Research Article

A New Approach for Solving Fractional Partial Differential Equations in the Sense of the Modified Riemann-Liouville Derivative

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Based on a fractional complex transformation, certain fractional partial differential equation in the sense of the modified Riemann-Liouville derivative is converted into another ordinary differential equation of integer order, and the exact solutions of the latter are assumed to be expressed in a polynomial in Jacobi elliptic functions including the Jacobi sine function, the Jacobi cosine function, and the Jacobi elliptic function of the third kind. The degree of the polynomial can be determined by the homogeneous balance principle. With the aid of mathematical software, a series of exact solutions for the fractional partial differential equation can be found. For demonstrating the validity of this approach, we apply it to solve the space fractional KdV equation and the space-time fractional Fokas equation. As a result, some Jacobi elliptic functions solutions for the two equations are obtained.

1. Introduction

Fractional differential equations involving fractional derivatives are generalizations of classical differential equations of integer order. Fractional derivative is useful in describing the memory and hereditary properties of materials and processes. Fractional partial differential equations (FPDEs) are widely used as models to express many important physical phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, and chemical physics. Recently, there have been a lot of achievements on the theory and applications of fractional partial differential equations and, among these investigations for the properties of fractional partial differential equations, the analytical solutions of FPDEs play an important role in nonlinear science and engineering and can help understand some physical phenomena better. So far many powerful methods have been proposed to seek analytical solutions or semianalytical solutions of fractional partial differential equations. For example, these methods include the simplest equation method [1], the Exp-function method [2], the variational iterative method [3–6], the Adomian decomposition

method [7, 8], the fractional subequation method [9–12], and the homotopy perturbation method [13–15].

In this paper, we present a new method to seek exact solutions in the forms of the Jacobi elliptic functions for fractional partial differential equations. The fractional partial differential equation is defined in the sense of the modified Riemann-Liouville derivative. This method belongs to the categories of the subequation methods. First by a fractional complex transformation, certain fractional partial differential equation is converted into another ordinary differential equation of integer order. The main point of this method lies in the fact that the solutions of the converted ordinary differential equation are supposed to be expressed in a polynomial in Jacobi elliptic functions, and the degree of the polynomial can be determined by the homogeneous balance principle. By substituting this polynomial into the ordinary differential equation, a series of algebraic equations can be derived. By solving these equations with mathematical software such as Maple, a series of exact solutions in the forms of the Jacobi elliptic functions can be obtained.

The modified Riemann-Liouville fractional derivative, defined by Jumarie in [16–19], has many excellent characters

in handling with many fractional calculus problems. We now list the definition for it as follows.

Definition 1. The modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1. \end{cases} \quad (1)$$

Some important properties for the modified Riemann-Liouville derivative are listed as follows (see [8–11, 15]):

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \quad (2)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t), \quad (4)$$

$$D_t^\alpha f[g(t)] = D_g^\alpha f[g(t)](g'(t))^\alpha. \quad (5)$$

We organize the rest of this paper as follows. In Section 2, we give the description of the present method for seeking Jacobi elliptic function solutions for fractional partial differential equations. Then in Section 3 we present some applications and apply this method to the space fractional KdV equation and the space-time fractional Fokas equation. Some concluding comments are presented in Section 4.

2. Summary of the Method

In this section, we give the description of the method for solving seeking Jacobi elliptic function solutions for fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, \dots, x_n$, is given by

$$P\left(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, \frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_k}{\partial x_1}, D_{x_2}^\beta u_1, \dots, D_{x_2}^\beta u_1, \dots, \frac{\partial u_1}{\partial x_{n-1}}, \dots, \frac{\partial u_k}{\partial x_{n-1}}, D_{x_n}^\gamma u_1, \dots, D_{x_n}^\gamma u_k, \dots\right) = 0, \quad (6)$$

where $u_i = u_i(t, x_1, x_2, \dots, x_n)$, $i = 1, \dots, k$, are unknown functions, P is a polynomial in u_i , and their various partial derivatives include fractional derivatives.

