Research Article

Fixed Points and Stability of a Class of Integrodifferential Equations

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We study a class of integrodifferential functional differential equations $\ddot{x} + f(t, x, \dot{x})\dot{x} + \sum_{j=1}^N \int_{t-r_j(t)}^t a_j(t, s)g_j(s, x(s))ds = 0$ with variable delay. By using the fixed point theory, we establish necessary and sufficient conditions ensuring that the zero solution of this equation is asymptotically stable.

1. Introduction

It is well known that differential equations have many applications in engineering, control theory, neural networks, biology, and so on. The stability of the solution of functional differential equations has drawn a lot of attention for many years. Liapunov's direct method has been widely applied to study the stability problems for a long time; see, for example, [1–3]. Nowadays, many authors have applied the fixed points theory to study the stability of solution of functional differential equations; some authors even use fixed point theory to study stability conditions for stochastic equations with delay; see, for example, [4–7] and the references therein.

Burton studied stability of a nonconvolution equation

$$\dot{x} = - \int_{t-r}^t a(t, s) g(x(s)) ds, \quad (1)$$

where r was a positive constant. He obtained sufficient conditions ensuring that the zero solution was asymptotically stable via fixed point theorem; see [8].

Becker and Burton studied the following differential equation:

$$\dot{x} = - \int_{t-r(t)}^t a(t, s) g(x(s)) ds \quad (2)$$

with variable delay and its special case

$$\dot{x} = - \int_{t-r}^t a(t-s) g(x(s)) ds, \quad (3)$$

for r , is a positive constant, $t \geq 0$, where function $r(t) : [0, \infty) \rightarrow [0, \infty)$. They found sufficient conditions ensuring that the zero solution was asymptotically stable by changing the supremum metric to an exponentially weighted metric. Moreover, they hoped to reduce condition that the function $t - r(t)$ was strictly increasing; see [9]. Jin and Luo reduced this condition in their paper. Moreover, they established necessary and sufficient conditions that could ensure that the zero solution of this equation is asymptotically stable; see, for example, [10].

Ardjouni et al. studied the stability of the following neutral integrodifferential equation:

$$\dot{x}(t) = - \sum_{j=1}^N \int_{t-\tau_j(t)}^t a_j(t, s) x(s) ds + \sum_{j=1}^N c_j(t) x'(t - \tau_j(t)) \quad (4)$$

via fixed point theorem. Also $m_j(0) = \inf\{t - \tau_j(t) : t \geq 0\}$. Their main result was as follows.

Theorem A. Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in \mathbb{R}^+$ and there exist continuous functions

$h_j : [m_j(0), \infty) \rightarrow \mathbb{R}$ for $j = 1, 2, \dots, N$ and a constant $\alpha \in (0, 1)$ such that, for $t \geq 0$,

$$(i) \liminf_{t \rightarrow \infty} \int_0^t H(s) ds > -\infty,$$

(ii)

$$\begin{aligned} & \sum_{j=1}^N \left| \frac{c_j(t)}{1 - \tau_j'(t)} \right| + \sum_{j=1}^N \int_{t-\tau_j(t)}^t |h_j(s) + B_j(t, s)| ds \\ & + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u) du} |h_j(s - \tau_j(s)) (1 - \tau_j'(s) - r_j(s))| ds \\ & + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u) du} |H(s)| \\ & \times \int_{s-\tau_j(s)}^s |h_j(u) + B_j(s, u)| du ds \leq \alpha, \end{aligned} \quad (5)$$

where

$$\begin{aligned} H(t) &= \sum_{j=1}^N h_j(t), \\ r_j(t) &= \frac{[c_j(t) H(t) + c_j'(t)] (1 - \tau_j'(t)) + c_j(t) \tau_j''(t)}{(1 - \tau_j'(t))^2}, \\ B_j(t, s) &= \int_t^s a_j(u, s) du, \\ B_j(t, t - \tau_j(t)) &= \int_t^{t-\tau_j(t)} a_j(u, t - \tau_j(t)) du. \end{aligned} \quad (6)$$

Then the zero solution is asymptotically stable if and only if

$$\int_0^t H(s) ds \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (7)$$

They improved the results of [9]; see [11]. Other results on fixed points and stability properties in equations with variable delays can be found in [5].

Recently, Zhao and Yuan investigate the stability of a generalized Volterra-Levin equation:

$$\begin{aligned} \dot{x} &= - \int_{t-L}^t p(s-t) g(s, x(s)) ds, \quad t > t_0, \\ x(t) &= \psi(t), \quad t_0 - L \leq t \leq t_0. \end{aligned} \quad (8)$$

They give new conditions on the stability of zero solution of this equation; their conditions are different from [5]; see, for example, [12].

The stability of second order equation has always been hot issue of research. Levin and Nohel [13] studied the global asymptotic stability of a class of nonlinear systems:

$$\ddot{x}(t) + h(t, x, \dot{x}) \dot{x} + f(x) = e(t). \quad (9)$$

They obtained asymptotic stability by constructing proper Lyapunov function.

Burton [14] considered the following equation with delay:

$$\ddot{x} + f(t, x, \dot{x}) \dot{x} + b(t) g(x(t-L)) = 0, \quad (10)$$

where L is a positive constant, and obtained some sufficient conditions under which each solution $x(t)$ satisfies $(x(t), \dot{x}(t)) \rightarrow 0$ via the fixed point theorem.

We generalized the above equation to an equation with a variable delay [15] as follows:

$$\ddot{x} + f(t, x, \dot{x}) \dot{x} + b(t) g(x(t - \tau(t))) = 0, \quad (11)$$

and obtained some results on asymptotic stability of the zero solution. Before we introduce our new results we recall the main results in [15]. There are *basic assumptions* on the delay function $\tau(t)$: $(\mathcal{A}) t - \tau(t)$ which is strictly increasing and $\lim_{t \rightarrow \infty} t - \tau(t) = \infty$. $p(t)$ is the inverse of $t - \tau(t)$. Moreover $0 \leq b(t) \leq M$ for some constant $M > 0$.

