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Research Article

An Improved Antiwindup Design Using an Anticipatory Loop and an Immediate Loop

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We present an improved antiwindup design for linear invariant continuous-time systems with actuator saturation nonlinearities. In the improved approach, two antiwindup compensators are simultaneously designed: one activated immediately at the occurrence of actuator saturation and the other activated in anticipatory of actuator saturation. Both the static and dynamic antiwindup compensators are considered. Sufficient conditions for global stability and minimizing the induced L_2 gain are established, in terms of linear matrix inequalities (LMIs). We also show that the feasibility of the improved antiwindup is similar to the traditional antiwindup. Benefits of the proposed approach over the traditional antiwindup and a recent innovative antiwindup are illustrated with well-known examples.

1. Introduction

Actuator saturation, which may cause loss in performance and even instability, is a ubiquitous and inevitable fact in any practical control systems. One general method to reduce adverse effects of saturation is the so-called antiwindup (AW). In AW design, a linear controller which does not take the saturation nonlinearity into account is first designed. Then, an AW compensator is added to ensure that stability is maintained (at least in some region near the origin) and that less performance degradation occurs than no AW is used [1]. Such an approach has received much attention in recent several decades due to its intuitive motivation and its effectiveness in practice (see [1–4] and references therein). The traditional AW scheme is depicted in Figure 1, where P, C, and AW are the plant, the linear controller, and the AW compensator, respectively. Typically, the linear controller can be designed using the well-established linear control theory, and various methods for designing the AW compensators have been proposed in the literature (see [5–8] for some representative examples).

One of the main features of the traditional AW is that the AW compensator activated as soon as the saturation is encountered (nearly all the AW designs were based on this paradigm). In a pair of recent papers [9–12], Sajjadi-Kia and Jabbari investigated the effects of deferring the activation of the AW compensator and a so-called delayed AW was proposed. Based on the assumption that the linear controller possesses a reasonable amount of performance robustness, the motivation of the delayed AW is to apply the AW compensator until the closed-loop performance faces substantial decrease. With several examples, the authors showed that the delayed AW renders better performance than the traditional AW. Motivated by Sajjadi-Kia and Jabbari's work and considering the dynamical nature of the system, Wu and Lin proposed a new AW which is called as anticipatory AW [13–16]. The anticipatory AW is opposite to the delayed AW, and its basic idea is to activate the AW compensator in anticipation of actuator saturation. It was shown in [14] that the anticipatory AW has the potential of leading to significant improvement in the closed-loop performance, in terms of both the transient quality in reference tracking and the region of stability.

A further modified AW is to simultaneously design two AW loops, one for immediate activation and the other for delayed activation [17, 18]. The main idea is to separate the

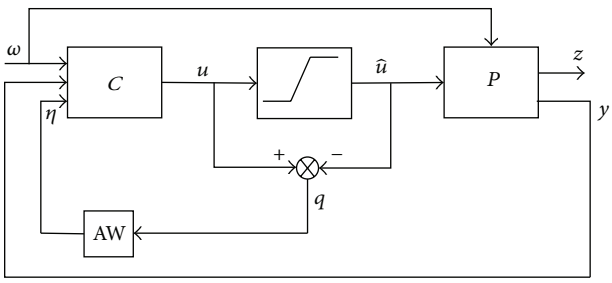

Figure 1: The traditional AW scheme.

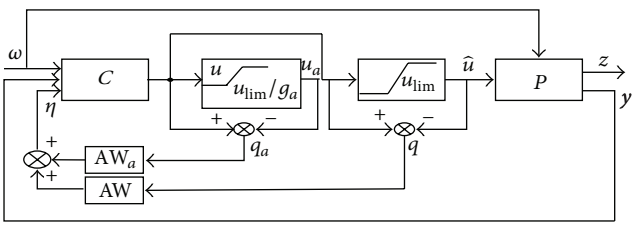

Figure 2: The improved AW scheme.

saturated zone and obtain more aggressive and effective AW gains in lower levels of saturation. Using such a multiloop AW, one can obtain better response for moderate levels of saturation, but for high saturation levels, the improvement diminishes (see the numerical example in [18]).

Motivated by Wu and Lin's work, we are hopeful to achieve a better performance by adding an anticipatory AW loop to the traditional AW scheme to take a "precautionary" action before actuator saturation occurs. It has been confirmed in [14] that a single anticipatory AW loop can work better than a single delayed AW loop. The main observation, here, is to combine an immediate (traditional) AW loop and an anticipatory AW loop to further improve the closed-loop performance. The proposed AW scheme is depicted in Figure 2, where AW_a is the anticipatory AW compensator. We show that the synthesis results can be cast as an optimization over LMIs, and the two sets of AW gains can be straightforward obtained. Since we focus on the global results, we will restrict ourselves to stable plants.

The rest of this paper is organized as follows. In Section 2, we provide a general description of the traditional AW and the proposed AW. In Section 3, we demonstrate the synthesis results in detail, with LMIs. We will first focus on static gains and then extend to dynamic gains. The feasibility of the resulting optimization problem will also be examined. In Section 4, we illustrate the benefits of the proposed AW through two examples. Finally, we conclude with Section 5.

Notation. The notation in this paper is standard. $\mathbf{R}$ is the set of real numbers. A^T is the transpose of a real matrix A. The matrix inequality $A > B$ ($A \geq B$) means that A and B are square Hermitian matrices and A-B is positive (semi-) definite. A block diagonal matrix with submatrices X_1, $X_2, \ldots, X_p$ in its diagonal will be denoted by $\operatorname{diag}\{X_1, X_2, \ldots, X_p\}$. I denotes the identity matrix of appropriate dimensions. To reduce clutter, off-diagonal entries in symmetric matrices are occasionally replaced by "$*$". The sector condition used in this paper is defined as follows.

Definition 1 (sector condition [19]). A function $f : \mathbf{R}^n \to \mathbf{R}^n$ is said to belong to the sector $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if $[f(\omega) - K_1\omega]^T[f(\omega) - K_2\omega] \leq 0$ for all $\omega \in \mathbf{R}^n$.

