Research Article

The CEV Model and Its Application in a Study of Optimal Investment Strategy

Aiyin Wang,^{1,2} Ls Yong,¹ Yang Wang,¹ and Xuanjun Luo¹

¹ The School of Finance, Southwestern University of Finance and Economics, Chengdu 610074, China

² Department of Mathematics, Xinjiang University of Finance and Economics, Urumqi 830012, China

Correspondence should be addressed to Aiyin Wang; way8848@sina.com

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The constant elasticity of variance (CEV) model is used to describe the price of the risky asset. Maximizing the expected utility relating to the Hamilton-Jacobi-Bellman (HJB) equation which describes the optimal investment strategies, we obtain a partial differential equation. Applying the Legendre transform, we transform the equation into a dual problem and obtain an approximation solution and an optimal investment strategies for the exponential utility function.

1. Introduction

The constant elasticity of variance (CEV) model is a natural extension of the geometric Brownian motion (GBM). The Merton's model and its extensions [1–3] were generally studied under the assumption that risky asset price was described by the GBM. The CEV model is originally proposed by Cox and Ross [4] as an alternative diffusion process for European option pricing [5–8]. Comparing with the GBM, we know that the advantages of the CEV model are that the volatility rate has correlation with the risky asset price and can explain the empirical bias such as volatility smile [9]. The CEV model was usually applied to calculate the theoretical price, sensitivities, and implied volatility of options [10–12]. In recent years, the optimal investment problem for a pension fund has become an important subject [13–17], in which the CEV model was applied to study the optimal investment strategy for a defined contribution (DC) pension plan. However, the application of the CEV model to the investment and consumption problem has not been widely reported in the existing academic articles.

In this paper, we introduce the CEV model into an investment and consumption problem and optimally allocate the wealth between one risk-free asset and one risky asset. We use CEV model to describe the risk asset's price and discuss personal optimal portfolio with consumption

under the framework of the classical Merton's portfolio optimization problems. Our goal is to choose an optimal investment strategy to maximize total expected utility of wealth. By using the method of stochastic optimal control, the corresponding Hamilton-Jacobi-Bellman (HJB) equation for the value function of the optimization problem is obtained. Applying the Legendre transform and dual theory [14, 15], we obtain an approximation solution and the optimal investment strategies for the exponential utility function. The novelty of this paper is different from those of [13–17], in which the CEV model is used to study the optimal investment strategy for pension fund, we use CEV model to study personal optimal portfolio with consumption factors.

This paper is organized as follows. We introduce the CEV model about the personal portfolio in Section 2. In Section 3, we turn our stochastic optimal problem into a corresponding Hamilton-Jacobi-Bellman (HJB) equation. We transform our problem into the dual problem by applying the Legendre transform and the dual theory in Section 4. In Section 5, we derive an approximation solution by choosing the CARA utility. Conclusions are given in Section 6.

2. The CEV Model

In this section, we consider that the market structure consists of a risk-free asset and a single risky asset which is described

by the *CEV* model. With personal consumption, the dynamical equation of personal portfolio is established.

We denote the price of the risk-free asset (i.e., the bank account) at time t by B_t , which satisfies the following formula:

$$dB_t = rB_t dt, \quad (1)$$

where r is a constant rate of interest.

We denote the price of the risky asset (hereinafter called “stock”) at time t by S_t , which satisfies the *CEV* model. Consider

$$\frac{dS_t}{S_t} = \mu dt + kS_t^\beta dW_t, \quad (2)$$

where μ is an expected instantaneous return of the stock and satisfies the general condition $\mu > \beta$. kS_t^β is the instantaneous volatility and β is the elasticity parameter and satisfies the general condition $\beta < 0$. $W_t : t \geq 0$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the risk-neutral probability. The filtration $\mathcal{F} = \{\mathcal{F}_t\}$ is a right continuous filtration of sigma-algebras on this space.