Step 1. For (6), suppose that $u_i(t, x_1, x_2, \dots, x_n) = U_i(\xi)$ and a fractional complex transformation for ξ as follows:

$$\xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + k_1 x_1 + \frac{k_2 x_2^\beta}{\Gamma(1+\beta)} + \dots + k_{n-1} x_{n-1} + \frac{k_n x_n^\gamma}{\Gamma(1+\gamma)} + \xi_0, \quad (7)$$

where $c, k_1, \dots, k_{n-1}, k_n, \xi_0$ are all nonzero constants. Based on the transformation above, for the terms in (6) containing fractional derivative, such as $D_t^\alpha u_1$, by using (2) and (4), one can obtain that

$$D_t^\alpha u_1 = D_t^\alpha U_1(\xi) = U_1'(\xi) D_t^\alpha \xi = c U_1'(\xi). \quad (8)$$

For the terms in (6) containing derivative of integer order, such as $\partial u_1 / \partial x_1$, one has

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial U_1}{\partial \xi} \xi'_{x_1} = k_1 U_1'(\xi). \quad (9)$$

So by this transformation for ξ , (6) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ :

$$\tilde{P}(U_1, \dots, U_k, U_1', \dots, U_k', U_1'', \dots, U_k'', \dots) = 0. \quad (10)$$

Step 2. Suppose that the solution of (10) can be expressed by a polynomial in Jacobi elliptic functions as follows:

$$U_j(\xi) = a_j^{(0)} + \sum_{n+p+q=1}^{m_j} a_j^{(n,p,q)} \text{sn}^n(\xi) \text{cn}^p(\xi) \text{dn}^q(\xi), \quad (11)$$

$$j = 1, 2, \dots, k,$$

where n, p , and q are nonnegative integers with $1 \leq n + p + q \leq m_j$, $a_j^{(0)}, a_j^{(n,p,q)}$, $j = 1, 2, \dots, k$, are constants to be determined later, the positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives, and nonlinear terms appearing in (10), $\text{sn}(\xi)$, $\text{cn}(\xi)$, and $\text{dn}(\xi)$, denote the Jacobi elliptic sine function, Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind, respectively.

For the Jacobi elliptic functions, one has

$$\begin{aligned} \text{sn}'(\xi) &= \text{cn}(\xi) \text{dn}(\xi), & \text{cn}'(\xi) &= -\text{sn}(\xi) \text{dn}(\xi), \\ \text{dn}'(\xi) &= -m^2 \text{sn}(\xi) \text{cn}(\xi), \end{aligned} \quad (12)$$

where m is the modulus, and

$$\begin{aligned} \text{cs}(\xi) &= \frac{\text{cn}(\xi)}{\text{sn}(\xi)}, & \text{sd}(\xi) &= \frac{\text{sn}(\xi)}{\text{dn}(\xi)}, & \text{dc}(\xi) &= \frac{\text{dn}(\xi)}{\text{cn}(\xi)}, \\ \text{sc}(\xi) &= \frac{1}{\text{cs}(\xi)}, & \text{ds}(\xi) &= \frac{1}{\text{sd}(\xi)}, & \text{cd}(\xi) &= \frac{1}{\text{dc}(\xi)}, \\ \text{nd}(\xi) &= \frac{1}{\text{dn}(\xi)}, & \text{ns}(\xi) &= \frac{1}{\text{sn}(\xi)}, & \text{nc}(\xi) &= \frac{1}{\text{cn}(\xi)}, \end{aligned}$$

