence $\{\omega_i\}$ is not obvious. Over the space span $\{\sin t, \cos t, \sinh t, \cosh t, 1, t, \dots, t^{n-5}\}$, $n \geq 5$. Xu and Wang [15] presented two new unified mathematics

models of conics and polynomial curves, called algebraic hyperbolic trigonometric (AHT) Bézier curves and nonuniform algebraic hyperbolic trigonometric (NUAHT) B-spline curves of order n , which share most of the properties as those of the Bézier curves and B-spline curves in polynomial space.

In this paper, we present a new geometric modeling method based on two kinds of TH-type uniform B-splines which are composed of hyperbolic and trigonometric functions. The introduced spline has the following features: (1) the new spline curves can be adjusted totally or locally. (2) The given curves can switch into T-type B-spline curves or H-type B-spline curves when the parameter is equal to 0 or 1. (3) Without solving the system of equations, the new curves can interpolate certain control points directly. (4) The TH-type B-spline curves can be used to represent some conics and transcendental curves with the parameters and control points chosen properly.

The rest of this paper is organized as follows. In Sections 2 and 3, the TH-type basis functions and corresponding TH-type curves are established and the properties of the basis functions are proved. In Section 4, some properties of the TH-type B-spline curves are discussed. It is pointed out in Section 5 that some transcendental curves can be represented precisely with the TH-type curves, and the applications of the curves are shown in Section 6.

2. Quadratic TH-Type B-Spline

Definition 1. Given $t \in [0, 1]$, the quadratic basis functions based on weighted trigonometric and hyperbolic polynomials are as follows:

$$\begin{aligned} \text{qth}_{0,2}(t; \lambda_i) &= \frac{1}{2} - \lambda_i \frac{2e \cosh(1-t) - e^2 - 1}{2(e-1)^2} \\ &\quad + \frac{1}{2} (\lambda_i - 1) \sin \frac{\pi t}{2}, \\ \text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1}) &= \frac{1}{2} (1 - \lambda_i) \sin \frac{\pi t}{2} \\ &\quad + \lambda_i \frac{2e (\cosh 1 - \cosh(1-t))}{(e-1)^2} \\ &\quad + \lambda_{i+1} \frac{2e (\cosh 1 - \cosh(t))}{(e-1)^2} \\ &\quad + \frac{1}{2} (1 - \lambda_{i+1}) \cos \frac{\pi t}{2}, \\ \text{qth}_{2,2}(t; \lambda_{i+1}) &= \lambda_{i+1} \frac{e (\cosh(t) - 1)}{2(e-1)^2} \\ &\quad + \frac{1}{2} (1 - \lambda_{i+1}) \left(1 - \cos \frac{\pi t}{2}\right), \end{aligned} \quad (1)$$

which are named the basis functions of quadratic TH-type B-spline.

Theorem 2. The above functions have the following properties.

- (i) *Partition of unity:* $\text{qth}_{0,2}(t; \lambda_i) + \text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1}) + \text{qth}_{2,2}(t; \lambda_{i+1}) = 1$.

- (ii) *Symmetry:* $\text{qth}_{0,2}(t; \lambda_i) = \text{qth}_{2,2}(1-t; \lambda_i)$, $\text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1}) = \text{qth}_{1,2}(1-t; \lambda_{i+1}, \lambda_i)$.

- (iii) *Nonnegativity:* if $(1 + \sqrt{e})^2 / ((1 - \sqrt{e})(1 + \sqrt{e})^2 + \sqrt{2e}) \leq \lambda_i$, $\lambda_{i+1} \leq (e-1)^2 \pi^2 / ((e-1)^2 \pi^2 - 8e)$, then $\text{qth}_{k,2}(t) \geq 0$, $k = 0, 1, 2$.

Proof. (i) and (ii) are easy to be proved by simple computation. Next we will prove (iii).

By direct computation, we have $\text{qth}_{0,2}(0; \lambda_i) = 1$, $\text{qth}_{0,2}(1; \lambda_i) = 0$. And since $0 \leq t \leq 1$, $\lambda_i \leq (e-1)^2 \pi^2 / ((e-1)^2 \pi^2 - 8e)$ and $\text{qth}'_{0,2}(t; \lambda_i) \leq 0$, then we have $\text{qth}_{0,2}(t; \lambda_i) \geq 0$. Evidenced by the same token, we have $\text{qth}_{2,2}(t; \lambda_{i+1}) \geq 0$.