The main results in [15] can be stated as follows.

Theorem B. Suppose $(\mathcal{A})$ and the following conditions hold.

- (i) There exists a constant $l > 0$ such that $g(x)$ satisfies the Lipschitz condition on $[-l, l]$. The function $g(x)$ is odd and is strictly increasing on $[-l, l]$, and $x - g(x)$ is nondecreasing on $[0, l]$.
- (ii) There exist an $\alpha \in (0, 1)$ and a continuous function $a(t) : [0, \infty) \rightarrow [0, \infty)$ such that $f(t, x, y) \geq a(t)$ for $t \geq 0, x \in \mathbb{R}, y \in \mathbb{R}, \int_0^\infty a(t) dt = \infty$, and

$$\begin{aligned} & 2 \sup_{t \geq 0} \int_t^{p(t)} \int_0^\infty e^{-\int_s^{w+s} a(v) dv} b(s) dw ds \\ & + 2 \sup_{t \geq 0} \int_0^t \int_{t-s}^\infty e^{-\int_s^{w+s} a(v) dv} b(s) dw ds \leq \alpha. \end{aligned} \quad (12)$$

- (iii) There exist constants $a_0 > 0$ and $Q > 0$ such that, for each $t \geq 0$, if $J \geq Q$, then

$$\int_t^{t+J} a(v) dv \geq a_0 J. \quad (13)$$

- (iv) There exist continuous functions $F : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ and $c(t) : [0, \infty) \rightarrow [0, \infty)$ such that, for all $t \geq 0, x \in \mathbb{R}, y \in \mathbb{R}, f(t, x, y) \leq F(x, y)c(t)$. The function $g'(x)$ is continuous on $[-l, l]$, $g'(0) \neq 0$.

If, for each $\gamma > 0, \iint_0^\infty e^{-\int_s^{w+s} \gamma c(v)} b(s) dw ds = \infty$, then the zero solution of (11) is asymptotically stable.

In [16], we have studied the following equation:

$$\ddot{x} + f(t, x, \dot{x}) \dot{x} + \int_{t-r(t)}^t a(t, s) g(x(s)) ds = 0 \quad (14)$$

with variable delay. By using the fixed point theory, we obtain conditions which ensure that the zero solution of this equation is stable under an exponentially weighted metric.

Moreover, we establish necessary and sufficient conditions ensuring that the zero solution is asymptotically stable.

In this paper we consider

$$\ddot{x} + f(t, x, \dot{x}) \dot{x} + \sum_{j=1}^N \int_{t-r_j(t)}^t a_j(t, s) g_j(s, x(s)) ds = 0 \quad (15)$$

for $t \geq 0$, where functions $r_j(t) : [0, \infty) \rightarrow [0, \infty)$, $a_j(t, s) : [0, \infty) \times [-r(0), \infty) \rightarrow R$, $g_j : [-r(0), \infty) \times R \rightarrow R$, and $f : R \times R \times R \rightarrow R^+$ are all continuous, where $R^+ = [0, +\infty)$. We assume that $t - r_j(t) \rightarrow \infty$ as $t \rightarrow \infty$, $j = 1, 2, \dots, N$.

For each $t_0 \geq 0$, define $m_j(t_0) = \inf\{s - r_j(s) : s \geq t_0\}$, $j = 1, 2, \dots, N$. Let $m(t_0) = \min\{m_1(t_0), \dots, m_N(t_0)\}$. Also $C(t_0) = C([m(t_0), t_0], R)$ is a function space endowed with function norm $\|\cdot\|$, where $\|\psi\| = \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}$. We will also use $\|\phi\|$ to express the supremum on $[m(t_0), \infty)$ later. It is well known that, in [2], for a given continuous function ϕ , there exists a solution of (15) on an interval $[t_0, T)$; if the solution remains bounded, then $T = \infty$. We denote by $(x(t), y(t))$ the solution $(x(t, t_0, \phi), y(t, t_0, \phi))$. Denote by $A(t) := f(t, x(t), y(t))$.

We can write (15) in an equivalent form as

$$\dot{x} = y,$$

$$\dot{y} = -A(t) y - \sum_{j=1}^N \int_{t-r_j(t)}^t a_j(t, s) g_j(s, x(s)) ds. \quad (16)$$

We will give a necessary and sufficient condition ensuring that the zero solution of this equation is asymptotically stable. To our knowledge, there are few results about its stability. This paper is organized as follows. In the next section we will state our main results. Their proofs will be given in Sections 3 and 4.

2. Statement of Main Results

First of all, we consider a special case of (15). We consider the following equation:

$$\ddot{x} + f(t, x, \dot{x}) \dot{x} + \int_{t-L}^t p(s-t) g(s, x(s)) ds = 0. \quad (17)$$

Rewrite this equation as

$$\dot{x} = y,$$

$$\dot{y} = -A(t) y - \int_{t-L}^t p(s-t) g(s, x(s)) ds. \quad (18)$$

Our main results are as follows.

Theorem 1. Assume that $h(t) \in C(R^+, R^+)$ and the following conditions hold.

- (i) Consider that $\int_{-L}^0 p(s) ds = 1$. There exist a constant $\lambda > 0$ and a function $Q(x) \in C(R, R^+)$ such that, for $\forall x, y_1, y_2 \in R$, $|g(x, y_1) - g(x, y_2)| \leq Q(x)|y_1 - y_2|$ and the inequality $\sup_{t \geq 0} \int_{-L}^0 |p(s)| \int_{t-L}^{t+s} Q(u) du ds \leq \lambda$ hold. There exists a constant $K > 0$ such that $Q(t) \leq K$.