$$\begin{aligned}
\operatorname{cn}^2(\xi) &= -\operatorname{sn}^2(\xi) + 1, & \operatorname{dn}^2(\xi) &= -m^2 \operatorname{sn}^2(\xi) + 1, \\
\operatorname{dn}^2(\xi) &= m^2 \operatorname{cn}^2(\xi) + 1 - m^2, & \operatorname{ns}^2(\xi) &= \operatorname{cs}^2(\xi) + 1, \\
\operatorname{ns}^2(\xi) &= \operatorname{ds}^2(\xi) + m^2, & \operatorname{ds}^2(\xi) &= \operatorname{cs}^2(\xi) + 1 - m^2, \\
\operatorname{nc}^2(\xi) &= \operatorname{sc}^2(\xi) + 1, & \operatorname{dc}^2(\xi) &= (1 - m^2) \operatorname{nc}^2(\xi) + m^2, \\
\operatorname{dc}^2(\xi) &= (1 - m^2) \operatorname{sc}^2(\xi) + 1, \\
\operatorname{cd}^2(\xi) &= \frac{m^2 - 1}{m^2} \operatorname{nd}^2(\xi) + \frac{1}{m^2}, \\
\operatorname{cd}^2(\xi) &= (m^2 - 1) \operatorname{sd}^2(\xi) + 1, \\
\operatorname{nd}^2(\xi) &= m^2 \operatorname{sd}^2(\xi) + 1.
\end{aligned} \tag{13}$$

Step 3. By substituting (11) into (10) and using (12), the left-hand side of (10) is converted into another polynomial in $\operatorname{sn}^n(\xi) \operatorname{cn}^p(\xi) \operatorname{dn}^q(\xi)$. Collecting all coefficients of the same power and equating them to zero yield a set of algebraic equations for $a_j^{(0)}, a_j^{(n,p,q)}, j = 1, 2, \dots, k$.

Step 4. By solving the equations system in Step 3, we can construct a variety of Jacobi elliptic function solutions for (6).

3. Application of the Present Method to Some Fractional Partial Differential Equations

3.1. Space Fractional KdV Equation. Consider the following space fractional KdV equation:

$$u_t + u D_x^\alpha u + \beta D_x^{3\alpha} u = 0, \quad 0 < \alpha \leq 1, \tag{14}$$

where $u = u(x, t)$ is the distribution function in terms of the space x and the time t , the concerned fractional derivative is defined by the modified Riemann-Liouville derivative, and β is a constant denoting the dispersion coefficient. When $\alpha = 1$, (14) becomes the known KdV equation of integer order, which can be used to describe the motions of waves in nonlinear optics, plasma, or fluids.

In the following, we will apply the described method in Section 2 to solve (14) and seek Jacobi elliptic function solutions for it. To this end, we suppose $u(x, t) = U(\xi)$, where $\xi = ct + (k/\Gamma(1 + \alpha))x^\alpha + \xi_0$, k, c, ξ_0 are all constants with $k, c \neq 0$. Then by the use of (2) and (4) one can deduce that

$$D_x^\alpha u = D_x^\alpha U(\xi) = U'(\xi) D_x^\alpha \xi = k_1 U'(\xi), \tag{15}$$

$$u_t = c U'(\xi),$$

and then (14) can be turned into the following form with respect to the new variable ξ :

$$U'(\xi) + k U(\xi) U'(\xi) + \beta k^3 U'''(\xi) = 0. \tag{16}$$

Suppose that the solution of (16) can be expressed by

$$U(\xi) = a^{(0)} + \sum_{n+p+q=1}^m a^{(n,p,q)} \operatorname{sn}^n(\xi) \operatorname{cn}^p(\xi) \operatorname{dn}^q(\xi). \tag{17}$$

By balancing the order of U''' and UU' in (16) one can obtain $m = 2$. So

$$\begin{aligned}
U(\xi) &= a^{(0)} + a^{(1,0,0)} \operatorname{sn}(\xi) + a^{(0,1,0)} \operatorname{cn}(\xi) + a^{(0,0,1)} \operatorname{dn}(\xi) \\
&\quad + a^{(2,0,0)} \operatorname{sn}^2(\xi) + a^{(0,2,0)} \operatorname{cn}^2(\xi) + a^{(0,0,2)} \operatorname{dn}^2(\xi) \\
&\quad + a^{(1,1,0)} \operatorname{sn}(\xi) \operatorname{cn}(\xi) + a^{(0,1,1)} \operatorname{cn}(\xi) \operatorname{dn}(\xi) \\
&\quad + a^{(1,0,1)} \operatorname{sn}(\xi) \operatorname{dn}(\xi).
\end{aligned} \tag{18}$$

Substituting (18) into (16), using (12) and collecting all the terms with the same power of $\operatorname{sn}^n(\xi) \operatorname{cn}^p(\xi) \operatorname{dn}^q(\xi)$ together, equating each coefficient to zero yields a set of algebraic equations. Solving these equations with the aid of mathematical software yields the following values, where i denoted the unit of the imaginary numbers.

