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Research Article

Stability Analysis of Fractional-Order Nonlinear Systems with Delay

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Stability analysis of fractional-order nonlinear systems with delay is studied. We propose the definition of Mittag-Leffler stability of time-delay system and introduce the fractional Lyapunov direct method by using properties of Mittag-Leffler function and Laplace transform. Then some new sufficient conditions ensuring asymptotical stability of fractional-order nonlinear system with delay are proposed firstly. And the application of Riemann-Liouville fractional-order systems is extended by the fractional comparison principle and the Caputo fractional-order systems. Numerical simulations of an example demonstrate the universality and the effectiveness of the proposed method.

1. Introduction

Fractional calculus is a 300-year-old mathematical topic. Although it has a long history, it has not been used in physics and engineering for many years. However, during the last twenty years or so, fractional calculus starts to attract increasing attentions of physicists and engineers from an application point of view [1, 2]. It was found that many systems in interdisciplinary fields could be elegantly described with the help of fractional derivatives. There are two essential differences between fractional-order derivation and integer-order derivation. Firstly, the fractional-order derivative is concerned with the whole time domain for a mechanical or physical process, while the integer-order derivative indicates a variation or certain attribute at particular time. Secondly, the fractional-order derivative is related to the whole space for a physical process, while the integer-order derivative describes the local properties of a certain position. It is the reason that many real world physical systems are well characterized by the fractional-order state equations [3–6], such as fractional-order Lotka-Volterra equation [3] in biological systems, fractional-order Schrödinger equation [4] in quantum mechanics, fractional-order Langevin equation [5] in anomalous diffusion, and fractional-order oscillator equation [6] in damping vibration.

Time delays are inherent phenomena in interconnected real systems or processes, including feedback control systems, due to transportation of material, energy, or information. The presence of time delays in a feedback control system leads to a closed-loop characteristic equation which involves the exponential type transcendental terms. The exponential transcendentality brings infinitely many isolated roots, and hence it makes the stability analysis of time-delay systems a challenging task. Integer-order delay systems have been thoroughly investigated during the past decades [7–12], but there is no general stability theory for fractional-order dynamic systems with delay [13]. The necessary and sufficient stability conditions for linear fractional-order differential equations and linear time-delayed fractional differential equations have already been obtained in [13–15]. Reference [16] investigated the stability of n-dimensional linear fractional-order differential systems with order $1 < \alpha < 2$. However, only under some special circumstances or in certain cases, the practical problems may be regarded as linear systems. Therefore, stability of nonlinear dynamic systems not only is of great significance but also has important value in application.

In [17], the authors proposed the finite-time stabilization of a class of multistate time delay of fractional nonlinear systems. In [18, 19], the authors studied the stability of fractional nonlinear dynamic systems using Lyapunov direct method with the introductions of Mittag-Leffler stability and generalized Mittag-Leffler stability notions.

Although there are some works about stability of fractional-order time-delay systems and Lyapunov's second method, few attempts were done in order to combine these two powerful concepts and to observe what the benefits of this combination are. The aim of this paper is to present several analytical and numerical approaches for the stability analysis of fractional nonlinear time-delay systems. For extending the application of fractional calculus in nonlinear systems, we introduce the fractional comparison principle and some properties of Mittag-Leffler function. By the properties of Mittag-Leffler function, Laplace transform, and some fractional inequalities, some sufficient conditions for the asymptotical stability of fractional nonlinear systems with delay are proposed firstly. And the application of Riemann-Liouville fractional-order time-delay systems is extended by using fractional comparison principle and Caputo fractional-order systems. The numerical example illustrates principal results of the paper.

This paper is organized as follows. In Section 2 some basic definitions of fractional calculus and properties of Mittag-Leffler functions are presented. Mittag-Leffler stability and fractional-order extension of Lyapunov direct method are proposed in Section 3. An illustrative example is given in Section 4. Finally, conclusion is in Section 5.

2. Fractional-Order Derivatives and Mittag-Leffler Functions

2.1. Definition of Fractional Derivatives and Mittag-Leffler Functions. Fractional calculus plays an important role in modern science [1, 20, 21]. There are some definitions for fractional derivatives. In this paper, we give three commonly used definitions [1]: Riemann-Liouville (RL), Grünwald-Letnikov (GL), and Caputo definition.