- (ii) There exist $\alpha \in (0, 1)$ and a continuous function $A_1(t) : [0, \infty) \rightarrow [0, \infty)$ such that $f(t, x, y) \geq A_1(t)$ for $t \geq 0$, $x \in R$, $y \in R$

$$\int_0^t e^{-\int_u^t h(s) ds} \int_0^u e^{-\int_s^u A_1(v) dv} Q(s-L) ds du + 2\lambda \int_0^t e^{-\int_u^t h(s) ds} du \leq \alpha. \quad (19)$$

- (iii) There exist constants $a_0 > 0$ and $Q_0 > 0$ such that, for each $t \geq 0$, if $J \geq Q_0$, then

$$\int_t^{t+J} A_1(s) ds \geq a_0 J. \quad (20)$$

Then the zero solution of (17) is stable.

In addition, we have the following.

Theorem 2. Assume that $\lim_{t \rightarrow \infty} t - r_j(t) = \infty$, $r_j(t)$ is differentiable, $j = 1, 2, \dots, N$, and there exist some functions $h_j(t) \in C(R^+, R^+)$ such that for $t \geq 0$ the following conditions hold.

- (i) There exist a constant $\Gamma > 0$ and some functions $Q_j(x) \in C(R, R^+)$ such that, for $\forall x, y_1, y_2 \in R$,

$$|g_j(x, y_1) - g_j(x, y_2)| \leq Q_j(x) |y_1 - y_2|, \quad j = 1, 2, \dots, N, \\ \sup_{t \geq 0} \sum_{j=1}^N \int_{t-r_j(t)}^t \left(\int_t^u |a_j(v, t)| dv \right) Q_j(u) du \leq \Gamma. \quad (21)$$

There exists a constant $K^* > 0$ such that $Q_j(t) \leq K^*$, $j = 1, 2, \dots, N$.

- (ii) Let $H(t) = \sum_{j=1}^N h_j(t)$. There exist a constant $\alpha \in (0, 1)$ and a continuous function $A_1(t) : R^+ \rightarrow R^+$ such that $f(t, x, y) \geq A_1(t)$. For $\forall t \geq 0$, $x \in R$, $y \in R$,

$$2 \sum_{j=1}^N \int_{t-r_j(t)}^t h_j(v) dv + 2\Gamma \int_0^t e^{-\int_u^t H(v) dv} du \\ + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u) du} h_j(s-r_j(s)) |1-r_j'(s)| ds \\ + \sum_{j=1}^N \int_0^t e^{-\int_u^t H(v) dv} \left(\int_0^u e^{-\int_s^u A_1(v) dv} K^* |W_j(s)| ds \right) du \leq \alpha, \quad (22)$$

where $W_j(s) = (1-r_j'(s)) \int_s^{s-r_j(s)} a_j(v, s-r_j(s)) dv$.

- (iii) There exist constants $a_0 > 0$ and $Q_0 > 0$ such that, for each $t \geq 0$, if $J \geq Q_0$, then

$$\int_t^{t+J} A_1(s) ds \geq a_0 J. \quad (23)$$

Then the zero solution of (15) is asymptotically stable if and only if

$$\int_0^t H(s) ds \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (24)$$

3. Proof of Theorem 1

In this section, we will prove Theorem 1 by applying the fixed point theory. We will give the expression of the solution of the related equation. The following result can be found in [12].

Lemma 3. Let the function $p : [-L, 0] \rightarrow R$, $g \in C(R \times R, R)$, $\int_{-L}^0 p(s)ds = 1$. Then

$$\dot{x}(t) = - \int_{t-L}^t p(s-t) g(s, x(s)) ds \quad (25)$$

is equivalent to

$$\begin{aligned} \dot{x}(t) = & -g(t-L, x(t-L)) \\ & + \frac{d}{dt} \int_{-L}^0 p(s) \int_{t+s}^{t-L} g(u, x(u)) du ds. \end{aligned} \quad (26)$$

Lemma 4. Let $\psi : [-L, 0] \rightarrow R$ be a given continuous function; if $(x(t), y(t))$ is the solution of (17) satisfying $x(t) = \psi(t)$, $t \in [-L, 0]$, and $y(0) = \dot{x}(0)$, then $x(t)$ is the solution of the following integral equation:

$$\begin{aligned} x(t) = & \psi(0) e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L)) ds \right) du \\ & + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \right. \\ & \quad \times \int_{-L}^0 p(v) \int_v^{-L} g(w, \psi(w)) dw dv \Big) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u+v}^{u-L} g(w, x(w)) dw dv \right) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ & \quad \times \left(\int_{-L}^0 p(v) \right. \\ & \quad \quad \times \left. \int_{s+v}^{s-L} g(w, x(w)) dw dv \right) ds \Big] du, \end{aligned} \quad (27)$$

where $B(t) = \dot{x}(0) e^{-\int_0^t A(s)ds}$.

Proof. We apply the variation of parameters formula to the second equation of (18); we obtain

$$\begin{aligned} \dot{x}(t) = & \dot{x}(0) e^{-\int_0^t A(s)ds} \\ & - \int_0^t e^{-\int_s^t A(v)dv} \left(\int_{s-L}^s p(v-s) g(v, x(v)) dv \right) ds. \end{aligned} \quad (28)$$

By Lemma 3, (28) can be written as

$$\begin{aligned} \dot{x}(t) = & B(t) - \int_0^t e^{-\int_s^t A(v)dv} g(s-L, x(s-L)) ds \\ & + \int_0^t e^{-\int_s^t A(v)dv} \\ & \quad \times \left(\frac{d}{ds} \int_{-L}^0 p(s) \int_{s+v}^{s-L} g(u, x(u)) du dv \right) ds; \end{aligned} \quad (29)$$

multiplying both sides of the above equation by $e^{\int_0^t h(s)ds}$, we see that

$$\begin{aligned} x(t) = & \psi(0) e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L)) ds \right) du \\ & + \int_0^t e^{-\int_u^t h(s)ds} \\ & \quad \times \left[\int_0^u e^{-\int_s^u A(v)dv} \right. \\ & \quad \times \left(\frac{d}{ds} \int_{-L}^0 p(s) \right. \\ & \quad \quad \times \left. \int_{s+v}^{s-L} g(w, x(w)) dw dv \right) ds \Big] du. \end{aligned} \quad (30)$$