Case 1. Consider

$$\begin{aligned}
a^{(0)} &= -\frac{c}{k}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, \\
a^{(0,0,1)} &= 0, & a^{(2,0,0)} &= -4\beta k^2 m^2, & a^{(0,2,0)} &= 4\beta k^2 m^2, \\
a^{(0,0,2)} &= 4\beta k^2, & a^{(1,1,0)} &= 0, \\
a^{(0,1,1)} &= 0, & a^{(1,0,1)} &= 0.
\end{aligned} \tag{19}$$

Case 2. Consider

$$\begin{aligned}
a^{(0)} &= -\frac{c}{k}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, & a^{(0,0,1)} &= 0, \\
a^{(2,0,0)} &= -4\beta k^2 m^2, & a^{(0,2,0)} &= \beta k^2 m^2, & a^{(0,0,2)} &= \beta k^2, \\
a^{(1,1,0)} &= 0, & a^{(0,1,1)} &= \pm 6\beta k^2 m, & a^{(1,0,1)} &= 0.
\end{aligned} \tag{20}$$

Case 3. Consider

$$\begin{aligned}
a^{(0)} &= -\frac{c}{k}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, & a^{(0,0,1)} &= 0, \\
a^{(2,0,0)} &= -\beta k^2 m^2, & a^{(0,2,0)} &= \beta k^2 m^2, & a^{(0,0,2)} &= 4\beta k^2, \\
a^{(1,1,0)} &= \pm 6m^2 k^2 \beta i, & a^{(0,1,1)} &= 0, & a^{(1,0,1)} &= 0.
\end{aligned} \tag{21}$$

Case 4. Consider

$$\begin{aligned}
a^{(0)} &= -\frac{c}{k}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, & a^{(0,0,1)} &= 0, \\
a^{(2,0,0)} &= -\beta k^2 m^2, & a^{(0,2,0)} &= 4\beta k^2 m^2, & a^{(0,0,2)} &= \beta k^2, \\
a^{(1,1,0)} &= \pm 6mk^2 \beta i, & a^{(0,1,1)} &= 0, & a^{(1,0,1)} &= 0.
\end{aligned} \tag{22}$$

Case 5. Consider

$$\begin{aligned} a^{(0)} &= -\frac{c}{k}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, & a^{(0,0,1)} &= 0, \\ a^{(2,0,0)} &= -\beta k^2 m^2, & a^{(0,2,0)} &= \beta k^2 m^2, & a^{(0,0,2)} &= \beta k^2, \\ a^{(1,1,0)} &= \pm 3m^2 k^2 \beta i, & a^{(0,1,1)} &= \pm 3\beta k^2 m, \\ a^{(1,0,1)} &= \pm 3\beta k^2 m i. \end{aligned} \quad (23)$$

Case 6. Consider

$$\begin{aligned} a^{(0)} &= a^{(0)}, & a^{(1,0,0)} &= 0, & a^{(0,1,0)} &= 0, & a^{(0,0,1)} &= 0, \\ a^{(2,0,0)} &= a^{(0,2,0)} + m^2 a^{(0,0,2)}, & a^{(0,2,0)} &= a^{(0,2,0)}, \\ a^{(0,0,2)} &= a^{(0,0,2)}, & a^{(1,1,0)} &= 0, & a^{(0,1,1)} &= 0, \\ a^{(1,0,1)} &= 0, \end{aligned} \quad (24)$$

where $a^{(0)}$, $a^{(0,2,0)}$, and $a^{(0,0,2)}$ are arbitrary constants.