2. Problem Formulation

Consider the following stable plant:

$$P : \begin{cases} \dot{x}_p = A_p x_p + B_1\omega + B_2\hat{u} \\ z = C_1 x_p + D_{11}\omega + D_{12}\hat{u} \\ y = C_2 x_p + D_{21}\omega + D_{22}\hat{u}, \end{cases} \tag{1}$$

where $x_p \in \mathbf{R}^{n_p}$ is the plant state, $\omega \in \mathbf{R}^{n_\omega}$ is the exogenous input (reference signals, disturbances, and noise), $\hat{u} \in \mathbf{R}^{n_u}$ is the control input, $z \in \mathbf{R}^{n_z}$ is the controlled output, $y \in \mathbf{R}^{n_y}$ is the measurement output, and $A_p, B_1, B_2, C_1, D_{11}, D_{12}, C_2, D_{21}$, and D_{22} are real constant matrices of appropriate dimensions. Pairs (A_p, B_2) and (C_2, A_p) are assumed to be controllable and observable, respectively. Without loss of generality, we will assume that $D_{22} = 0$ henceforth.

Considering plant P, we assume that an n_cth-order linear dynamic controller

$$C : \begin{cases} \dot{x}_c = A_c x_c + B_{cy} y + B_{c\omega}\omega \\ u = C_c x_c + D_{cy} y + D_{c\omega}\omega \end{cases} \tag{2}$$

has been designed to guarantee that the closed-loop system is stable and achieve some performance specifications in the absence of actuator saturation. Here, $x_c \in \mathbf{R}^{n_c}$ is the controller state, $u \in \mathbf{R}^{n_u}$ is the controller output, and $A_c, B_{cy}, B_{c\omega}, C_c, D_{cy}$, and $D_{c\omega}$ are real constant matrices of appropriate dimensions.

In the absence of actuator saturation, the unconstrained interconnection between the plant P and linear controller C is given by

$$\hat{u} = u. \tag{3}$$

If saturation is present at the input of the plant, the unconstrained interconnection (3) is no longer guaranteed and it will be replaced by

$$\hat{u} = \operatorname{sat}(u), \tag{4}$$

where $\operatorname{sat}(\cdot)$ is the standard decentralized saturation function defined as

$$\operatorname{sat}(u) = \left[\operatorname{sat}(u_1), \ldots, \operatorname{sat}(u_{n_u})\right]^T \tag{5}$$

with $\operatorname{sat}(u_i) = \operatorname{sign}(u_i)\min\{|u_i|, u_{\lim}^i\}$; here $u_{\lim}^i$ is the saturation bound for the ith input.

In order to mitigate the undesirable effects caused by actuator saturation, a correction term proportional to $q = u - \hat{u}$ is added to the linear controller; that is,

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_{cy} y + B_{c\omega}\omega + \eta_1, \\ u &= C_c x_c + D_{cy} y + D_{c\omega}\omega + \eta_2, \end{aligned} \tag{6}$$

where

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = -\Lambda q = -\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} q \tag{7}$$

is the traditional static AW compensator. Defining $x = [x_p^T \;\; x_c^T]^T$, the traditional AW closed-loop system can be written as

$$\begin{aligned} \dot{x} &= Ax + B_\omega \omega + \left(B_q - B_\eta \Lambda\right) q, \\ z &= C_z x + D_{z\omega}\omega + \left(D_{zq} - D_{z\eta}\Lambda\right) q, \\ u &= C_u x + D_{u\omega}\omega - D_{u\eta}\Lambda q, \end{aligned} \tag{8}$$

where

$$\begin{gathered} A = \begin{bmatrix} A_p + B_2 D_{cy} C_2 & B_2 C_c \\ B_{cy} C_2 & A_c \end{bmatrix}, \\ B_\omega = \begin{bmatrix} B_1 + B_2 D_{c\omega} + B_2 D_{cy} D_{21} \\ B_{c\omega} + B_{cy} D_{21} \end{bmatrix}, \\ B_q = \begin{bmatrix} -B_2 \\ 0 \end{bmatrix}, \qquad B_\eta = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \\ C_z = \begin{bmatrix} C_1 + D_{12} D_{cy} C_2 & D_{12} C_c \end{bmatrix}, \\ D_{z\omega} = D_{11} + D_{12} D_{cy} D_{21} + D_{12} D_{c\omega}, \\ D_{zq} = -D_{12}, \qquad D_{z\eta} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \\ C_u = \begin{bmatrix} D_{cy} C_2 & C_c \end{bmatrix}, \qquad D_{u\omega} = D_{cy} D_{21} + D_{c\omega}, \\ D_{u\eta} = [0 \;\; I]. \end{gathered} \tag{9}$$

In the proposed AW scheme, an artificial saturation element with a lower saturation bound $u_{\lim}/g_a$ is added; here $g_a > 1$ is a design variable specified by designer. We note that, when the magnitude of the controller output u satisfies $u_{\lim}/g_a \le u < u_{\lim}$, only the added (anticipatory) AW loop is activated, and if the controller output u goes beyond $u_{\lim}$, both the immediate AW loop and the anticipatory AW loop are activated. In Figure 2, signals to motivate the immediate AW loop and the anticipatory AW loop are $q = u - \hat{u}$ and $q_a = u - u_a$, respectively. We first assume that all the AW gains are static; that is,

$$\mathrm{AW} := -\Lambda q, \qquad \mathrm{AW}_a := -\Lambda_a q_a. \tag{10}$$

Then, the closed-loop system depicted in Figure 2 can be written into the following equivalent state-space form:

$$\begin{aligned} \dot{x} &= Ax + B_\omega \omega + \left(B_q - B_\eta \Lambda\right) q - B_\eta \Lambda_a q_a, \\ z &= C_z x + D_{z\omega}\omega + \left(D_{zq} - D_{z\eta}\Lambda\right) q - D_{z\eta}\Lambda_a q_a, \\ u &= C_u x + D_{u\omega}\omega - D_{u\eta}\Lambda q - D_{u\eta}\Lambda_a q_a. \end{aligned} \tag{11}$$

Consider that the anticipatory AW loop has dynamic gains; that is,

$$\begin{aligned} \dot{x}_a &= A_a x_a + B_a q_a, \\ \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix} &= C_a x_a + D_a q_a, \end{aligned} \tag{12}$$

where $x_a \in \mathbf{R}^{n_a}$ is the dynamic AW compensator state and A_a, B_a, C_a, and D_a are real constant matrices of appropriate dimensions. Define $\tilde{x} = [x_p^T \;\; x_c^T \;\; x_a^T]^T$. Then, the closed-loop system can be described as