If we integrate the last term by parts, we have

$$\begin{aligned} x(t) = & \psi(0) e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_0^u e^{-\int_s^u A(v)dv} g(s-L, x(s-L)) ds \right) du \\ & + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \right. \\ & \quad \times \int_{-L}^0 p(v) \int_v^{-L} g(w, \psi(w)) dw dv \Big) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u+v}^{u-L} g(w, x(w)) dw dv \right) du \\ & - \int_0^t e^{-\int_u^t h(s)ds} \\ & \quad \times \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ & \quad \times \left(\int_{-L}^0 p(v) \right. \\ & \quad \quad \times \left. \int_{s+v}^{s-L} g(w, x(w)) dw dv \right) ds \Big] du. \end{aligned} \quad (31)$$

□

Let $(C, \|\cdot\|)$ be the Banach space of bounded continuous functions on $[-L, \infty)$ with the supremum norm. For a given continuous initial function $\psi : [-L, 0] \rightarrow R$, define the set $C_\psi \subset C$ by

$$C_\psi = \{\phi : [-L, \infty) \rightarrow R \mid \phi \in C, \phi(t) = \psi(t), t \in [-L, 0]\} \quad (32)$$

and its subset

$$C_\psi^l = \{\phi : [-L, \infty) \rightarrow R \mid \phi \in C, \phi(t) = \psi(t), \\ t \in [-L, 0], |\phi(t)| \leq l, t > 0\}, \quad (33)$$

where $\psi : [-L, 0] \rightarrow [-l, l]$ is a given initial function and l is a positive constant. Let P_1 be a mapping defined on C_ψ^l as follows: for $\phi \in C_\psi^l$, if $t \in [-L, 0]$, $(P_1\phi)(t) = \psi(t)$. If $t > 0$,

$$\begin{aligned} (P_1\phi)(t) &= \psi(0) e^{-\int_0^t h(s)ds} + \int_0^t e^{-\int_u^t h(s)ds} B(u) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \int_0^u e^{-\int_s^u A(v)dv} g(s-L, \phi(s-L)) ds du \\ &\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(e^{-\int_0^u A(v)dv} \right. \\ &\quad \times \left. \int_{-L}^0 p(v) \int_v^{-L} g(w, \psi(w)) dw dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u+v}^{u-L} g(w, \phi(w)) dw dv \right) du \\ &\quad - \int_0^t e^{-\int_u^t h(s)ds} \left[\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ &\quad \times \left(\int_{-L}^0 p(v) \right. \\ &\quad \times \left. \left. \int_{s+v}^{s-L} g(w, \phi(w)) dw dv \right) ds \right] du. \end{aligned} \quad (34)$$

Easy calculation shows that

$$\begin{aligned} |(P_1\phi)(t) - (P_1\eta)(t)| &\leq \int_0^t e^{-\int_u^t h(s)ds} \int_0^u e^{-\int_s^u A(v)dv} |g(s-L, \phi(s-L)) \\ &\quad - g(s-L, \eta(s-L))| ds du \\ &\quad + \int_0^t e^{-\int_u^t h(s)ds} \left(\int_{-L}^0 p(v) \int_{u+v}^{u-L} |g(w, \phi(w)) \right. \\ &\quad \left. - g(w, \eta(w))| dw dv \right) du \end{aligned}$$

$$\begin{aligned} &+ \int_0^t e^{-\int_u^t h(s)ds} \\ &\quad \times \left(\int_0^u e^{-\int_s^u A(v)dv} A(s) \right. \\ &\quad \times \left(\int_{-L}^0 p(v) \right. \\ &\quad \times \left. \left. \int_{s+v}^{s-L} |g(w, \phi(w)) \right. \right. \\ &\quad \left. \left. - g(w, \eta(w))| dw dv \right) ds \right) du. \end{aligned} \quad (35)$$

We can easily obtain that $(P_1\phi)(t)$ is a contraction mapping by using condition (ii). We continue to prove Theorem 1. Choose $\psi : [-L, 0] \rightarrow R$ and $\dot{x}(0)$ such that

$$\left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) |\dot{x}(0)| + \delta + \lambda \delta \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \leq (1-\alpha)l. \quad (36)$$

Condition (i) implies that $g(t, 0) = 0$, since $g(t, x)$ satisfies the Lipschitz condition with respect to x , such that δ exists and $\delta < l$. By the expression of $(P_1\phi)(t)$, condition (ii), and condition (iii) of Theorem 1, we have

$$\begin{aligned} \int_0^t e^{-\int_0^u A(s)ds} du &= \int_0^{Q_0} e^{-\int_0^u A(s)ds} du + \int_{Q_0}^t e^{-\int_0^u A(s)ds} du \\ &\leq \int_0^{Q_0} du + \int_{Q_0}^t e^{-a_0 u} du \\ &\leq Q_0 + \frac{e^{-a_0 Q_0}}{a_0}. \end{aligned} \quad (37)$$

According to condition (ii) and (36), we have

$$|(P_1\phi)(t)| \leq (1-\alpha)l + \alpha l = l. \quad (38)$$

Note that if $t \in [-L, 0]$, then $(P_1\phi)(t) = \psi(t)$. We obtain that $|(P_1\phi)(t)| \leq l$, $t \in [-L, \infty)$. Thus $P_1 : C_\psi^l \rightarrow C_\psi^l$. Since we have proved that P_1 is a contraction mapping, hence P_1 has a unique fixed point $x(t)$ and $|x(t)| \leq l$.

Recalling (28), we have

$$\begin{aligned} |y(t)| &\leq |\dot{x}(0)| e^{-\int_0^t A(s)ds} \\ &\quad + \int_0^t e^{-\int_s^t A(v)dv} \int_{s-L}^s |p(s-v)| |g(v, x(v))| dv ds; \end{aligned} \quad (39)$$

then

$$\begin{aligned} |y(t)| &\leq |\dot{x}(0)| + Kl \cdot \int_0^t e^{-\int_u^t A(s)ds} du \\ &< l \left(1 + K \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \right). \end{aligned} \quad (40)$$

It follows that

$$|x(t)| + |y(t)| < l \left(2 + K \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \right). \quad (41)$$

For any given $\epsilon > 0$; we only need to replace ϵ by l . Then we obtain the stability of zero solution. This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section, we will prove Theorem 2. First of all, we will obtain a useful expression for the solution of (15) in the following proof.