By substituting the results above into (18) we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for (14), where $\xi = ct + (k/\Gamma(1+\alpha))x^\alpha + \xi_0$.

Family 1. Consider

$$\begin{aligned} u_1(x, t) &= -\frac{c}{k} - 4\beta k^2 m^2 \text{sn}^2(\xi) + 4\beta k^2 m^2 \text{cn}^2(\xi) \\ &\quad + 4\beta k^2 \text{dn}^2(\xi). \end{aligned} \quad (25)$$

Family 2. Consider

$$\begin{aligned} u_2(x, t) &= -\frac{c}{k} - 4\beta k^2 m^2 \text{sn}^2(\xi) + \beta k^2 m^2 \text{cn}^2(\xi) \\ &\quad + \beta k^2 \text{dn}^2(\xi) \pm 6\beta k^2 m \text{cn}(\xi) \text{dn}(\xi). \end{aligned} \quad (26)$$

Family 3. Consider

$$\begin{aligned} u_3(x, t) &= -\frac{c}{k} - \beta k^2 m^2 \text{sn}^2(\xi) + \beta k^2 m^2 \text{cn}^2(\xi) \\ &\quad + 4\beta k^2 \text{dn}^2(\xi) \pm 6m^2 k^2 \beta \text{isn}(\xi) \text{cn}(\xi). \end{aligned} \quad (27)$$

Family 4. Consider

$$\begin{aligned} u_4(x, t) &= -\frac{c}{k} - \beta k^2 m^2 \text{sn}^2(\xi) + 4\beta k^2 m^2 \text{cn}^2(\xi) \\ &\quad + \beta k^2 \text{dn}^2(\xi) \pm 6mk^2 \beta \text{isn}(\xi) \text{cn}(\xi). \end{aligned} \quad (28)$$

Family 5. Consider

$$\begin{aligned} u_5(x, t) &= -\frac{c}{k} - \beta k^2 m^2 \text{sn}^2(\xi) + \beta k^2 m^2 \text{cn}^2(\xi) \\ &\quad + \beta k^2 \text{dn}^2(\xi) \pm 3m^2 k^2 \beta \text{isn}(\xi) \text{cn}(\xi) \\ &\quad \pm 3\beta k^2 m \text{cn}(\xi) \text{dn}(\xi) \pm 3\beta k^2 m \text{isn}(\xi) \text{dn}(\xi). \end{aligned} \quad (29)$$

Family 6. Consider

$$\begin{aligned} u_6(x, t) &= a^{(0)} + [a^{(0,2,0)} + m^2 a^{(0,0,2)}] \text{sn}^2(\xi) + a^{(0,2,0)} \text{cn}^2(\xi) \\ &\quad + a^{(0,0,2)} \text{dn}^2(\xi), \end{aligned} \quad (30)$$

where $a^{(0)}$, $a^{(0,2,0)}$, and $a^{(0,0,2)}$ are arbitrary constants.

Remark 2. We note that the Jacobi elliptic function solutions established in (25)–(30) for the space fractional KdV equation (14) are new exact solutions so far in the literature.

3.2. *Space-Time Fractional Fokas Equation.* Consider the following space-time fractional Fokas equation [8, 20, 21]:

$$\begin{aligned} 4 \frac{\partial^{2\alpha} q}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha} q}{\partial x_1^{3\alpha} \partial x_2^\alpha} + \frac{\partial^{4\alpha} q}{\partial x_2^{3\alpha} \partial x_1^\alpha} + 12 \frac{\partial^\alpha q}{\partial x_1^\alpha} \frac{\partial^\alpha q}{\partial x_2^\alpha} \\ + 12q \frac{\partial^{2\alpha} q}{\partial x_1^\alpha \partial x_2^\alpha} - 6 \frac{\partial^{2\alpha} q}{\partial y_1^\alpha \partial y_2^\alpha} = 0, \quad 0 < \alpha \leq 1. \end{aligned} \quad (31)$$