Definition 1 (see [1]). The fractional integral ${}_aD_t^{-\alpha}$ of function $f(t)$ is defined as follows:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{1} $$

where fractional-order $\alpha > 0$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function.

Definition 2 (see [1]). The Riemann-Liouville derivative with order α of function $f(t)$ is defined as

$$ \begin{aligned} {}_a^{\mathrm{RL}}D_t^{\alpha} f(t) &= \frac{d^n}{dt^n}\, {}_aD_t^{-(n-\alpha)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \end{aligned} \tag{2} $$

where $n-1 < \alpha < n, n \in Z^+$.

Also, there are other definitions of fractional derivative introduced by Caputo and Grünwald-Letnikov.

Definition 3 (see [1]). The Caputo derivative with order α of function $f(t)$ is given as

$$ \begin{aligned} {}_a^{C}D_t^{\alpha} f(t) &= {}_aD_t^{-(n-\alpha)} \frac{d^n}{dt^n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \end{aligned} \tag{3} $$

where $n-1 < \alpha < n, n \in Z^+$.

The formulas for Laplace transform of Riemann-Liouville derivative ${}_a^{\mathrm{RL}}D_t^{\alpha} f(t)$ and Caputo fractional derivative ${}_a^{C}D_t^{\alpha} f(t)$ have the following forms [1]:

$$ \begin{aligned} \mathfrak{L}t\left\{{}_a^{\mathrm{RL}}D^{\alpha} f(t); s\right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left[{}_a^{\mathrm{RL}}D_t^{\alpha-k-1} f(t)\right]_{t=0}, \\ \mathfrak{L}\left\{{}_a^{C}D_t^{\alpha} f(t); s\right\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \end{aligned} \tag{4} $$

where $n-1 \le \alpha < n$ and $F(s) = \mathfrak{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt$.

Definition 4. The Grünwald-Letnikov derivative with order α of function $f(t)$ is defined as

$$ \begin{aligned} {}_a^{\mathrm{GL}}D_t^{\alpha} f(t) &= \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} \\ &\quad + \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(m+1)}(\tau)\, d\tau, \end{aligned} \tag{5} $$

where $m < \alpha < m+1$.

Throughout the papers about these definitions, we obtain the following conclusions. Grünwald-Letnikov definition is suitable for numerical calculations, Riemann-Liouville definition plays an important role in theory analysis, and Caputo definition is well used since its Laplace transform allows for initial conditions taking the same forms as those for integer-order derivatives, which have clear physical interpretations and have a wide range of applications in the process of factual modeling. As a generalization of the exponential function which is frequently used in the solutions of integer-order systems, the Mittag-Leffler function is frequently used in the solutions of the fractional differential equations. The definition and properties are given in the following.

Definition 5 (see [1]). The Mittag-Leffler function is given as

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{6} $$

where $\alpha > 0$ and $z \in \mathbb{C}$.

The generalization of Mittag-Leffler function with two parameters is widely used and defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{7}$$

where $\alpha > 0$, $\beta > 0$ and $z \in \mathbb{C}$.

Remark 6. If $\beta = 1$, we have $E_{\alpha,1}(z) = E_\alpha(z)$, especially $E_{1,1}(z) = E_1(z) = e^z$.

2.2. Properties of Fractional Derivatives and Mittag-Leffler Functions. In this section, some important properties of the fractional derivatives and the Mittag-Leffler functions are given, which are used in this paper.

Lemma 7. *Let $\alpha \in (0,1)$ and let $f(0) \geq 0$; then*

$${}_0^C D_t^\alpha f(t) \leq {}_0^{RL} D_t^\alpha f(t), \tag{8}$$

where ${}_0^C D_t^\alpha$ and ${}_0^{RL} D_t^\alpha$ are the Caputo and the Riemann-Liouville fractional derivatives.