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{A}\tilde{x} + \widetilde{B}_\omega \omega + \left(\widetilde{B}_q - \widetilde{B}_\eta \Lambda\right) q - \widetilde{B}_{\tilde{\eta}} \widetilde{\Lambda}_a q_a, \\ z &= \widetilde{C}_z \tilde{x} + D_{z\omega}\omega + \left(D_{zq} - D_{z\eta}\Lambda\right) q - D_{z\tilde{\eta}} \widetilde{\Lambda}_a q_a, \\ u &= \widetilde{C}_u \tilde{x} + D_{u\omega}\omega - D_{u\eta}\Lambda q - D_{u\tilde{\eta}} \widetilde{\Lambda}_a q_a, \end{aligned} \tag{13}$$

where

$$\begin{gathered} \widetilde{A} = \begin{bmatrix} A & B_\eta C_a \\ 0 & A_a \end{bmatrix}, \qquad \widetilde{B}_\omega = \begin{bmatrix} B_\omega \\ 0 \end{bmatrix}, \\ \widetilde{B}_q = \begin{bmatrix} B_q \\ 0 \end{bmatrix}, \qquad \widetilde{B}_\eta = \begin{bmatrix} B_\eta \\ 0 \end{bmatrix}, \\ \widetilde{\Lambda}_a = -\begin{bmatrix} B_a \\ D_a \end{bmatrix}, \qquad \widetilde{B}_{\tilde{\eta}} = \begin{bmatrix} 0 & B_\eta \\ I & 0 \end{bmatrix}, \qquad \widetilde{C}_z = \begin{bmatrix} C_z & D_{z\eta} C_a \end{bmatrix}, \\ D_{z\tilde{\eta}} = D_{12} D_{u\tilde{\eta}}, \qquad \widetilde{C}_u = \begin{bmatrix} C_u & D_{u\eta} C_a \end{bmatrix}, \\ D_{u\tilde{\eta}} = \begin{bmatrix} 0 & D_{u\eta} \end{bmatrix}. \end{gathered} \tag{14}$$

In this paper, the objective of AW design is to compute the AW gains to meet some performance requirements. Similar to much of the AW design techniques, we choose the induced L_2 gain as the performance index. The induced L_2 gain from the exogenous input ω to the output z is defined as [20]

$$\sup_{\|\omega\|_2 \neq 0} \frac{\|z\|_2}{\|\omega\|_2}. \tag{15}$$

3. Main Results

For simplicity, we will first consider the single actuator system. The results can be readily expanded to the multiactuator case, and it will be discussed later.

3.1. Static AW Gains. We state the following lemma that obtains the stabilizing gains Λ and Λ_a.

Lemma 2. *The closed-loop system depicted in Figure 2 is stable and the L_2 gain from ω to z is less than γ if there exist positive scalars M and M_a, symmetric matrix $Q > 0$, and matrices X, X_a such that the following LMI holds:*

$$\begin{bmatrix} \Phi_{11} & * & * & * & * \\ B_\omega^T & -\gamma I & * & * & * \\ C_z Q & D_{z\omega} & -\gamma I & * & * \\ \Phi_{41} & D_{u\omega} & \Phi_{43} & \Phi_{44} & * \\ \Phi_{51} & D_{u\omega} & \Phi_{53} & \Phi_{54} & \Phi_{55} \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{gathered} \Phi_{11} = QA^T + AQ, \qquad \Phi_{41} = -X_a^T B_\eta^T + C_u Q, \\ \Phi_{43} = -X_a^T D_{z\eta}^T, \qquad \Phi_{44} = -D_{u\eta} X_a - X_a^T D_{u\eta}^T - 2M_a, \end{gathered}$$

$$\Phi_{51} = MB_q^T - X^T B_\eta^T + C_u Q, \qquad \Phi_{53} = MD_{zq}^T - X^T D_{z\eta}^T,$$
$$\Phi_{54} = -D_{u\eta} X - X^T D_{u\eta}^T, \qquad \Phi_{55} = -D_{u\eta} X - X^T D_{u\eta}^T - 2M. \tag{17}$$

An upper bound on the induced L_2 gain can be obtained by minimizing γ subject to LMI (16). *If the optimization problem is feasible, then the static AW gains Λ and Λ_a can be calculated from $\Lambda = XM^{-1}$ and $\Lambda_a = X_a M_a^{-1}$.*

Proof. Define $q = \varphi(u)$ and $q_a = \varphi_a(u)$. Note that φ and φ_a are the dead-zone nonlinearity function. It is straightforward to show that

$$\varphi \in [0 \quad 1], \qquad \varphi_a \in [0 \quad 1]. \tag{18}$$

Thus, we have

$$q^T (q - u) \le 0, \qquad q_a^T (q_a - u) \le 0. \tag{19}$$

We construct a quadratic Lyapunov function in the form $V = x^T Q^{-1} x$, where $Q > 0$. To guarantee stability of the closed-loop system and estimate the induced L_2 gain from ω to z, we require [21]

$$\frac{d}{dt}\left(x^T Q^{-1} x\right) + \gamma^{-1} z^T z - \gamma \omega^T \omega < 0. \tag{20}$$

Invoking S-procedure for some positive scalars W and W_a (for multiactuator case, W and W_a are some positive definite matrices), we get the following sufficient condition to guarantee (20):

$$\begin{aligned} &\frac{d}{dt}\left(x^T Q^{-1} x\right) + \gamma^{-1} z^T z - \gamma \omega^T \omega - 2q^T W (q - u) \\ &\quad - 2q_a^T W_a (q_a - u) < 0. \end{aligned} \tag{21}$$