Equation (31) is a transformed generalization of the following $(4+1)$ -dimensional Fokas equation:

$$\frac{\partial^2 q}{\partial t \partial x_1} = \frac{1}{4} \frac{\partial^4 q}{\partial x_1^3 \partial x_2} - \frac{1}{4} \frac{\partial^4 q}{\partial x_2^3 \partial x_1} - \frac{3}{2} q \frac{\partial^2 q^2}{\partial x_1 \partial x_2} + \frac{3}{2} \frac{\partial^2 q}{\partial y_1 \partial y_2}, \quad (32)$$

which is one of the new high-dimensional nonlinear wave equations Fokas recently obtained by extending the integrable KP equation and DS equation. In [8], El-Sayed et al. obtained some exact analytical solutions for (31) including the generalized hyperbolic function solutions, generalized trigonometric function solutions, and rational solutions, while in [20, 21] the authors obtained some other new exact solutions using the Riccati equation method and the $(D^\alpha G/G)$ method.

In the following we will apply the proposed method in Section 2 to solve (31).

Suppose $q(t, x_1, x_2, y_1, y_2) = Q(\xi)$, where $\xi = k_1 x_1^\alpha / \Gamma(1+\alpha) + k_2 x_2^\alpha / \Gamma(1+\alpha) + l_1 y_1^\alpha / \Gamma(1+\alpha) + l_2 y_2^\alpha / \Gamma(1+\alpha) + ct^\alpha / \Gamma(1+\alpha) + \xi_0$; $k_1, k_2, l_1, l_2, c, \xi_0$ are all constants with $k_1, k_2, l_1, l_2, c \neq 0$. Then by the use of (2) and (4), and similar to the process of (15) and (16), (31) can be converted into the following form:

$$\begin{aligned} 4ck_1 U'' - k_1^3 k_2 U^{(4)} + k_2^3 k_1 U^{(4)} + 12k_1 k_2 U'^2 \\ + 12k_1 k_2 U U'' - 6l_1 l_2 U'' = 0. \end{aligned} \quad (33)$$

Suppose that the solution of (33) can be expressed by

$$Q(\xi) = a^{(0)} + \sum_{n+p+q=1}^m a^{(n,p,q)} \text{sn}^n(\xi) \text{cn}^p(\xi) \text{dn}^q(\xi). \quad (34)$$

By balancing the order of $U^{(4)}$ and UU'' in (33) one can obtain $m = 2$. So

$$\begin{aligned} Q(\xi) = & a^{(0)} + a^{(1,0,0)} \operatorname{sn}(\xi) + a^{(0,1,0)} \operatorname{cn}(\xi) \\ & + a^{(0,0,1)} \operatorname{dn}(\xi) + a^{(2,0,0)} \operatorname{sn}^2(\xi) + a^{(0,2,0)} \operatorname{cn}^2(\xi) \\ & + a^{(0,0,2)} \operatorname{dn}^2(\xi) + a^{(1,1,0)} \operatorname{sn}(\xi) \operatorname{cn}(\xi) \\ & + a^{(0,1,1)} \operatorname{cn}(\xi) \operatorname{dn}(\xi) + a^{(1,0,1)} \operatorname{sn}(\xi) \operatorname{dn}(\xi). \end{aligned} \quad (35)$$

Substituting (35) into (33), using (12) and collecting all the terms with the same power of $\operatorname{sn}^n(\xi) \operatorname{cn}^p(\xi) \operatorname{dn}^q(\xi)$ together, equating each coefficient to zero yields a set of algebraic equations. Solving these equations yields the following values.