Proof. By using the definitions of fractional derivatives, we have ${}_0^{RL} D_t^\alpha f(t) = {}_0^C D_t^\alpha f(t) + ((f(0)t^{-\alpha})/(\Gamma(1-\alpha)))$. Since $\alpha \in (0,1)$ and $f(0) \geq 0$, we have the conclusion ${}_0^C D_t^\alpha f(t) \leq {}_0^{RL} D_t^\alpha f(t)$. □

Lemma 8. *If ${}_0^C D_t^\alpha x(t) \geq 0$ and $x(0) \geq 0$, $0 < \alpha < 1$, then $x(t) \geq 0$.*

Proof. Suppose that ${}_0^C D_t^\alpha x(t) = f(t,x) \geq 0$. Using the equivalent Volterra integral equation [14]

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau, \tag{9}$$

since $t - \tau \geq 0$, $\Gamma(\alpha) > 0$, and $f(t,x) > 0$, we can get $x(t) \geq x(0) \geq 0$; that is, $x(t) \geq 0$.

According to the properties of the fractional derivatives and Lemma 8, we obtain the comparison theorem of the fractional derivatives. □

Theorem 9 (comparison theorem). *Let $0 < \alpha < 1$ and let $x(0) = y(0)$; then $x(t) \geq y(t)$, if ${}_0^C D_t^\alpha x(t) \geq {}_0^C D_t^\alpha y(t)$.*

Proof. The fractional differentiation and fractional integration are linear operations; then ${}_0^C D_t^\alpha (x(t) - y(t)) \geq 0$. By the Lemma 8, we can easily get $x(t) - y(t) \geq 0$; that is, $x(t) \geq y(t)$. □

Lemma 10 (see [1]). *Consider the Laplace transform of Mittag-Leffler function with two parameters; one has*

$$\mathfrak{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \left(\Re(s) > |\lambda|^{1/\alpha}\right), \tag{10}$$

where t and s are, respectively, the variables in the time domain and the Laplace domain, $\Re(s)$ stands for the real part of s, $\lambda \in \mathbb{R}$, and $\mathfrak{L}\{\cdot\}$ denotes the Laplace transform.

Proof. By the definitions of Laplace transform and Mittag-Leffler function, we have

$$\begin{aligned}
\mathfrak{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\} &= \int_0^\infty e^{-st} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k t^{\alpha k}}{\Gamma(k\alpha+\beta)} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\Gamma(k\alpha+\beta)} \int_0^\infty e^{-st} t^{\alpha k+\beta-1} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{\Gamma(k\alpha+\beta)} \frac{\Gamma(\alpha k+\beta)}{s^{\alpha k+\beta}} \\
&= \sum_{k=0}^{\infty} (-1)^k s^{-\beta} \left(\frac{\lambda}{s^\alpha}\right)^k \\
&= \frac{s^{\alpha-\beta}}{s^\alpha + \lambda},
\end{aligned} \tag{11}$$

where $\Re(s) > |\lambda|^{1/\alpha}$. □

3. Fractional-Order Extension of Lyapunov Direct Method

3.1. Fractional-Order Nonlinear Systems with Delay and Mittag-Leffler Stability. We also use functional differential equations to describe time-delay systems [22–24]. To formally introduce the concept of functional differential equations, let $\mathscr{C} = \mathscr{C}([a,b], \mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a maximum time-delay d of a system. In this case, we are often interested in the set of continuous functions mapping $[-d,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $\mathscr{C} = \mathscr{C}([-d,0], \mathbb{R}^n)$. For any $T > 0$ and any continuous function of time $x \in \mathscr{C}([t_0 - d, t_0 + T], \mathbb{R}^n)$, and $t_0 \leq t \leq t_0 + T$, let $x_t \in \mathscr{C}$ be a segment of the function x defined as $x_t(\theta) = x(t+\theta)$ and $-d \leq \theta \leq 0$.

Firstly, the Caputo fractional nonlinear time-delay system is given by

$${}_{t_0}^C D_t^\alpha x(t) = f(t, x_t), \tag{12}$$

where $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$, and $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$. In equation (12), the Caputo derivative of the state variable x is on $[t_0, t]$ and $x(\tau)$ is for $t - d \leq \tau \leq t$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length d, from $t_0 - d$ to t_0; that is,

$$x_{t_0}(\theta) = x(t_0 + \theta) = \varphi(\theta), \quad \forall \theta \in [-d, 0], \tag{13}$$

where $\varphi(\cdot)$ is an element of the Banach space $\mathscr{C}$ with the standard uniform norm: $\|\varphi\|_\infty = \max_{\theta\in[-d,0]} \|\varphi(\theta)\|$. And the vector norm $\|\cdot\|$ represents the 2-norm $\|\cdot\|_2$.