If all the AW gains are static, then, in view of the closed-loop system equation (11), inequality (21) can be expanded as

$$\begin{bmatrix} \Omega_{11} & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{bmatrix} < 0, \tag{22}$$

where

$$\begin{aligned} \Omega_{11} &= A^T Q^{-1} + Q^{-1} A + \gamma^{-1} C_z^T C_z, \\ \Omega_{21} &= B_\omega^T Q^{-1} + \gamma^{-1} D_{z\omega}^T C_z, \\ \Omega_{22} &= \gamma^{-1} D_{z\omega}^T D_{z\omega} - \gamma, \\ \Omega_{31} &= \left(B_q - B_\eta \Lambda\right)^T Q^{-1} + \gamma^{-1}\left(D_{zq} - D_{z\eta}\Lambda\right)^T C_z \\ &\quad + WC_u, \\ \Omega_{32} &= \gamma^{-1}\left(D_{zq} - D_{z\eta}\Lambda\right)^T D_{z\omega} + WD_{u\omega}, \\ \Omega_{33} &= \gamma^{-1}\left(D_{zq} - D_{z\eta}\Lambda\right)^T \left(D_{zq} - D_{z\eta}\Lambda\right) - WD_{u\eta}\Lambda \\ &\quad - \Lambda^T D_{u\eta}^T W - 2W, \\ \Omega_{41} &= -\Lambda_a^T B_\eta^T Q^{-1} - \gamma^{-1} \Lambda_a^T D_{z\eta}^T C_z + W_a C_u, \\ \Omega_{42} &= -\gamma^{-1} \Lambda_a^T D_{z\eta}^T D_{z\omega} + W_a D_{u\omega}, \\ \Omega_{43} &= -\gamma^{-1} \Lambda_a^T D_{z\eta}^T \left(D_{zq} - D_{z\eta}\Lambda\right) - W_a D_{u\eta}\Lambda \\ &\quad - \Lambda_a^T D_{u\eta}^T W, \\ \Omega_{44} &= \gamma^{-1} \Lambda_a^T D_{z\eta}^T D_{z\eta} \Lambda_a - W_a D_{u\eta} \Lambda_a - \Lambda_a^T D_{u\eta}^T W_a^T \\ &\quad - 2W_a. \end{aligned} \tag{23}$$

Applying the Schur complement and the congruence transformation $\mathrm{diag}\{Q, I, I, I, I\}$ then a subsequent congruence transformation $\mathrm{diag}\{I, I, I, W_a^{-1}, W^{-1}\}$ and finally with the change of variables $M = W^{-1}$, $M_a = W_a^{-1}$, $X = \Lambda M$, and $X_a = \Lambda_a M_a$ leads to (16). □

One can obtain a more aggressive and effective anticipatory AW compensator by the following approach. We are now assuming that the controller output satisfies $|u| < u_{\lim}$. Under such an assumption, the immediate AW loop will never be activated, and the closed-loop system can be relative written as

$$\begin{aligned} \dot{x} &= Ax + B_\omega \omega - B_\eta \Lambda_a q_a, \\ z &= C_z x + D_{z\omega}\omega - D_{z\eta}\Lambda_a q_a, \\ u &= C_u x + D_{u\omega}\omega - D_{u\eta}\Lambda_a q_a. \end{aligned} \tag{24}$$

Define $q_a = \tilde{\varphi}_a(u)$ and $s_a = 1 - 1/g_a$. As depicted in Figure 3, when the magnitude of the controller output is bound with $u_{\lim}$, the artificial saturation element satisfies the following sector condition:

$$\tilde{\varphi}_a \in [0, s_a]. \tag{25}$$

Thus we now have

$$q_a^T (q_a - s_a u) \le 0. \tag{26}$$

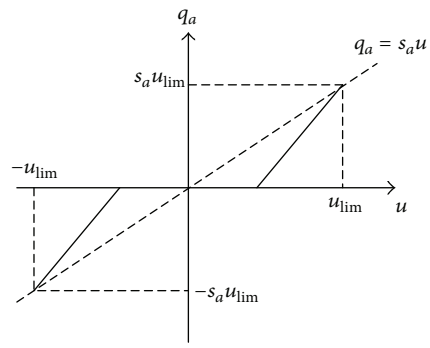

Figure 3: Sector condition for the artificial saturation element.

We then invoke S-procedure for some positive scalars W_a (for multiactuator case, W_a is some positive definite matrices) to obtain

$$\frac{d}{dt}\left(x^T Q^{-1} x\right) + \gamma_a^{-1} z^T z - \gamma_a \omega^T \omega - 2q_a^T W_a \left(q_a - s_a u\right) < 0 \tag{27}$$

which is a sufficient condition for inequality (20).

Expanding inequality (27) with the closed-loop system matrices in (24), followed by application of Schur complement and then congruent transformations diag$\{Q, I, I, I, I\}$ and diag$\{I, I, I, W_a^{-1}\}$, and finally with the substitution of $M_a = W_a^{-1}$ and $X_a = \Lambda_a M_a$, we arrive at the following equivalent LMI:

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ B_\omega^T & -\gamma_a I & * & * \\ C_z Q & D_{z\omega} & -\gamma_a I & * \\ \Psi_{41} & s_a D_{u\omega} & \Phi_{43} & \Psi_{44} \end{bmatrix} < 0, \tag{28}$$

where $\Psi_{41} = -X_a^T B_\eta^T + s_a C_u Q$ and $\Psi_{44} = -s_a D_{u\eta} X_a - s_a X_a^T D_{u\eta}^T - 2M_a$.

To guarantee the uniqueness of Λ_a, we use the same X_a and M_a in (22) and (28); though, to reduce conservatism, we can use different Lyapunov matrix (denoted by Q_a) in (28). Thus, LMI (28) can be rewritten as

$$\begin{bmatrix} \overline{\Phi}_{11} & * & * & * \\ B_\omega^T & -\gamma_a I & * & * \\ C_z Q_a & D_{z\omega} & -\gamma_a I & * \\ \overline{\Psi}_{41} & s_a D_{u\omega} & \Phi_{43} & \Psi_{44} \end{bmatrix} < 0, \tag{29}$$

where $\overline{\Phi}_{11}$ and $\overline{\Psi}_{41}$ are counterparts of Φ_{11} and Ψ_{41} when Q is replaced by Q_a.

Based on the above analysis, we arrive at the following theorem that guarantees global stability and characterizes the induced L_2 gain of the closed-loop system.

Theorem 3. *The closed-loop system depicted in Figure 2 is stable and the L_2 gain from ω to z is less than γ if there exist positive scalars M and M_a, symmetric matrices $Q > 0$ and $Q_a > 0$, and matrices X, X_a such that LMI* (16) *and* (29) *hold. The optimization problem is*

$$\begin{aligned} &\min \quad c\gamma + c_a\gamma_a \quad (\text{with } c > 0,\ c_a > 0) \\ &\text{subject to} \quad \text{LMI (16) and (29)}. \end{aligned} \tag{30}$$

If the optimization problem is feasible, then the static AW gains Λ and Λ_a can be calculated from $\Lambda = XM^{-1}$ and $\Lambda_a = X_a M_a^{-1}$.