We apply the variation of parameters formula to the second equation of (15); then

$$\begin{aligned} \dot{x}(t) = B_1(t) - \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t A(v)dv} \\ \times \left(\int_{s-r_j(s)}^s a_j(s, v) g_j(v, x(v)) dv \right) ds, \end{aligned} \quad (42)$$

where $B_1(t) = \dot{x}(t_0)e^{-\int_{t_0}^t A(s)ds}$.

Lemma 5. *The equation*

$$\dot{x}(t) = - \sum_{j=1}^N \int_{t-r_j(t)}^t a_j(t, s) g_j(s, x(s)) ds \quad (43)$$

is equivalent to

$$\begin{aligned} \dot{x}(t) &= \sum_{j=1}^N C_j(t, t-r_j(t)) (1-r_j'(t)) g_j(t-r_j(t), x(t-r_j(t))) \\ &+ \sum_{j=1}^N \frac{d}{dt} \int_{t-r_j(t)}^t C_j(t, s) g_j(s, x(s)) ds, \end{aligned} \quad (44)$$

where

$$\begin{aligned} C_j(t, s) &= \int_t^s a_j(u, s) du, \\ C_j(t, t-r_j(t)) &= \int_t^{t-r_j(t)} a_j(u, t-r_j(t)) du. \end{aligned} \quad (45)$$

Proof. Differentiating the integral term in (44), we have

$$\begin{aligned} &\frac{d}{dt} \int_{t-r_j(t)}^t C_j(t, s) g_j(s, x(s)) ds \\ &= \int_{t-r_j(t)}^t \frac{\partial}{\partial t} C_j(t, s) g_j(s, x(s)) ds + C_j(t, t) g_j(t, x(t)) \\ &\quad - C_j(t, t-r_j(t)) g_j(t-r_j(t), x(t-r_j(t))) \\ &\quad \times (1-r_j'(t)). \end{aligned} \quad (46)$$

It follows that if $C_j(t, t) = 0$, $(\partial/\partial t)C_j(t, s) = -a_j(t, s)$, then (43) is equivalent to (44). Easy calculation shows that

$$\begin{aligned} C_j(t, s) &= \int_t^s a_j(u, s) du, \\ C_j(t, t-r_j(t)) &= \int_t^{t-r_j(t)} a_j(u, t-r_j(t)) du. \end{aligned} \quad (47)$$

□

By Lemma 5, (42) can be written as

$$\begin{aligned} \dot{x}(t) = B_1(t) + \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t A(v)dv} \\ \times \left(\frac{d}{ds} \int_{s-r_j(s)}^s C_j(s, v) \right. \\ \left. \times g_j(v, x(v)) dv \right) ds \\ + \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t A(v)dv} \left(C_j(s, s-r_j(s)) (1-r_j'(s)) \right. \\ \left. \times g_j(s-r_j(s), x(s-r_j(s))) \right) ds. \end{aligned} \quad (48)$$

We define

$$\begin{aligned} C_\psi^0 = \{ \phi : [m(t_0), \infty) \longrightarrow R \mid \phi \in C, \phi(t) = \psi(t), \\ t \in [m(t_0), t_0], \phi(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \}. \end{aligned} \quad (49)$$

Multiplying both sides of the above equation by $e^{\int_{t_0}^t H(u)du}$, easy calculation shows that

$$\begin{aligned} x(t) &= \psi(t_0) e^{-\int_{t_0}^t H(u)du} + \int_{t_0}^t e^{-\int_s^t H(u)du} B_1(s) ds \\ &\quad + \int_{t_0}^t e^{-\int_s^t H(u)du} H(s) x(s) ds \\ &\quad + \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v)dv} \\ &\quad \times \left(\int_{t_0}^u e^{-\int_s^u A(v)dv} \frac{d}{ds} \right. \\ &\quad \times \int_{s-r_j(s)}^s C_j(s, v) g_j(v, x(v)) dv \Big) ds du \\ &\quad + \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v)dv} \left(\int_{t_0}^u e^{-\int_s^u A(v)dv} W_j(s) \right. \\ &\quad \times g_j(s-r_j(s), x(s-r_j(s))) ds \Big) du, \end{aligned} \quad (50)$$

where $W_j(s) = C_j(s, s - r_j(s))(1 - r'_j(s))$. Performing an integration by parts, then

$$\begin{aligned}
 x(t) &= \psi(t_0) e^{-\int_{t_0}^t H(s) ds} + \int_{t_0}^t e^{-\int_s^t H(u) du} B_1(s) ds \\
 &+ \sum_{j=1}^N \int_{t-r_j(t)}^t h_j(v) x(v) dv \\
 &- \sum_{j=1}^N \int_{t_0-r_j(t_0)}^{t_0} h_j(u) \psi(u) du \cdot e^{-\int_{t_0}^t H(u) du} \\
 &- \sum_{j=1}^N \int_{t_0}^t \left(\int_{s-r_j(s)}^s h_j(v) x(v) dv \right) e^{-\int_s^t H(u) du} H(s) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t H(u) du} h_j(s - r_j(s)) x(s - r_j(s)) \\
 &\quad \times (1 - r'_j(s)) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \left(\int_{u-r_j(u)}^u C_j(u, v) g_j(v, x(v)) dv \right) du \\
 &- \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \left[\int_{t_0}^u e^{-\int_{t_0}^u A(v) dv} \right. \\
 &\quad \times \left(\int_{t_0-r_j(t_0)}^{t_0} C_j(t_0, v) \right. \\
 &\quad \times g_j(v, \psi(v)) dv \left. \right) ds \left. \right] du \\
 &- \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t H(u) du} \left(\int_{t_0}^s e^{-\int_v^s A(r) dr} A(v) \right. \\
 &\quad \times \int_{v-r_j(v)}^v C_j(v, u) \\
 &\quad \times g_j(u, x(u)) du dv \left. \right) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \left(\int_{t_0}^u e^{-\int_s^u A(v) dv} W_j(s) \right. \\
 &\quad \times g_j(s - r_j(s), x(s - r_j(s))) ds \left. \right) du.
 \end{aligned} \tag{51}$$