Case 1. Consider

$$\begin{aligned} a^{(0)} = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2}, \quad a^{(1,0,0)} = 0, \quad a^{(0,1,0)} = 0, \\ a^{(0,0,1)} = & 0, \quad a^{(2,0,0)} = \frac{1}{3}m^2(k_1^2 - k_2^2), \\ a^{(0,2,0)} = & \frac{1}{3}m^2(k_2^2 - k_1^2), \quad a^{(0,0,2)} = \frac{1}{3}(k_2^2 - k_1^2), \\ a^{(1,1,0)} = & 0, \quad a^{(0,1,1)} = 0, \quad a^{(1,0,1)} = 0. \end{aligned} \quad (36)$$

Case 2. Consider

$$\begin{aligned} a^{(0)} = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2}, \quad a^{(1,0,0)} = 0, \quad a^{(0,1,0)} = 0, \\ a^{(0,0,1)} = & 0, \quad a^{(2,0,0)} = \frac{1}{3}m^2(k_1^2 - k_2^2), \\ a^{(0,2,0)} = & \frac{1}{12}m^2(k_2^2 - k_1^2), \quad a^{(0,0,2)} = \frac{1}{12}(k_2^2 - k_1^2), \\ a^{(1,1,0)} = & 0, \quad a^{(0,1,1)} = \pm \frac{1}{2}m(k_1^2 - k_2^2), \quad a^{(1,0,1)} = 0. \end{aligned} \quad (37)$$

Case 3. Consider

$$\begin{aligned} a^{(0)} = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2}, \quad a^{(1,0,0)} = 0, \quad a^{(0,1,0)} = 0, \\ a^{(0,0,1)} = & 0, \quad a^{(2,0,0)} = \frac{1}{12}m^2(k_1^2 - k_2^2), \\ a^{(0,2,0)} = & \frac{1}{12}m^2(k_2^2 - k_1^2), \quad a^{(0,0,2)} = \frac{1}{3}(k_2^2 - k_1^2), \\ a^{(1,1,0)} = & \pm \frac{i}{2}(k_1^2 - k_2^2)m^2, \quad a^{(0,1,1)} = 0, \quad a^{(1,0,1)} = 0. \end{aligned} \quad (38)$$

Case 4. Consider

$$\begin{aligned} a^{(0)} = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2}, \quad a^{(1,0,0)} = 0, \quad a^{(0,1,0)} = 0, \\ a^{(0,0,1)} = & 0, \quad a^{(2,0,0)} = \frac{1}{12}m^2(k_1^2 - k_2^2), \\ a^{(0,2,0)} = & \frac{1}{3}m^2(k_2^2 - k_1^2), \quad a^{(0,0,2)} = \frac{1}{12}(k_2^2 - k_1^2), \\ a^{(1,1,0)} = & 0, \quad a^{(0,1,1)} = 0, \quad a^{(1,0,1)} = \pm \frac{i}{2}m(k_1^2 - k_2^2). \end{aligned} \quad (39)$$

Case 5. Consider

$$\begin{aligned} a^{(0)} = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2}, \quad a^{(1,0,0)} = 0, \quad a^{(0,1,0)} = 0, \\ a^{(0,0,1)} = & 0, \quad a^{(2,0,0)} = \frac{1}{12}m^2(k_1^2 - k_2^2), \\ a^{(0,2,0)} = & \frac{1}{12}m^2(k_2^2 - k_1^2), \quad a^{(0,0,2)} = \frac{1}{12}(k_2^2 - k_1^2), \\ a^{(1,1,0)} = & \pm \frac{i}{4}(k_1^2 - k_2^2)m^2, \quad a^{(0,1,1)} = \mp \frac{1}{4}m(k_1^2 - k_2^2), \\ a^{(1,0,1)} = & \pm \frac{i}{4}m(k_1^2 - k_2^2). \end{aligned} \quad (40)$$

By substituting the results above into (35) we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for (31), where $\xi = k_1 x_1^\alpha / \Gamma(1 + \alpha) + k_2 x_2^\alpha / \Gamma(1 + \alpha) + l_1 y_1^\alpha / \Gamma(1 + \alpha) + l_2 y_2^\alpha / \Gamma(1 + \alpha) + ct^\alpha / \Gamma(1 + \alpha) + \xi_0$.