As the lemma suggests, using LMI (16) alone guarantees stability and a L_2 performance γ. LMI (29) is used to obtain a more aggressive and effective anticipatory AW compensator. We note that γ is the only L_2 performance of the closed-loop system. The gain γ_a is best described as a measure of the aggressiveness and effectiveness of the anticipatory AW loop.

3.2. Combination of Static Immediate AW and Dynamic Anticipatory AW. The synthesis after letting the anticipatory AW to be dynamic is parallel to that in previous section. To convexify and simplify the synthesis results, we use the same change of variable approach in [18], which results in a dynamic AW with $n_a = n_p + n_c$.

Theorem 4. *The closed-loop system depicted in Figure 2 is stable and the L_2 gain from ω to z is less than γ if there exist positive scalars M and M_a, symmetric matrices $Y > 0$, $Y_a > 0$, and $U > 0$, and matrices X, F_1, F_2, F_3, and F_4 such that the following LMI holds:*

$$\begin{bmatrix} \Delta_{11} & * & * & * & * & * \\ \Delta_{21} & F_1 + F_1^T & * & * & * & * \\ B_\omega^T & 0 & -\gamma I & * & * & * \\ \Delta_{41} & \Delta_{42} & D_{z\omega} & -\gamma I & * & * \\ \Delta_{51} & \Delta_{52} & D_{u\omega} & \Delta_{54} & \Delta_{55} & * \\ \Delta_{61} & \Delta_{62} & D_{u\omega} & \Delta_{64} & \Delta_{65} & \Delta_{66} \end{bmatrix} < 0, \tag{31}$$

where

$$\begin{gathered} \Delta_{11} = AY + YA^T + B_\eta F_2 + F_2^T B_\eta^T, \\ \Delta_{21} = UA^T + F_1 + F_2^T B_\eta^T, \\ \Delta_{41} = C_z Y + D_{z\eta} F_2, \qquad \Delta_{42} = C_z U + D_{z\eta} F_2, \\ \Delta_{51} = F_4^T B_\eta^T + C_u Y + D_{u\eta} F_2, \\ \Delta_{52} = F_3^T + C_u U + D_{u\eta} F_2, \\ \Delta_{54} = F_4 D_{u\eta}^T D_{12}^T, \\ \Delta_{55} = D_{u\eta} F_4 + F_4^T D_{u\eta}^T - 2M_a, \\ \Delta_{61} = MB_q^T - X^T B_\eta^T + C_u Y + D_{u\eta} F_2, \\ \Delta_{62} = C_u U + D_{u\eta} F_2, \qquad \Delta_{64} = MD_{zq}^T - X^T D_{z\eta}^T, \\ \Delta_{65} = D_{u\eta} F_4 - X^T D_{u\eta}^T, \qquad \Delta_{66} = -D_{u\eta} X - X^T D_{u\eta}^T - 2M. \end{gathered} \tag{32}$$

The optimization problem is

$$\begin{aligned} &\min \quad c\gamma + c_a\gamma_a \quad (\text{with } c > 0,\ c_a > 0) \\ &\text{subject to} \quad \text{LMI (31)}, \\ &\begin{bmatrix} \overline{\Delta}_{11} & * & * & * & * \\ \Delta_{21} & F_1 + F_1^T & * & * & * \\ B_\omega^T & 0 & -\gamma_a I & * & * \\ \overline{\Delta}_{41} & \Delta_{42} & D_{z\omega} & -\gamma_a I & * \\ \overline{\Gamma}_{51} & \Gamma_{52} & s_a D_{u\omega} & \Delta_{54} & \Gamma_{55} \end{bmatrix} < 0, \end{aligned} \tag{33}$$

where $\overline{\Delta}_{11}$ and $\overline{\Delta}_{41}$ are counterparts of Δ_{11} and Δ_{41} when Y is replaced with Y_a, $\overline{\Gamma}_{51} = F_4^T B_\eta^T + s_a C_u Y_a + s_a D_{u\eta} F_2$, $\Gamma_{52} = F_3^T + s_a C_u U + s_a D_{u\eta} F_2$, and $\Gamma_{55} = s_a D_{u\eta} F_4 + s_a F_4^T D_{u\eta}^T - 2M_a$. If the optimization problem is feasible, then the AW gains can be calculated from $\Lambda = XM^{-1}$, $A_a = F_1 U^{-1}$, $B_a = F_3 M_a^{-1}$, $C_a = F_2 U^{-1}$, and $D_a = F_4 M_a^{-1}$.

Proof. We rely on the Lyapunov matrix $V = \tilde{x}^T Q^{-1} \tilde{x}$ with $Q > 0$ and partition Q as follows:

$$Q = \begin{bmatrix} Y & U \\ U & U \end{bmatrix}, \tag{34}$$

where $Y = Y^T \in \mathbf{R}^{n_a \times n_a}$ and $U = U^T \in \mathbf{R}^{(n_p+n_c)\times(n_p+n_c)}$. Define a set of variables $F_1 = A_a U$, $F_2 = C_a U$, $F_3 = B_a M_a$, $F_4 = D_a M_a$, and $X = \Lambda M$. Expanding (21) (where x is replaced by $\tilde{x}$) in terms of closed-loop system matrices in (13) and with some proper congruence transformations leads to (31).