From (ii), we have $\text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1}) = 1 - \text{qth}_{0,2}(t; \lambda_i) - \text{qth}_{2,2}(t; \lambda_{i+1})$. Obviously, if we can prove $\text{qth}_{0,2}(t; \lambda_i) + \text{qth}_{2,2}(t; \lambda_{i+1}) \leq 1$, we can prove $\text{qth}_{1,2}(t; \lambda_{i+1}) \geq 0$.

Let $f(t; \lambda_i, \lambda_{i+1}) = \text{qth}_{0,2}(t; \lambda_i) + \text{qth}_{2,2}(t; \lambda_{i+1})$; we have $f(0; \lambda_i, \lambda_{i+1}) = f(1; \lambda_i, \lambda_{i+1}) = 1$. Thus, when $\lambda_i, \lambda_{i+1} \geq (1 + \sqrt{e})^2 / ((1 - \sqrt{e})(1 + \sqrt{e})^2 + \sqrt{2e})$, we can get

$$f'(t; \lambda_i, \lambda_{i+1}) = \begin{cases} < 0, & t \in [0, 0.5), \\ = 0, & t = 0.5, \\ > 0, & t \in (0.5, 1]. \end{cases} \quad (2)$$

So the maximum value of the function $f(t; \lambda_i, \lambda_{i+1})$ equals 1. That is, $\text{qth}_{0,2}(t; \lambda_i) + \text{qth}_{2,2}(t; \lambda_{i+1}) \leq 1$, which means $\text{qth}_{1,2}(t; \lambda_{i+1}) \geq 0$. $\square$

Definition 3. Given control points $P_i \in R^d$ ($d = 2, 3, i = 0, 1, \dots, n$), the curves

$$\begin{aligned} \text{QTH}_{i,2}(t; \lambda_i, \lambda_{i+1}) &= P_{i-1} \text{qth}_{0,2}(t; \lambda_i) \\ &\quad + P_i \text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1}) + P_{i+1} \text{qth}_{2,2}(t; \lambda_{i+1}), \\ t &\in [0, 1], \quad i = 1, 2, \dots, n-1 \end{aligned} \quad (3)$$

are defined quadratic TH-type B-spline curve segments with shape parameters λ_i and λ_{i+1} , where $\text{qth}_{0,2}(t; \lambda_i)$, $\text{qth}_{1,2}(t; \lambda_i, \lambda_{i+1})$, and $\text{qth}_{2,2}(t; \lambda_{i+1})$ are the bases of quadratic TH-type B-spline.

3. Cubic TH-Type B-Spline

By a similar method, we may define the bases of cubic TH-type B-spline.

Definition 4. For $(e-1)^2 / ((e-1)^2 - \pi) \leq \lambda_i \leq \lambda_{i+1} \leq (e-1)^2 \pi^2 / ((e-1)^2 \pi^2 - 8e)$ and $t \in [0, 1]$, the following functions,

$$\begin{aligned} \text{cth}_{0,3}(t; \lambda_i) &= \frac{1}{\pi} (\lambda_i - 1) \cos \frac{\pi t}{2} + \frac{1}{(e-1)^2} \\ &\quad \times ((2(e-1)^2 - (1+e^2)) \lambda_i \\ &\quad \times (1-t) - 2e \lambda_i \sinh(1-t)), \end{aligned}$$