For given initial conditions of the form (13), let $y(t) = y(t, t_0, \varphi(\cdot))$ be a solution of the fractional-order time-delay system (12). The stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $y(t, t_0, \varphi)$. In the following, without loss of generality, we will

assume that the functional differential equation (12) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. Indeed, if it is desirable to study the stability of a nontrivial solution $y(t)$, then we may resort to the variable transformation $z(t) = x(t) - y(t)$, so that the new system

$$ {}_{t_0}^{C}D_t^{\alpha} z(t) = f(t, z_t + y_t) - f(t, y_t) \tag{14} $$

has the trivial solution $z(t) = 0$.

Definition 11 (see [24]). The system (12) is said to be stable if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\varphi(\cdot)\|_\infty < \delta$, then $\|x(t, t_0, \varphi)\| < \epsilon$ for all $t \geq t_0$.

Definition 12 (see [24]). If the value δ from the above definition may be chosen such that in addition $x(t, t_0, \varphi(\cdot)) \to 0$, when $t \to \infty$, then the system (12) is said to be asymptotically stable.

Definition 13 (exponential stability [23, 25]). The system (12) is said to be exponentially stable if there exist $\alpha > 0$ and $\mu \geq 0$ such that for every solution $x(t, t_0, \varphi(\cdot))$ of system satisfy the following inequality:

$$ \|x(t, t_0, \varphi(\cdot))\| \leq \mu \|\varphi\|_\infty e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0. \tag{15} $$

Definition 14 (Mittag-Leffler stability). The solution (12) is said to be Mittag-Leffler stable if there exist $\lambda \geq 0$ and $b > 0$ such that

$$ \|x(t, t_0, \varphi(\cdot))\| \leq \left\{ m(\varphi) E_\alpha\left(-\lambda (t - t_0)^\alpha\right)\right\}^b, \tag{16} $$

where $\alpha \in (0, 1)$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally the Lipschitz on $x \in \mathscr{C}$ with the Lipschitz constant m_0.

Lemma 15. *Consider the real-valued continuous $f(t, x)$ in system* (12)*, one obtains*

$$ \|{}_aD_t^{-\alpha} f(t, x)\| \leq {}_aD_t^{-\alpha} \|f(t, x)\|, \tag{17} $$

where $\alpha > 0$ and $\|\cdot\|$ denotes an arbitrary norm.

Proof. It follows from the definition of fractional integral (1) that

$$ \begin{aligned} \|{}_aD_t^{-\alpha} f(t, x(t))\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left\| \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} \right\| d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\|f(\tau, x(\tau))\|}{(t-\tau)^{1-\alpha}} d\tau \\ &= {}_aD_t^{-\alpha} \|f(t, x(t))\|. \end{aligned} \tag{18} $$

□

3.2. Fractional-Order Extension of Lyapunov Direct Method. It is well known that Lyapunov stability provides an important tool for stability analysis in nonlinear systems. We primarily study the Lyapunov direct method which involves finding a Lyapunov function candidate for a given nonlinear system. If there exists such function, the system is stable. Applying Lyapunov direct method is to search for an appropriate function. However, Lyapunov direct method is a sufficient condition. It means that if one cannot find a Lyapunov function, the system may still be stable and one cannot claim the system is not stable. In the following, we extend the Lyapunov direct method to the fractional-order nonlinear time-delay system and propose some sufficient conditions of stability for the fractional-order time-delay system.

Theorem 16. *Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in system* (12) *takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $V(t, x(t, 0, \varphi)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and locally Lipschitz with respect to x such that*

$$ \alpha_1 \|x(t)\|^a \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^{ab} \tag{19} $$

and the fractional derivative of V along the solution $x(t) = x(t, 0, \varphi)$ of system (12) *satisfies*

$$ {}_0^{C}D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \|x(t)\|^{ab}, \tag{20} $$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $0 < \beta < 1$, and α_i $(i = 1, 2, 3)$, a and b are arbitrary positive constants. Then system is asymptotically stable.