As before, we hope to obtain a more aggressive and effective anticipatory AW compensator. Assuming that $|u| < u_{\lim}$, the closed-loop system can be written as

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{A}\tilde{x} + \widetilde{B}_\omega \omega - \widetilde{B}_{\tilde{\eta}} \widetilde{\Lambda}_a q_a, \\ z &= \widetilde{C}_z \tilde{x} + D_{z\omega}\omega - D_{z\tilde{\eta}} \widetilde{\Lambda}_a q_a, \\ u &= \widetilde{C}_u \tilde{x} + D_{u\omega}\omega - D_{u\tilde{\eta}} \widetilde{\Lambda}_a q_a. \end{aligned} \tag{35}$$

Note that the sector condition (26) is also guaranteed if the magnitude of the controller output satisfies $|u| < u_{\lim}$. Thus, inequality (27) (where x is replaced by $\tilde{x}$) still hold true. Followed by the substitution of the closed-loop system matrices in (35), application of some proper congruence transformations yields

$$\begin{bmatrix} \Delta_{11} & * & * & * & * \\ \Delta_{21} & F_1 + F_1^T & * & * & * \\ B_\omega^T & 0 & -\gamma_a I & * & * \\ \Delta_{41} & \Delta_{42} & D_{z\omega} & -\gamma_a I & * \\ \Gamma_{51} & \Gamma_{52} & s_a D_{u\omega} & \Delta_{54} & \Gamma_{55} \end{bmatrix} < 0, \tag{36}$$

where $\Gamma_{51} = F_4^T B_\eta^T + s_a C_u Y + s_a D_{u\eta} F_2$. To reduce conservative, we use a different Y in (36) (denoted by Y_a). Thus, we complete the proof of Theorem 4. □

As before, LMI (31) alone guarantees stability and a L_2 performance γ. LMI (33) is used to obtain a more aggressive and effective dynamic AW compensator.

Remark 5. The parameters g_a, c, and c_a will indeed affect the obtained closed-loop performance (in general, we can fix $c = 1$ and adjust c_a). It is straightforward to see that a larger c_a will lead to a smaller γ_a. The values of g_a and c_a can be determined by a trial and error procedure, based on the computational results. Take g_a as an example; we set the initial value of g_a as $1 + \delta$ for a small scalar $\delta > 0$. Then adjust δ to 1.1δ or 0.1δ iteratively until a desired closed-loop performance is achieved [14].

Remark 6. The results can be readily extended to multi-input plants. In the multi-input case, the design variable g_a is replaced by a diagonal matrix $G_a = \mathrm{diag}\{g_{a_i}\}$; here, g_{a_i} is the design point chosen for the ith input. In addition, the positive scalars W, W_a, M, and M_a in single-input case are now diagonal positive definite matrices.

3.3. Feasibility of the Improved AW. In [18], the authors pointed out that their modified AW, which contains an immediate activation compensator and a delayed activation compensator, is feasible if the traditional AW has a solution. In this subsection, we will show that similar feasibility condition can be obtained for the modified AW proposed in this paper.

In traditional AW synthesis, the condition for stability and a L_2 performance level of γ is as follows [22]:

$$\frac{d}{dt}\left(x^T Q^{-1} x\right) + \gamma^{-1} z^T z - \gamma \omega^T \omega - 2q^T W (q - u) < 0. \tag{37}$$

We assume that there exists a pair of solutions (Q^*, γ^*, W^*) satisfying the above inequality; that is,

$$\frac{d}{dt}\left(x^T Q^{*-1} x\right) + \gamma^{*-1} z^T z - \gamma^* \omega^T \omega - 2q^T W^* (q - u) < 0. \tag{38}$$

We can always find a small enough W_a^* ($W_a^* > 0$) such that (20) holds:

$$\begin{aligned} &\frac{d}{dt}\left(x^T Q^{*-1} x\right) + \gamma^{*-1} z^T z - \gamma^* \omega^T \omega - 2q^T W^* (q - u) \\ &\quad - 2q_a^T W_a^* (q_a - u) < 0. \end{aligned} \tag{39}$$

Now consider (27). As $0 < s_a < 1$, a sufficient condition to ensure (27) is

$$\frac{d}{dt}\left(x^T Q^{-1} x\right) + \gamma_a^{-1} z^T z - \gamma_a \omega^T \omega - 2q_a^T W_a (q_a - u) < 0. \tag{40}$$

We note that if (39) holds then (Q^*, γ^*, W_a^*) is a solution of (40) (i.e., a solution of (27)). Thus, (Q^*, γ^*, W^*, W_a^*) is a solution of inequalities (20) and (27). Finally, we can conclude that the proposed AW is guaranteed to have a solution if the traditional AW counterpart is feasible.

4. Numerical Examples

For ease of comparison, we choose the two examples that are also considered in [18].

Example 1. The plant and linear controller are as follows:

$$\left[\begin{array}{c|c|c} A_p & B_2 & B_1 \\ \hline C_1 & D_{12} & D_{11} \\ \hline C_2 & D_{22} & D_{21} \end{array}\right] = \left[\begin{array}{ccc|c|c} -10.6 & -6.09 & -0.9 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & -1 \\ \hline -1 & -11 & -30 & 0 & 0 \end{array}\right],$$
$$\left[\begin{array}{c|c|c} A_c & B_{cy} & B_{c\omega} \\ \hline C_c & D_{cy} & D_{c\omega} \end{array}\right] = \left[\begin{array}{cc|c|c} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array}\right]. \tag{41}$$

The saturation limit is $u_{\lim} = 1$.

In [10], a static AW compensator

$$\Lambda = [-0.1968 \quad 0.0025 \quad -0.9860]^T \tag{42}$$

which guarantees a performance level of $\gamma = 85.78$ was designed through traditional approach for this example. In [18], two AW compensators

$$\begin{aligned} \Lambda &= [-1512.0 \quad 18.9 \quad 79.5]^T, \\ \widetilde{\Lambda} &= [-97.3641 \quad 1.2207 \quad 0.9435]^T \end{aligned} \tag{43}$$