$$\begin{aligned}
\text{cth}_{1,3}(t; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)^2 - (1+e^2)\lambda_{i+1}}{2(e-1)^2} \\
&\quad - \frac{(e-1)^2 - (1+e^2)(2\lambda_i + \lambda_{i+1})}{2(e-1)^2} \\
&\quad \times (1-t) + \frac{2}{\pi}(1-\lambda_i)\cos\frac{\pi t}{2} \\
&\quad - \frac{1}{\pi}(1-\lambda_{i+1})\sin\frac{\pi t}{2} \\
&\quad + \frac{(e+1)\lambda_{i+1}}{2(e-1)}\cosh(1-t) \\
&\quad - \frac{(1+e^2)\lambda_{i+1} + 4e\lambda_i}{(e-1)^2\pi}\sinh(1-t), \\
\text{cth}_{2,3}(t; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)^2 - (1+e^2)\lambda_i}{2(e-1)^2} \\
&\quad - \frac{(e-1)^2 - (1+e^2)(\lambda_i + 2\lambda_{i+1})}{2(e-1)^2}t \\
&\quad + \frac{2}{\pi}(1-\lambda_{i+1})\sin\frac{\pi t}{2} \\
&\quad - \frac{1}{\pi}(1-\lambda_i)\cos\frac{\pi t}{2} + \frac{(e+1)\lambda_i}{2(e-1)}\cosh t \\
&\quad - \frac{(1+e^2)\lambda_i + 4e\lambda_{i+1}}{(e-1)^2}\sinh t, \\
\text{cth}_{3,3}(t; \lambda_{i+1}) &= \frac{1}{\pi}(\lambda_{i+1} - 1)\sin\frac{\pi t}{2} + \frac{1}{2(e-1)^2} \\
&\quad \times \left(((e-1)^2 - (1+e^2)\lambda_{i+1})t \right. \\
&\quad \left. + 2e\lambda_{i+1}\sinh t \right),
\end{aligned} \tag{4}$$

are called basis functions of cubic TH-type B-spline with shape parameters λ_i and λ_{i+1} .

It is easy to prove that the basis functions of cubic TH-type B-spline have the same properties: nonnegativity, partition of unity, and symmetry.

Definition 5. Given control points $P_i \in R^d$ ($d = 2, 3, i = 0, 1, \dots, n$), the curves

$$\begin{aligned}
\text{CTH}_{i,3}(t; \lambda_i, \lambda_{i+1}) &= P_{i-1}\text{cth}_{0,3}(t; \lambda_i) + P_i\text{cth}_{1,3}(t; \lambda_i, \lambda_{i+1}) \\
&\quad + P_{i+1}\text{cth}_{2,3}(t; \lambda_i, \lambda_{i+1}) + P_{i+2}\text{cth}_{3,3}(t; \lambda_{i+1}), \\
t &\in [0, 1], \quad i = 1, 2, \dots, n-1
\end{aligned} \tag{5}$$

are defined cubic TH-type B-spline curve segments with shape parameters λ_i and λ_{i+1} , where $\text{cth}_{0,3}(t; \lambda_i)$,

$\text{cth}_{1,3}(t; \lambda_i, \lambda_{i+1})$, $\text{cth}_{2,3}(t; \lambda_i, \lambda_{i+1})$, and $\text{cth}_{3,3}(t; \lambda_{i+1})$ are the basis functions of cubic TH-type B-spline.

4. The Properties of the TH-Type B-Spline Curves

According to the properties of the basis functions and definition, it is easy to get the following properties of curves (3) and (5).

(i) *Continuity*

Theorem 6. For the uniform knots, the curves (3) are C^1 continuous and the curves (5) are C^2 continuous.

Proof. For the curve (3), we can get

$$\begin{aligned}
\text{QTH}_{i,2}(0; \lambda_i, \lambda_{i+1}) &= \frac{1}{2}(P_{i-1} + P_i), \\
\text{QTH}_{i,2}(1; \lambda_i, \lambda_{i+1}) &= \frac{1}{2}(P_i + P_{i+1}), \\
\text{QTH}'_{i,2}(0; \lambda_i, \lambda_{i+1}) &= \frac{(1-e)\pi + (e(\pi-2) - 2 - \pi)\lambda_i}{4(e-1)}(P_i - P_{i-1}), \\
\text{QTH}'_{i,2}(1; \lambda_i, \lambda_{i+1}) &= \frac{(1-e)\pi + (e(\pi-2) - 2 - \pi)\lambda_{i+1}}{4(e-1)}(P_{i+1} - P_i).
\end{aligned} \tag{6}$$

Thus, we obtain $\text{QTH}_{i-1,2}^{(k)}(1; \lambda_i, \lambda_{i+1}) = \text{QTH}_{i,2}^{(k)}(0; \lambda_i, \lambda_{i+1})$ ($k = 0, 1$); that is to say, the curves (3) are C^1 continuous.