Let P_2 be a mapping defined on C_ψ^0 as follows: for $\phi \in C_\psi$, if $t \in [m(t_0), t_0]$, $(P_2\phi)(t) = \psi(t)$. If $t > t_0$, we define

$$\begin{aligned}
 (P_2\phi)(t) &= \psi(t_0) e^{-\int_{t_0}^t H(s) ds} + \int_{t_0}^t e^{-\int_s^t H(u) du} B_1(s) ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j=1}^N \int_{t-r_j(t)}^t h_j(v) \phi(v) dv \\
 &- \sum_{j=1}^N \int_{t_0-r_j(t_0)}^{t_0} h_j(u) \psi(u) du \cdot e^{-\int_{t_0}^t H(u) du} \\
 &- \sum_{j=1}^N \int_{t_0}^t \left(\int_{s-r_j(s)}^s h_j(v) \phi(v) dv \right) e^{-\int_s^t H(u) du} H(s) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t H(u) du} h_j(s - r_j(s)) \phi(s - r_j(s)) \\
 &\quad \times (1 - r'_j(s)) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \left(\int_{u-r_j(u)}^u C_j(u, v) g_j(v, \phi(v)) dv \right) du \\
 &- \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \\
 &\quad \times \left[\int_{t_0}^u e^{-\int_{t_0}^u A(v) dv} \right. \\
 &\quad \times \left(\int_{t_0-r_j(t_0)}^{t_0} C_j(t_0, v) \right. \\
 &\quad \times g_j(v, \psi(v)) dv \left. \right) ds \left. \right] du \\
 &- \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t H(u) du} \\
 &\quad \times \left(\int_{t_0}^s e^{-\int_v^s A(r) dr} A(v) \right. \\
 &\quad \times \int_{v-r_j(v)}^v C_j(v, u) g_j(u, \phi(u)) du dv \left. \right) ds \\
 &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t H(v) dv} \\
 &\quad \times \left(\int_{t_0}^u e^{-\int_s^u A(v) dv} \right. \\
 &\quad \times W_j(s) g_j(s - r_j(s), \phi(s - r_j(s))) ds \left. \right) du.
 \end{aligned} \tag{52}$$

If $\phi \in C_\psi^0$, since $\int_0^t H(s) ds \rightarrow \infty$ as $t \rightarrow \infty$, $t_0 \geq 0$, hence $\int_{t_0}^t H(s) ds \rightarrow \infty$ as $t \rightarrow \infty$. The first term and fourth term of $(P_2\phi)(t) \rightarrow 0$, as $t \rightarrow \infty$.

Note that

$$\int_{t_0}^t e^{-\int_u^t H(s) ds} B_1(u) du = \dot{x}(t_0) \int_{t_0}^t e^{-\int_u^t H(s) ds} e^{-\int_{t_0}^u A(s) ds} du. \tag{53}$$

Since $A(t) = f(t, x(t), y(t)) \geq A_1(t) \geq 0$, then

$$\int_{t_0}^t e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^u A(s)ds} du \leq \int_{t_0}^t e^{-\int_{t_0}^u A(s)ds} du. \quad (54)$$

For a given $\epsilon > 0$, there exists $T_1 > Q_0 + t_0$ such that $e^{a_0 t_0 - a_0 T} < a_0 \epsilon$. For $T_1 < t$ and $t_0 < Q_0 < T_1 \leq u$, we have

$$\begin{aligned} \int_{T_1}^t e^{-\int_{t_0}^u A(s)ds} du &\leq \int_{T_1}^t e^{-a_0(u-t_0)} du = \frac{e^{a_0 t_0}}{a_0} (e^{-a_0 T_1} - e^{-a_0 t}) \\ &< \frac{e^{a_0 t_0 - a_0 T_1}}{a_0} < \epsilon. \end{aligned} \quad (55)$$

For $t > T_1$, we have

$$\begin{aligned} \int_{t_0}^{T_1} e^{-\int_u^t H(s)ds} e^{-\int_{t_0}^u A(s)ds} du \\ = e^{-\int_{T_1}^t H(s)ds} \cdot \int_{t_0}^{T_1} e^{-\int_u^{T_1} H(s)ds} e^{-\int_{t_0}^u A(s)ds} du. \end{aligned} \quad (56)$$

The second term of $(P_2 \phi)(t) \rightarrow 0$, as $t \rightarrow \infty$.

Note that $\int_{t_0-r_j(t_0)}^{t_0} C_j(t_0, v) g_j(v, \psi(v)) dv$ is bounded; we can check the eighth term of $(P_2 \phi)(t) \rightarrow 0$, as $t \rightarrow \infty$ similarly.

Then we can easily check that P_2 is a contraction mapping on C_ψ^0 by using condition (ii). By the contraction mapping principle, P_2 has a unique fixed point $x(t)$ in C_ψ^0 . Also $x(t) \rightarrow 0$ as $t \rightarrow +\infty$.