Proof. By (19) and (20), we can easily get ${}_0^{C}D_t^{\beta} V(t, x(t)) \leq -\alpha_3 \alpha_2^{-1} V(t, x(t))$. So there is a nonnegative function $W(t)$ satisfying

$$ {}_0^{C}D_t^{\beta} V(t, x(t)) + W(t) = -\alpha_3 \alpha_2^{-1} V(t, x(t)). \tag{21} $$

Taking the Laplace transform of (21), we have

$$ s^\beta V(s) - V(\varphi) s^{\beta-1} + W(s) = -\alpha_3 \alpha_2^{-1} V(s), \tag{22} $$

where nonnegative constant $V(\varphi) = V(0, x(0)) = V(0, \varphi(\cdot))$ and $V(s) = \mathfrak{L}\{V(t, x(t)); s\}$. Then

$$ V(s) = \frac{V(\varphi) s^{\beta-1} - W(s)}{s^\beta + (\alpha_3/\alpha_2)}. \tag{23} $$

If $x(0) = \varphi = 0$, namely, $V(0) = 0$, the solution of (12) is $x = 0$. If $x(0) \neq 0$, then, $V(\varphi) > 0$. Since $V(t, x(t)) > 0$ is Lipschitz with respect to x, it follows from the fractional uniqueness and the existence theorem [13] and the inverse Laplace transform that the unique solution of (23) is

$$ \begin{aligned} V(t, x(t)) &= V(\varphi) E_\beta\left(-\frac{\alpha_3}{\alpha_2} t^\beta\right) \\ &\quad - W(t) * \left[t^{\beta-1} E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2} t^\beta\right)\right]. \end{aligned} \tag{24} $$