For the curves (5), we get

$$\begin{aligned}
\text{CTH}_{i,3}(0; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)^2(\pi + 2\lambda_i - 2) - 2\pi\lambda_i}{2(e-1)^2\pi}(P_{i-1} + P_{i+1}) \\
&\quad + \frac{2(e-1)^2(\pi - \lambda_i - 2) + 2\pi\lambda_i}{(e-1)^2\pi}P_i, \\
\text{CTH}_{i,3}(1; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)^2(\pi + 2\lambda_{i+1} - 2) - 2\pi\lambda_{i+1}}{2(e-1)^2\pi}(P_i + P_{i+2}) \\
&\quad + \frac{2(e-1)^2(\pi - \lambda_{i+1} - 2) + 2\pi\lambda_{i+1}}{(e-1)^2\pi}P_{i+1},
\end{aligned}$$

$$\text{CTH}'_{i,3}(0; \lambda_i, \lambda_{i+1}) = \frac{1}{2}(P_{i+1} - P_{i-1}),$$

$$\begin{aligned}
CTH'_{i,3}(1; \lambda_i, \lambda_{i+1}) &= \frac{1}{2} (P_{i+2} - P_i), \\
CTH''_{i,3}(0; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)\pi + ((e-1)\pi - 2(e+1))\lambda_i}{4(e-1)} \\
&\quad \times (P_{i-1} - 2P_i + P_{i+1}), \\
CTH''_{i,3}(1; \lambda_i, \lambda_{i+1}) &= \frac{(e-1)\pi + ((e-1)\pi - 2(e+1))\lambda_{i+1}}{4(e-1)} \\
&\quad \times (P_i - 2P_{i+1} + P_{i+2}).
\end{aligned} \tag{7}$$

So, we have $CTH_{i-1,3}^{(k)}(1; \lambda_i, \lambda_{i+1}) = CTH_{i,3}^{(k)}(0; \lambda_i, \lambda_{i+1})$ ($k = 0, 1, 2$). This implies that curves (5) are C^2 continuous.

This implies the theorem. $\square$

(ii) *Local Adjustable Properties.* From formulas (3) and (5), the parameter λ_i only affects two curve segments without altering the remainder. Figure 1 shows local adjustable quadratic uniform TH-type spline curves, where all parameters $\lambda_i = 0.5$ in the solid curves and all parameters $\lambda_i = 0.5$ except $\lambda_3 = -1$ in the dotted curves. The parameter only affects the 2th and the 3th curve segment. Figure 2 shows the local adjustable cubic uniform TH-type spline curves, where all the parameters are equal to 0.5 in the solid curves, and all the parameters are equal to 0.5 except $\lambda_5 = -1$ in the dotted curves. The parameter λ_5 only affects the 4th and 5th curve segment.

Obviously, when all parameters λ_i are the same, the curves can be adjusted totally.

(iii) *Local Interpolating Properties.* For the curve (3), letting $\lambda_i = \lambda_{i+1} = (\sqrt{e} + 1)^2 / (1 - \sqrt{2e} + e) \doteq 5.05952$, then $qth_{i,2}(0.5) = P_i$; that is, the curve interpolates the point P_i . For the curve (5), when $\lambda_i = (2 - \pi)(e - 1)^2 / (2(e - 1)^2 - 2\pi) \doteq 8.91206$, $cth_{i,3}(0) = P_i$, $\lambda_{i+1} = (2 - \pi)(e - 1)^2 / (2(e - 1)^2 - 2\pi) \doteq 8.91206$, $cth_{i,3}(1) = P_{i+1}$; that is, the curve interpolates the points P_i and P_{i+1} . Figure 3 shows local interpolating quadratic TH-type spline curves, where the curve interpolates the point P_5 when the parameter $\lambda_5 = \lambda_6 = 5.05952$. The local interpolating cubic TH-type spline curves are showed in Figure 4, where the curves interpolate the point P_5 when the parameter $\lambda_5 = 8.91206$.