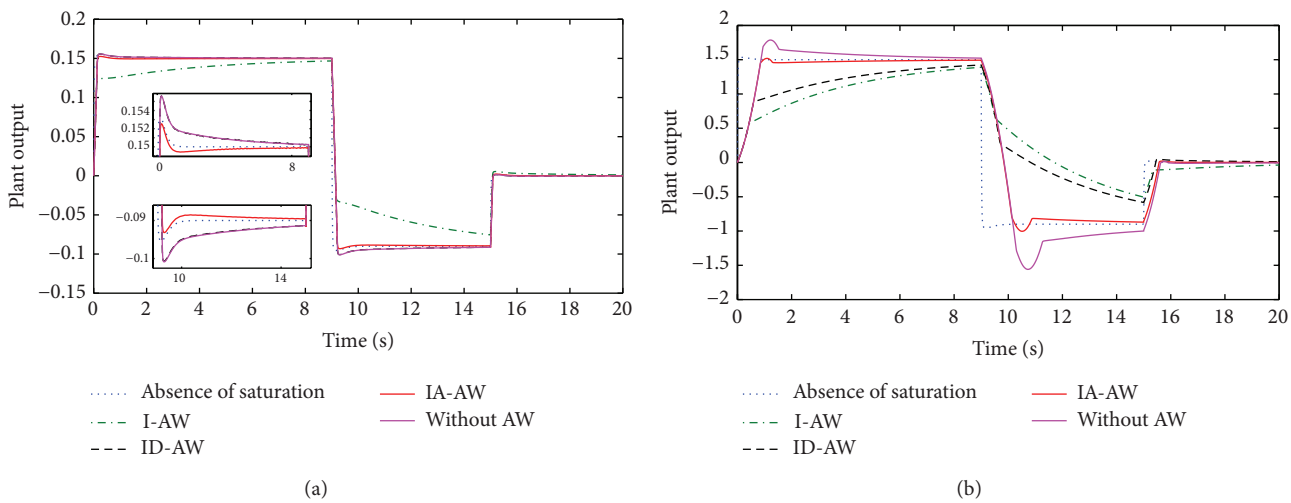

Figure 4: Transient performance: (a) small reference signal and (b) large reference signal.

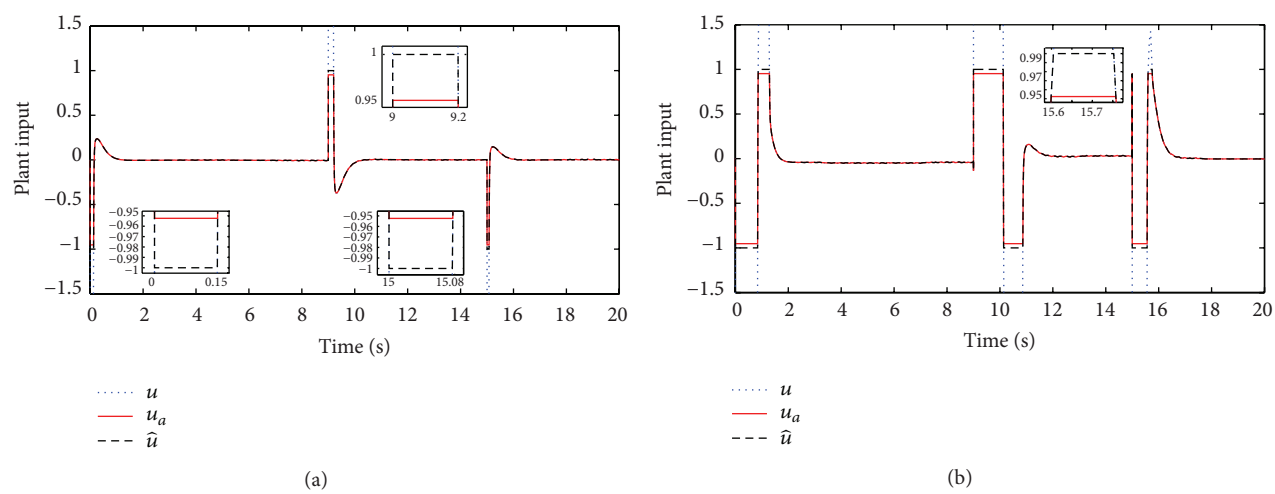

Figure 5: Time history of control commands: (a) small reference signal and (b) large reference signal.

were designed for the immediate AW loop and the delayed AW loop, respectively. We select $g_a = 1.05$ for this example. Using the aforementioned technique with $c = 1$ and $c_a = 20$, we obtain $\gamma = 85.1305$ and $\gamma_a = 1.0002$, and the resulting AW compensators are as follows:

$$\Lambda = \begin{bmatrix} -1.3340 \times 10^{0} & 1.6833 \times 10^{-2} & -2.2484 \times 10^{-1} \end{bmatrix}^{T},$$
$$\Lambda_a = \begin{bmatrix} 6.3135 \times 10^{-1} & -7.8859 \times 10^{-3} & 6.4954 \times 10^{-4} \end{bmatrix}^{T}. \tag{44}$$

Simulation results for a small and a large reference signals (which are the same as that in [18]) are depicted in Figure 4. We note that I-AW represents the traditional AW which has a single immediate AW loop, ID-AW represents the modified AW which consists of an immediate AW loop and a delayed AW loop, and IA-AW represents the proposed modified AW which consists of an immediate AW loop and an anticipatory AW loop. As Figure 4 suggests, system with IA-AW achieves the best performance, no matter the reference signal is small or large. In addition, the performance obtained by IA-AW is much better than that obtained by a single delayed AW loop or a single anticipatory AW loop (see the numerical example in [14]). The time histories of u, u_a, and $\hat{u}$ are illustrated in Figure 5. We can see that both the anticipatory AW loop and the immediate AW loop are activated.

Example 2. Consider the following example taken from [22] with actuator saturation $u_{\lim} = 5$:

$$\left[\begin{array}{c|c|c} A_p & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cccc|c|c} 0 & 1 & 0 & 0 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 & 0 & 2.71762 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -812.61 & -29.87 & -30.10 & 0 & -15.61 & 6.68268 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]. \tag{45}$$

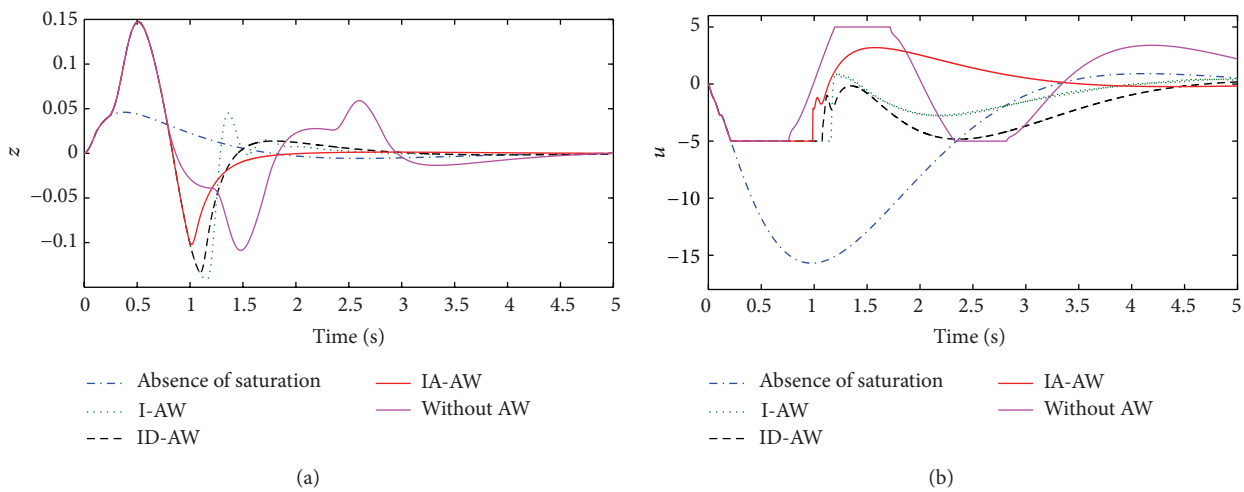

Figure 6: System response: (a) controlled output and (b) control command.