In order to obtain the asymptotic stability, we still need to show that the zero solution is stable. Let $\epsilon > 0$ be given, we choose $\psi : [m(t_0), t_0] \rightarrow R$ and $\dot{x}(t_0)$ such that

$$\begin{aligned} \delta + \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) |\dot{x}(t_0)| \\ + \sum_{j=1}^N \int_{t_0-r_j(t_0)}^{t_0} h_j(s) ds \cdot \delta + \alpha \Gamma \left(Q + \frac{e^{-a_0 Q}}{a_0} \right) \delta \\ \leq (1 - \alpha) \epsilon. \end{aligned} \quad (57)$$

Then we have

$$\begin{aligned} |x(t)| \leq \delta + \int_{t_0}^t e^{-\int_u^t H(s)ds} |\dot{x}(t_0)| e^{-\int_{t_0}^u A(s)ds} du \\ + \sum_{j=1}^N \int_{t_0-r_j(t_0)}^{t_0} h_j(s) ds \cdot \delta + \alpha \Gamma \left(Q + \frac{e^{-a_0 Q}}{a_0} \right) \delta + \alpha \epsilon. \end{aligned} \quad (58)$$

Easy calculation shows that

$$\begin{aligned} \int_{t_0}^t e^{-\int_{t_0}^u A(s)ds} du &\leq \int_{t_0}^{t_0+Q_0} du + \int_{t_0+Q_0}^t e^{-a_0(u-t_0)} du \\ &\leq Q_0 + \frac{e^{-a_0 Q_0}}{a_0}. \end{aligned} \quad (59)$$

Hence

$$|x(t)| \leq \epsilon. \quad (60)$$

Therefore

$$\begin{aligned} |\dot{x}(t)| \\ \leq |\dot{x}(t_0)| + \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t A(v)dv} \\ \times \left(\int_{s-r_j(s)}^s |a_j(s, v) g_j(v, x(v))| dv \right) ds. \end{aligned} \quad (61)$$

It follows that

$$|x(t)| + |y(t)| < \epsilon \left(2 + NK^* \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \right). \quad (62)$$

Therefore the zero solution is stable; since we have obtained that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, it follows that the zero solution is asymptotically stable.

A necessary condition is as follows. For each $t_0 \geq 0$, we denote that $M = \sup_{t \geq 0} \{e^{-\int_0^t H(s)ds}\}$. We will prove that $\int_0^\infty H(s)ds = \infty$ by way of contradiction. If $\int_0^\infty H(s)ds < \infty$, since $H(t) \geq 0$, $\exists$ being a sequence $\{t_n\}$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \int_0^{t_n} H(s)ds = \sigma$, where σ is a finite number. Choose η such that $-\eta \leq \int_0^{t_n} H(s)ds \leq \eta$ holds, for all $n \geq 1$.

Denote that

$$\begin{aligned} \omega(u) \triangleq H(u) \sum_{j=1}^N \int_{u-r_j(u)}^u |h_j(v)| dv \\ + \sum_{j=1}^N |h_j(s-r_j(s)) (1-r_j'(s))| \\ + \sum_{j=1}^N \int_{t_0}^u e^{-\int_v^u A(r)dr} A(v) \\ \times \int_{v-r_j(v)}^v |C_j(v, w)| Q_j(w) dw dv \\ + \sum_{j=1}^N \int_0^t e^{-\int_u^t H(v)dv} \left(\int_0^u e^{-\int_s^u A_1(v)dv} K^* \right. \\ \times |W_j(s)| ds \Big) du \\ + \sum_{j=1}^N \int_{u-r_j(u)}^u |C_j(u, v)| Q_j(v) dv du, \end{aligned} \quad (63)$$

where $W_j(s) = (1-r_j'(s)) \int_s^{s-r_j(s)} a_j(v, s-r_j(s)) dv$.

By conditions of Theorem 2, we have

$$0 \leq \int_0^{t_n} \omega(u) e^{-\int_u^{t_n} H(s)ds} du \leq \alpha. \quad (64)$$

Then

$$\int_0^{t_n} \omega(u) e^{\int_0^u H(s) ds} du \leq \alpha e^{\int_0^{t_n} H(s) ds} \leq \alpha e^\eta. \quad (65)$$

Thus the sequence $\{\int_0^{t_n} \omega(u) e^{\int_0^u H(s) ds} du\}$ is bounded; there exists a convergent subsequence; we assume that $\lim_{n \rightarrow \infty} \int_0^{t_n} \omega(u) e^{\int_0^u H(s) ds} du = I, I \geq 0$. We can choose a positive integer $\bar{k}$ large enough such that

$$\int_{t_{\bar{k}}}^{t_n} \omega(u) e^{\int_0^u H(s) ds} du < \frac{\delta_0}{8M} \quad (66)$$

for $\forall n \geq \bar{k}$, where $0 < \delta_0$ ($\delta_0 < \epsilon$) is satisfying

$$\begin{aligned} & \delta_0 + \max \left\{ Q_0 + \frac{e^{-a_0 Q_0}}{a_0}, 1 \right\} \delta_0 + \delta_0 \sum_{j=1}^N \int_{t_{\bar{k}}-r_j(t_{\bar{k}})}^{t_{\bar{k}}} h_j(s) ds \\ & + \alpha \Gamma \left(Q_0 + \frac{e^{-a_0 Q_0}}{a_0} \right) \delta_0 \leq (1 - \alpha). \end{aligned} \quad (67)$$

Now we consider the solution $x(t) = x(t, \psi, \dot{x}(t_{\bar{k}}))$ of (11) which satisfies

$$\begin{aligned} \psi(t_{\bar{k}}) &= \frac{3\delta_0}{4}, \quad \dot{x}(t_{\bar{k}}) = \frac{\delta_0}{4}, \\ | \psi(s) | + | \dot{x}(s) | &\leq \delta_0, \quad s \leq t_{\bar{k}}. \end{aligned} \quad (68)$$

We can obtain that $|x(t)| \leq 1$ by a similar argument with (60) if we replace ϵ by 1. Consider

$$\begin{aligned} & \psi(t_{\bar{k}}) - \sum_{j=1}^N \int_{t_{\bar{k}}-r_j(t_{\bar{k}})}^{t_{\bar{k}}} h_j(s) \psi(s) ds \\ & \geq \psi(t_{\bar{k}}) - \sum_{j=1}^N \int_{t_{\bar{k}}-r_j(t_{\bar{k}})}^{t_{\bar{k}}} h_j(s) |\psi(s)| ds \\ & \geq \frac{3\delta_0}{4} - \frac{\delta_0}{2} = \frac{\delta_0}{4}. \end{aligned} \quad (69)$$