Family 1. Consider

$$\begin{aligned} q_1(t, x_1, x_2, y_1, y_2) = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2} + \frac{1}{3}m^2(k_1^2 - k_2^2) \operatorname{sn}^2(\xi) \\ & + \frac{1}{3}m^2(k_2^2 - k_1^2) \operatorname{cn}^2(\xi) + \frac{1}{3}(k_2^2 - k_1^2) \operatorname{dn}^2(\xi). \end{aligned} \quad (41)$$

Family 2. Consider

$$\begin{aligned} q_2(t, x_1, x_2, y_1, y_2) = & -\frac{2ck_1 - 3l_1l_2}{6k_1k_2} + \frac{1}{3}m^2(k_1^2 - k_2^2) \operatorname{sn}^2(\xi) \\ & + \frac{1}{12}m^2(k_2^2 - k_1^2) \operatorname{cn}^2(\xi) + \frac{1}{12}(k_2^2 - k_1^2) \operatorname{dn}^2(\xi) \\ & \pm \frac{1}{2}m(k_1^2 - k_2^2) \operatorname{cn}(\xi) \operatorname{dn}(\xi). \end{aligned} \quad (42)$$

Family 3. Consider

$$\begin{aligned} q_3(t, x_1, x_2, y_1, y_2) &= -\frac{2ck_1 - 3l_1l_2}{6k_1k_2} + \frac{1}{12}m^2(k_1^2 - k_2^2)\text{sn}^2(\xi) \\ &\quad + \frac{1}{12}m^2(k_2^2 - k_1^2)\text{cn}^2(\xi) + \frac{1}{3}(k_2^2 - k_1^2)\text{dn}^2(\xi) \\ &\quad \pm \frac{i}{2}(k_1^2 - k_2^2)m^2\text{sn}(\xi)\text{cn}(\xi). \end{aligned} \quad (43)$$

Family 4. Consider

$$\begin{aligned} q_4(t, x_1, x_2, y_1, y_2) &= -\frac{2ck_1 - 3l_1l_2}{6k_1k_2} + \frac{1}{12}m^2(k_1^2 - k_2^2)\text{sn}^2(\xi) \\ &\quad + \frac{1}{3}m^2(k_2^2 - k_1^2)\text{cn}^2(\xi) + \frac{1}{12}(k_2^2 - k_1^2)\text{dn}^2(\xi) \\ &\quad \pm \frac{i}{2}m(k_1^2 - k_2^2)\text{sn}(\xi)\text{dn}(\xi). \end{aligned} \quad (44)$$

Family 5. Consider

$$\begin{aligned} q_5(t, x_1, x_2, y_1, y_2) &= -\frac{2ck_1 - 3l_1l_2}{6k_1k_2} + \frac{1}{12}m^2(k_1^2 - k_2^2)\text{sn}^2(\xi) \\ &\quad + \frac{1}{12}m^2(k_2^2 - k_1^2)\text{cn}^2(\xi) + \frac{1}{12}(k_2^2 - k_1^2)\text{dn}^2(\xi) \\ &\quad \pm \frac{i}{4}(k_1^2 - k_2^2)m^2\text{sn}(\xi)\text{cn}(\xi) \\ &\quad \mp \frac{1}{4}m(k_1^2 - k_2^2)\text{cn}(\xi)\text{dn}(\xi) \\ &\quad \pm \frac{i}{4}m(k_1^2 - k_2^2)\text{sn}(\xi)\text{dn}(\xi). \end{aligned} \quad (45)$$

Remark 3. The Jacobi elliptic function solutions established in (41)–(45) are new exact solutions to the space-time fractional Fokas equation (31).

4. Conclusions

In this paper, we have proposed a new approach for seeking Jacobi elliptic function solutions to fractional partial differential equations. Based on a fractional complex transformation, certain fractional partial differential equation is converted into another ordinary differential equation of integer order, which permits the exact solutions with the forms of Jacobi elliptic functions, and such solutions can be obtained by use of the concept of the subequation method. So in this way, this method belongs to the categories of subequation methods. As for applications, we apply it to seek Jacobi elliptic

function solutions for the space fractional KdV equation and the space-time fractional Fokas equation. With the aid of mathematical software, a series of exact solutions in the forms of Jacobi elliptic functions for them are successfully found.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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