For this plant, static AW is not feasible. In [22], a dynamic AW with $n_a = 4$ has been designed. In [18], an immediate static AW compensator and a delayed dynamic AW compensator were simultaneously designed. Using Theorem 4 with $c = 1$, $c_a = 20$, and $g_a = 1.1$, we obtain the immediate static AW compensator and the anticipatory dynamic AW compensator as follows:

$$\Lambda = \begin{bmatrix} -3.2177\times10^{0} & 6.9510\times10^{1} & 2.2583\times10^{0} & 1.2101\times10^{3} & 1.0537\times10^{2} \end{bmatrix}^{T},$$

$$A_a = \begin{bmatrix} 9.4734\times10^{0} & 1.8697\times10^{1} & 1.0377\times10^{1} & -6.2122\times10^{0} & -7.2457\times10^{-1} & 3.2341\times10^{-1} & 5.1008\times10^{0} & 1.2722\times10^{-1} \\ -7.7848\times10^{2} & -6.1629\times10^{2} & -3.2932\times10^{2} & 2.0814\times10^{2} & 1.7320\times10^{1} & -1.3155\times10^{1} & -1.9018\times10^{2} & -5.1783\times10^{0} \\ 3.2894\times10^{1} & 5.1818\times10^{1} & -2.7946\times10^{1} & -1.7101\times10^{1} & -2.0763\times10^{0} & 9.7751\times10^{-1} & 1.5257\times10^{1} & 3.8451\times10^{-1} \\ -1.9162\times10^{3} & -1.5092\times10^{3} & -8.2887\times10^{2} & 5.0887\times10^{2} & 4.2449\times10^{1} & -3.2502\times10^{1} & -4.6938\times10^{2} & -1.2794\times10^{1} \\ -4.1933\times10^{1} & -6.2229\times10^{1} & -7.1688\times10^{2} & -1.0255\times10^{2} & -7.8405\times10^{2} & -6.5210\times10^{1} & 4.5811\times10^{1} & -2.5154\times10^{1} \\ -1.2631\times10^{3} & -1.8707\times10^{3} & -2.4179\times10^{3} & 5.8860\times10^{2} & -6.0947\times10^{2} & -2.8116\times10^{2} & -1.8892\times10^{3} & -1.1105\times10^{2} \\ -2.5919\times10^{3} & -3.2742\times10^{3} & -2.6531\times10^{3} & 1.2413\times10^{3} & -2.1609\times10^{2} & -4.2016\times10^{2} & -5.1487\times10^{3} & -1.6566\times10^{2} \\ -6.3470\times10^{2} & -1.9625\times10^{3} & -2.2511\times10^{3} & 6.5502\times10^{2} & -1.6360\times10^{2} & -1.8693\times10^{2} & -1.8126\times10^{3} & -7.6273\times10^{1} \end{bmatrix},$$

$$B_a = \begin{bmatrix} -7.3023\times10^{-5} & 1.8710\times10^{-3} & -2.3074\times10^{-4} & 4.6813\times10^{-3} & 4.6430\times10^{-3} & 6.6275\times10^{-3} & 7.1269\times10^{-3} & 2.4370\times10^{-3} \end{bmatrix}^{T},$$

$$C_a = \begin{bmatrix} 3.2144\times10^{1} & 1.3465\times10^{2} & -6.1002\times10^{2} & -1.7056\times10^{2} & -7.2678\times10^{2} & -6.1800\times10^{1} & 1.0299\times10^{2} & -2.3419\times10^{1} \\ -1.3026\times10^{3} & 6.4224\times10^{3} & -4.6565\times10^{1} & -2.3379\times10^{3} & 8.1685\times10^{2} & 2.4871\times10^{2} & 2.0358\times10^{3} & 9.7648\times10^{1} \\ -2.7566\times10^{3} & -3.4909\times10^{3} & -2.9198\times10^{3} & 1.3161\times10^{3} & -2.2227\times10^{2} & -4.2649\times10^{2} & -5.0892\times10^{3} & -1.6916\times10^{2} \\ -2.7529\times10^{4} & -3.7637\times10^{4} & -3.1202\times10^{4} & 1.2980\times10^{4} & 8.7481\times10^{1} & 2.6140\times10^{2} & 3.3648\times10^{3} & 1.0113\times10^{2} \\ -1.8205\times10^{3} & -1.4185\times10^{3} & -1.0956\times10^{3} & 4.8082\times10^{2} & -7.3344\times10^{1} & 1.2916\times10^{2} & 3.7709\times10^{2} & 5.2704\times10^{1} \end{bmatrix},$$

$$D_a = \begin{bmatrix} 4.2089\times10^{-3} & -8.1195\times10^{-3} & 8.1115\times10^{-3} & -1.0155\times10^{-2} & -4.8913\times10^{-3} \end{bmatrix}^{T}. \tag{46}$$

The two compensators guarantee performance levels $\gamma = 186.3811$ and $\gamma_a = 23.2318$. Simulation results for a step input of duration 0.1 s and magnitude 0.5 are depicted in Figure 6. We can see that the proposed improved AW achieves the best system response by forcing the system to leave the saturation zone earlier than both the I-AW and the ID-AW.

5. Conclusions

We have proposed an improved AW design approach for stable linear systems subject to actuator saturation. In the proposed approach, two AW compensators were simultaneously computed, one for immediate activation at the occurrence of saturation and the other for anticipatory activation. Using

the induced L_2 gain as the performance index, the synthesis results were formulated and solved as optimization problems over LMIs. Numerical examples confirmed the effectiveness of the proposed AW design method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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