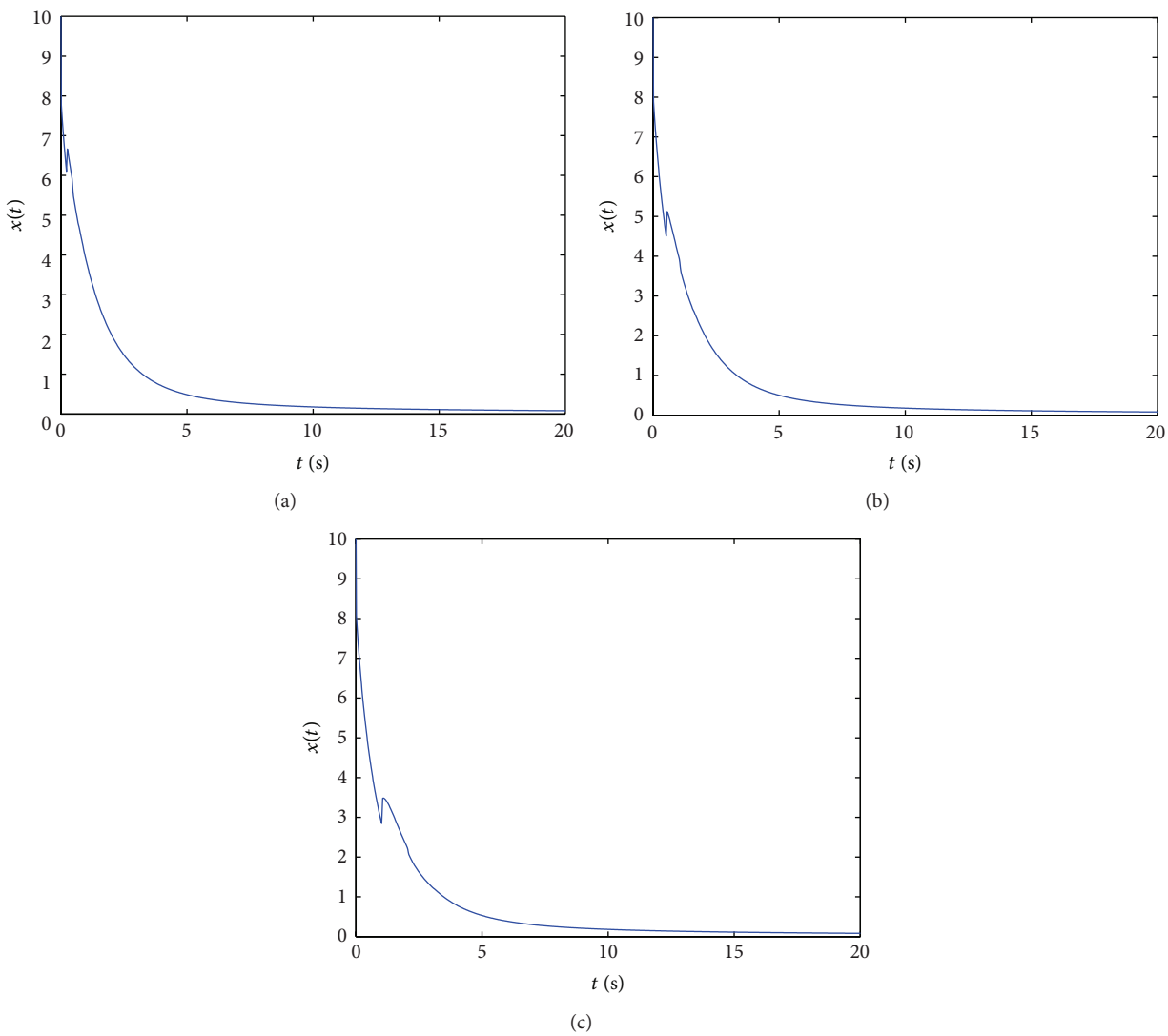

Figure 1: Time waveforms of numerical solutions $x(t)$ with $\alpha = 0.9$ and $\varphi(t) = t + 10$. (a), (b), and (c) show the solutions with $d = 0.2$ (a), $d = 0.5$ (b), and $d = 1$ (c), respectively.

Because both $W(t)$ and $t^{\beta-1}E_{\beta,\beta}(-(\alpha_3/\alpha_2)t^{\beta})$ are nonnegative functions, we have

$$V(t,x) \le V(\varphi) E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right). \tag{25}$$

Then we substitute (25) into (19),

$$\|x(t)\| \le \left(\frac{V(\varphi)}{\alpha_1}E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right)^{1/a}. \tag{26}$$

Let $m(\varphi) = (V(\varphi)/\alpha_1)^{1/a} \ge 0$; then

$$\|x(t)\| \le \left(m(\varphi) E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right)^{1/a}, \tag{27}$$

where $m(\varphi) = 0$ holds if and only if $\varphi = 0$. Because $V(t, x(t))$ is Lipschitz with respect to x and $V(0, x(0)) = 0$ if and only if $x(0) = 0$, $m = (V(0, x(0)))/\alpha_1$ is also Lipschitz with respect to $x(0)$ and $m(0) = 0$, which imply the Mittag-Leffler stability of system (12). □

Theorem 17. *Suppose $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ in system (12) takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $V(t, x(t, 0, \varphi)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1 \|x(t)\|^a \le V(t, x(t)) \le \alpha_2 \|x(t)\| \tag{28}$$

and the derivative of V along the solution $x(t, t_0, \varphi)$ of system (12) satisfies

$$\frac{dV(t, x(t))}{dt} \le -\alpha_3 \|x(t)\|, \tag{29}$$

where $t \ge 0$, α_i $(i = 1, 2, 3)$, and a are arbitrary positive constants. Then $\|x(t)\| \le (m(\varphi)E_{\beta}(-(\alpha_3/\alpha_2)t^{1-\alpha}))^{1/a}$; that is, system (12) is asymptotically stable.