5. The Representations of Some Known Curves

When the parameters $\lambda_i = \lambda_{i+1} = 0$, the curves (3) and (5) are T-type uniform B-spline curves. If the parameters $\lambda_i = \lambda_{i+1} = 1$, the curves (3) and (5) become H-type uniform B-splines.

5.1. The Representation of the Conic Curves. The ellipse and hyperbola are the most common in the conic curve. If the

FIGURE 1: Local adjustable quadratic uniform TH-type spline curves.

FIGURE 2: Local adjustable cubic uniform TH-type spline curves.

FIGURE 3: Local interpolating quadratic TH-type spline curves.

FIGURE 4: Local interpolating cubic uniform TH-type spline curves.

control points and the parameters are selected properly, the curves (3) and (5) can represent them precisely.

Given the uniform knots, for the quadratic T-type B-spline curve, we take the coordinates of the points P_{i-1} , P_i , and P_{i+1} as follows:

$$\begin{aligned} P_{i-1} &= (m - a, n + b), \\ P_i &= (m + a, n + b), \\ P_{i+1} &= (m + a, n - b), \end{aligned} \quad (8)$$

$(ab \neq 0).$

For the cubic T-type B-spline curve, we take

$$\begin{aligned} P_{i-1} &= \left(m, n - \frac{\pi}{2}b\right), \\ P_i &= \left(m + \frac{\pi}{2}a, n\right), \\ P_{i+1} &= \left(m, n - \frac{\pi}{2}b\right), \\ P_{i+2} &= \left(m - \frac{\pi}{2}a, n\right), \end{aligned} \quad (9)$$

$(ab \neq 0).$

Then, when $t \in [0, 1]$ and $\lambda_i = \lambda_{i+1} = 0$, we obtain a parametric equation as follows:

$$\begin{aligned} x(t) &= m + a \cos \frac{\pi}{2}t, \\ y(t) &= n + b \sin \frac{\pi}{2}t. \end{aligned} \quad (10)$$

It is the parametric form of the ellipse; see Figure 5. In order to represent the hyperbola, for the quadratic H-type uniform B-spline curves, the control points are taken as follows:

$$\begin{aligned} P_{i-1} &= \left(m + a, n + \frac{1-e}{1+e}b\right), \\ P_i &= \left(m + a, n \frac{1-e}{1+e}b\right), \\ P_{i+1} &= \left(m + \frac{e^2 - e + 1}{e}a, n + \frac{e^3 - 1}{e^2 + e}b\right), \end{aligned} \quad (11)$$

$(ab \neq 0).$

For the cubic H-type uniform B-spline curves, we take $P_{i-1} = (m + ((e^2 + 1)/e)a, n - ((e^4 + 1)/(e^3 - e))b)$, $P_i = (m + a, n + ((e^2 + 1)/(e^2 - 1))b)$, $P_{i+1} = (m, n + (2e/(e^2 - 1))b)$, $P_{i+2} = (m - a, ((e^2 + 1)/(e^2 - e))b)$ ($ab \neq 0$) as control points. So we get a parametric equation as follows:

$$\begin{aligned} x(t) &= m + a \cosh t, \\ y(t) &= n + b \sinh t, \end{aligned} \quad (12)$$

which represents an arc of the hyperbola; see Figure 6.

5.2. The Representation of the Transcendental Curves. In this section, we can represent the transcendental curves with the uniform TH-type B-splines, such as cycloid and catenary.

When parameters $\lambda_i = \lambda_{i+1} = 0$, control points are taken as follows:

$$\begin{aligned} P_{i-1} &= \left(\frac{\pi - 2}{4}a, \frac{4 + \pi}{4}a\right), \\ P_i &= \left(\frac{2 - \pi}{4}a, \frac{4 - \pi}{4}a\right), \\ P_{i+1} &= \left(\frac{6 - 3\pi}{4}a, \frac{\pi + 4}{4}a\right), \\ P_{i+2} &= \left(\frac{10 - \pi}{4}a, \frac{4 + 3\pi}{4}a\right), \end{aligned} \quad (13)$$

$(a \neq 0).$

So we obtain the parametric equation as follows:

$$\begin{aligned} x(t) &= a \left(t - \sin \frac{\pi}{2}t\right), \\ y(t) &= a \left(1 - \cos \frac{\pi}{2}t\right), \end{aligned} \quad (14)$$

which represents an arc of a cycloid; see Figure 7.