By (27) and $|x(t)| \leq 1$, if $t_n \geq t_{\bar{k}}$, we have

$$\begin{aligned} & \left| x(t_n) - \sum_{j=1}^N \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \right| \\ & \geq \left| \psi(t_{\bar{k}}) e^{-\int_{t_{\bar{k}}}^{t_n} H(s) ds} + \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} H(s) ds} B(u) du \right. \\ & \quad \left. - \sum_{j=1}^N \int_{t_{\bar{k}}-r_j(t_{\bar{k}})}^{t_{\bar{k}}} h_j(s) \psi(s) ds \cdot e^{-\int_{t_{\bar{k}}}^{t_n} H(u) du} \right| \\ & \quad - \left| \int_{t_{\bar{k}}}^{t_n} e^{-\int_u^{t_n} H(s) ds} \omega(u) du \right|. \end{aligned} \quad (70)$$

This implies that

$$\begin{aligned} & \left| x(t_n) - \sum_{j=1}^N \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \right| \\ & \geq e^{-\int_{t_{\bar{k}}}^{t_n} H(s) ds} \left[\left| \frac{\delta_0}{4} + \int_{t_{\bar{k}}}^{t_n} e^{\int_{t_{\bar{k}}}^u H(s) ds} \dot{x}(t_{\bar{k}}) e^{\int_{t_{\bar{k}}}^u A(s) ds} du \right| \right. \\ & \quad \left. - \int_{t_{\bar{k}}}^{t_n} \omega(u) e^{\int_{t_{\bar{k}}}^u H(s) ds} du \right]. \end{aligned} \quad (71)$$

Note that $\dot{x}(t_{\bar{k}}) = \delta_0/4 > 0$; it follows that

$$\begin{aligned} & \left| x(t_n) - \sum_{j=1}^N \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \right| \\ & \geq e^{-\int_{t_{\bar{k}}}^{t_n} h(s) ds} \left[\frac{\delta_0}{4} - e^{-\int_0^{t_{\bar{k}}} H(s) ds} \right. \\ & \quad \left. \cdot \int_{t_{\bar{k}}}^{t_n} \omega(u) e^{\int_0^u H(s) ds} du \right] \\ & \geq \frac{\delta_0}{8} e^{-\int_{t_{\bar{k}}}^{t_n} H(s) ds} \geq \frac{\delta_0}{8} e^{-2\eta} > 0. \end{aligned} \quad (72)$$

If the zero solution $x(t)$ is asymptotically stable, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$. By the mean value theorem and condition (ii), we have

$$\begin{aligned} & \left| \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \right| = \left| (x(\bar{\omega})) \int_{t_n-r_j(t_n)}^{t_n} h_j(s) ds \right| \\ & \leq |x(\bar{\omega})|. \end{aligned} \quad (73)$$

Thus

$$\sum_{j=1}^N \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \rightarrow 0 \quad \text{as } t_n \rightarrow \infty. \quad (74)$$

Consequently,

$$\lim_{t_n \rightarrow \infty} \left(x(t_n) - \sum_{j=1}^N \int_{t_n-r_j(t_n)}^{t_n} h_j(s) x(s) ds \right) = 0. \quad (75)$$

This is a contradiction to (72).

Hence

$$\int_0^\infty H(s) ds = \infty. \quad (76)$$

This completes the proof of Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

- [1] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-Differential Equations*, Academic Press, Orlando, Fla, USA, 1985.
- [2] J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 1977.
- [3] J. Levin and A. Nohel, "On a nonlinear delay equation," *Journal of Mathematical Analysis and Applications*, vol. 8, pp. 31–44, 1964.
- [4] A. Ardjouni and A. Djoudi, "Fixed points and stability in linear neutral differential equations with variable delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 6, pp. 2062–2070, 2011.
- [5] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, NY, USA, 2006.
- [6] C. H. Jin and J. W. Luo, "Fixed points and stability in neutral differential equations with variable delays," *Proceedings of the American Mathematical Society*, vol. 136, no. 3, pp. 909–918, 2008.
- [7] D. Zhao, S. Yuan, and T. Zhang, "Improved stability conditions for a class of stochastic Volterra-Levin equations," *Applied Mathematics and Computation*, vol. 231, pp. 39–47, 2014.
- [8] T. A. Burton, "Fixed points and stability of a nonconvolution equation," *Proceedings of the American Mathematical Society*, vol. 132, no. 12, pp. 3679–3687, 2004.
- [9] L. C. Becker and T. A. Burton, "Stability, fixed points and inverses of delays," *Proceedings of the Royal Society of Edinburgh A: Mathematics*, vol. 136, no. 2, pp. 245–275, 2006.
- [10] C. H. Jin and J. W. Luo, "Stability of an integro-differential equation," *Computers & Mathematics with Applications*, vol. 57, no. 7, pp. 1080–1088, 2009.
- [11] A. Ardjouni, A. Djoudi, and I. Soualhia, "Stability for linear neutral integro-differential equations with variable delays," *Electronic Journal of Differential Equations*, vol. 2012, pp. 1–14, 2012.
- [12] D. Zhao and S. Yuan, "3/2-stability conditions for a class of Volterra-Levin equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 94, pp. 1–11, 2014.
- [13] J. Levin and A. Nohel, "Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics," *Archive for Rational Mechanics and Analysis*, vol. 5, pp. 194–211, 1960.
- [14] T. A. Burton, "Fixed points, stability, and exact linearization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 5, pp. 857–870, 2005.
- [15] D. Pi, "Study the stability of solutions of functional differential equations via fixed points," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 2, pp. 639–651, 2011.
- [16] D. Pi, "Stability conditions of second order integrodifferential equations with variable delay," *Abstract and Applied Analysis*, vol. 2014, Article ID 371639, 11 pages, 2014.