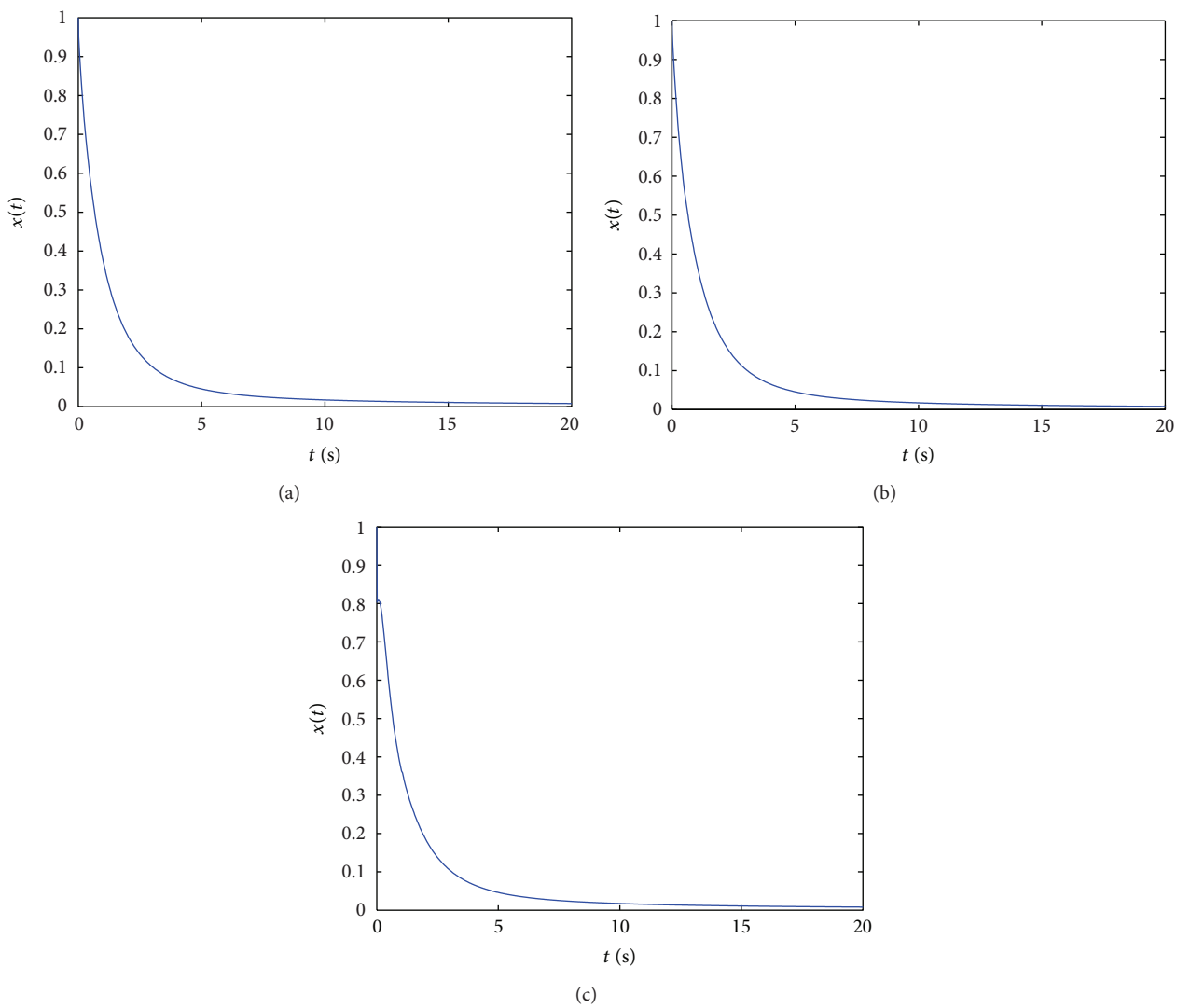

Figure 2: Time waveforms of numerical solutions $x(t)$with $\alpha = 0.8$ and $\varphi(t) = 2t^3 + 1$. (a), (b), and (c) show the solutions with $d = 0.2$ (a), $d = 0.5$ (b), and $d = 1$ (c), respectively.

Proof. It follows from the properties of Caputo derivative and Lemma 15 that

$$\begin{aligned} {}_0^C D_t^{1-\alpha} V(t, x(t)) &= {}_0 D_t^{-\alpha} \frac{dV(t, x(t))}{dt} \\ &\le -\alpha_3 {}_0 D_t^{-\alpha} \|x(t)\| \\ &\le -\frac{\alpha_3}{L} {}_0 D_t^{-\alpha} \|f(t, x(t))\| \\ &\le -\frac{\alpha_3}{L} \| {}_0 D_t^{-\alpha} f(t, x(t))\| \\ &\le -\frac{\alpha_3}{L} \|x(t)\|, \end{aligned} \tag{30}$$

where $[{}_0 D_t^{-(1-\alpha)} x(t)]_{t=0} = 0$. Let $\beta = 1 - \alpha$, let $\alpha_3' = \alpha_3 / L$, and let $b = a^{-1}$; the conclusion can be easily obtained by using Theorem 16.