Similarly, when taking $P_{i-1} = (m + 2a, n + ((e^4 + 1)/(e^3 - e))b)$, $P_i = (m + a, n + ((e^2 + 1)/(e^2 - 1))b)$, $P_{i+1} = (m, n + (2e/(e^2 - 1))b)$, and $P_{i+2} = (m - a, n + ((e^2 + 1)/(e^2 - e))b)$ ($ab \neq 0$) as control points, the parameters $\lambda_i = \lambda_{i+1} = 1$. By formula (5), we have the following equation:

$$\begin{aligned} x(t) &= m + at, \\ y(t) &= n + b \cosh t, \end{aligned} \quad (15)$$

which is the parametric equation of the catenary; see Figure 8.

FIGURE 5: The representation of ellipse with quadratic (a) and cubic (b) T-type B-spline curves.

FIGURE 6: The representation of hyperbola with quadratic (a) and cubic (b) H-type B-spline curves.

FIGURE 7: The representation of cycloid with cubic T-type B-spline curves.

FIGURE 8: The representation of catenary with cubic H-type B-spline curves.

6. The Applications of the TH-Type Splines

From the last section, we see, letting the parameter be equal to 0 or 1, the types of the curves can be switched easily. So, by selecting control points and parameters properly, we can represent different type curve segments among a blending curve. In Figure 9, a closed C^1 blending curve is composed of different type curves with the quadratic TH-type B-splines, where the coordinates of control points are $P_0 = P_6 = (-3, (e-1)/(e+1))$, $P_1 = P_7 = (-3, (1-e)/(e+1))$, $P_2 = (3, (e-1)/(e+1))$, $P_3 = (3, (1-e)/(e+1))$, $P_4 = (4-e-(1/e), (e^3-1)/(e^2+e))$, $P_5 = (e-4+(1/e), (e^3-1)/(e^2+e))$ and the parameters

$\lambda_i = (0, 0, 1, 1, 5.05952, 5.05952, 0)$ ($i = 1, 2, \dots, 7$). The 1st segment is a trigonometric curve, which is a quarter of a parabola. The 3rd segment is a hyperbola arc. The blending curve interpolates the point P_5 in that the parameters $\lambda_5 = \lambda_6 = 5.05952$.

Figure 10 shows an open C^2 blending curve represented by the cubic TH-type B-splines. The control points are taken

FIGURE 9: A closed C^1 blending curve with quadratic TH-type B-splines.

FIGURE 10: An open C^2 blending curve with cubic TH-type B-splines.

as follows: $P_0 = ((\pi - 2)/2, 1)$, $P_1 = (0, (2 - \pi)/2)$, $P_2 = ((2 - \pi)/2, 1)$, $P_3 = (2, (2 + \pi)/2)$, $P_4 = (2, (e^4 + e^3 - e + 1)/(e^3 - e))$, $P_5 = (1, 2e^2/(e^2 - 1))$, $P_6 = (0, (e^2 + 2e - 1)/(e^2 - 1))$, $P_7 = (-1, 2e^2/(e^2 - 1))$, $P_8 = (-(e^2 + 1)/e, (e^4 + e^3 - e + 1)/(e^3 - e))$, $P_9 = (-(\pi + 4)/2, 0)$, $P_{10} = (-2, -\pi/4)$, $P_{11} = ((\pi - 4)/2, 0)$, $P_{12} = (-2, \pi/4)$, where the parameters $\lambda_i = (0, 0, 8.91206, 8.91206, 1, 1, 1, 1, 0.5, 0)$ ($i = 1, 2, \dots, 11$). The 1st segment of the bending curve is a trigonometric curve, which is a part of the cycloid. The 5th and 6th segment are the catenary and hyperbola, respectively. The 10th segment is the parabola arc. Since the parameters $\lambda_3 = \lambda_4 = 12.0938$, the blending curve interpolates the points P_4 and P_5 .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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