In the following, we study the Riemann-Liouville fractional dynamic system

$$ {}_{t_0}^{\mathrm{RL}} D_t^{\alpha} x(t) = f(t, x_t) \tag{31}$$

with the initial condition $x_{t_0} = \varphi(\theta)$, where $\alpha \in (0, 1)$ and f is piecewise continuous in t and locally Lipschitz in x. □

Theorem 18. *Let the assumptions in Theorem 16 be satisfied except for replacing ${}_0^C D_t^{\alpha}$ by ${}_0^{RL} D_t^{\alpha}$; then one has the same conclusion $\|x(t)\| \le (m(\varphi) E_{\beta}(-(\alpha_3/\alpha_2) t^{\beta}))^{1/a}$; that is, system* (31) *is asymptotically stable.*

Proof. It follows from $V(t, x(t)) \ge 0$ and Lemma 7 that ${}_0^C D_t^{\alpha} V(t, x(t)) \le {}_0^{\mathrm{RL}} D_t^{\alpha} V(t, x(t))$. Then ${}_0^C D_t^{\alpha} V(t, x(t)) \le {}_0^{\mathrm{RL}} D_t^{\alpha} V(t, x(t)) \le -\alpha_3 \|x(t)\|$. Therefore the conclusion can be obtained by Theorem 16. □

4. An Illustrative Example

In this section an illustrative example is used as proofs of concept. And the numerical simulations of a fractional-order delay-time system demonstrate the universality and the effectiveness of the proposed method.

Example 19. For a fractional-order delay-time nonlinear system,

$$ {}_{0}^{\mathrm{RL}}D_{t}^{\alpha}\left(x(t)\operatorname{sgn}(x(t))\right) = -x(t)\operatorname{sgn}(x(t)) - x^{2}(t-d), \tag{32} $$

with the initial condition $x_{t_0} = \varphi(t)$ $(-d \le t \le 0)$, where $0 < \alpha < 1$ and $\operatorname{sgn}(\cdot)$ is the sign function. Choose the Lyapunov function $V(t, x(t)) = x\operatorname{sgn}(x)$; then ${}_{0}^{\mathrm{RL}}D_{t}^{\alpha}V(t, x(t)) = -x(t)\operatorname{sgn}(x(t)) - x^{2}(t-d) \le 0$. When selecting $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $a = b = 1$, it follows from Theorem 18 that $\|x(t)\| \le m(\varphi)E_{\alpha}(-t^{\alpha})$; that is, system (32) is asymptotically stable. In the following, we give the numerical simulations of system (32).

(a) When we select $\alpha = 0.9$ and $\varphi(t) = t + 10$, the numerical solutions of the fractional differential equations (32) are shown in Figure 1, which demonstrates the applicability of the proposed approach. In Figures 1(a)–1(c), the time waveforms of numerical solutions are given with $d = 0.2$ s, $d = 0.5$ s, and $d = 1$ s, respectively.

(b) When we select $\alpha = 0.8$ and $\varphi(t) = 4t^3 + 1$, the numerical solutions of the fractional differential equations (32) are shown in Figure 2, which demonstrates the applicability of the proposed approach. In Figures 2(a)–2(c), the time waveforms of numerical solutions are given with $d = 0.2$ s, $d = 0.5$ s, and $d = 1$ s, respectively.

Furthermore, when selecting other α and $\varphi(t)$, the fractional nonlinear time-delay system (32) is also asymptotically stable, which demonstrates the universality and the effectiveness of the proposed method.

5. Conclusion

Stability analysis of time-delay systems is one of the most fundamental and important issues for control systems. And stability of the nonlinear systems is important for scientists and engineers. Fractional dynamic systems were used intensively during the last decade in order to describe the behavior of complex systems in physics and engineering. In this paper the stabilization of fractional-order nonlinear time-delay systems is studied. We discussed the properties of the Caputo and Riemann-Liouville derivatives and the comparison theorem. And by the properties of Mittag-Leffler function and Laplace transform, we proposed the extending Lyapunov direct method which is the sufficient condition of stability for fractional-order time-delay systems. This enriches the knowledge of both the system theory and the fractional calculus. We partly extended the application of Riemann-Liouville fractional-order systems by fractional comparison theorem and Caputo fractional-order systems. And the numerical simulations of a fractional-order delay-time system demonstrate the universality and the effectiveness of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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