For personal portfolio, we denote personal consumption at time t by λ_t which is described by an arithmetic Brownian motion as follows:

$$d\lambda_t = bdt + adM_t, \quad (3)$$

where M_t is a standard Brownian motion, a and b are constants. Generally, b is written as a function of t and λ_t , that is, $b = b(\lambda_t, t)$, and a is written as a function of t . Here, for simplicity, a and b are regarded as constants. Assume that the Brownian motion W_t is correlated with M_t and the correlation coefficient is ρ , that is, $dW_t dM_t = \rho$.

Let V_t denote one's disposable wealth at time $t \in [0, T]$, π_t and let $1 - \pi_t$ denote the proportion of one's disposable wealth V_t invested in the risky asset and the risk-free asset, respectively. The dynamical equation of personal portfolio is given by

$$dV_t = \pi_t V_t (\mu dt + kS_t^\beta dW_t) + (1 - \pi_t) V_t rdt - \lambda_t dt. \quad (4)$$

To seek optimal investment strategy π_t , we maximize the expected utility of the terminal wealth

$$\max_{\pi_t} E \{U(V_T) \mid \mathcal{F}_t\}, \quad (5)$$

where $U(\cdot)$ is increasing and concave.

3. The Hamilton-Jacobi-Bellman Equation

In this section, we obtain the form of optimal investment strategy π_t and the Hamilton-Jacobi-Bellman equation of the optimal problem (5) by using the dynamic programming approach.

Define the value function for the optimal problem (5) as

$$H(t, S_t, V_t, \lambda_t) = \sup_{\pi_t \in \mathcal{A}} E \{U(V_T) \mid \mathcal{F}_t\}, \quad 0 < t < T, \quad (6)$$

where $\mathcal{A}$ is the set of admissible trading strategies. Then, the wealth process becomes Markov process. Consider

$$dV_t = (\pi_t V_t \mu + (1 - \pi_t) r V_t - \lambda_t) dt + \pi_t V_t k S_t^\beta dW_t. \quad (7)$$

We define the value function

$$\begin{aligned} H^\pi(t, S_t, V_t, \lambda_t) &= E \{U(V_T) \mid \mathcal{F}_t\} \\ &= E \{U(V_T) \mid S_t = s, V_t = v, \lambda_t = \lambda\}. \end{aligned} \quad (8)$$

Using the Markov property and the law of iterated expectation, we know that $H^\pi(t, s, v, \lambda)$ is a martingale. The value function $H(t, s, v, \lambda) = \sup_{\pi_t \in \mathcal{A}} H^\pi(t, s, v, \lambda)$ is also a martingale.

Using Itô formula, we have

$$\begin{aligned} H(t, S_t, V_t, \lambda_t) &= \sup_{\pi_t \in \mathcal{A}} E_t \{H(t + \Delta t, S_{t+\Delta t}, V_{t+\Delta t}, \lambda_{t+\Delta t})\} \\ &= \sup_{\pi_t \in \mathcal{A}} E_t \left\{ H(t, S_t, V_t, \lambda_t) + H_{S_t} dS_t \right. \\ &\quad + H_{\lambda_t} d\lambda_t + \frac{1}{2} H_{S_t S_t} dS_t dS_t \\ &\quad + \frac{1}{2} H_{V_t V_t} dV_t dV_t + \frac{1}{2} H_{\lambda_t \lambda_t} d\lambda_t d\lambda_t \\ &\quad + H_{S_t V_t} dS_t dV_t + H_{\lambda_t V_t} d\lambda_t dV_t \\ &\quad \left. + H_{S_t \lambda_t} dS_t d\lambda_t \right\}. \end{aligned} \quad (9)$$

Considering the equation $H(T, s, v, \lambda) = U(V_T)$, we have the following Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem

$$\begin{aligned} H_t + \mu s H_s + (rv - \lambda) H_v + b H_\lambda + \frac{1}{2} k^2 s^{2\beta+2} H_{ss} \\ + \frac{1}{2} a^2 H_{vv} - a k s^{\beta+1} \rho H_{sv} \\ + \sup_{[\pi]} \left\{ \frac{1}{2} \pi^2 v^2 k^2 s^{2\beta} H_{vv} + \pi v (\mu - r) H_v \right. \\ \left. + \pi v k^2 s^{2\beta+1} H_{sv} + \pi v a \rho k s^\beta H_{v\lambda} \right\} = 0, \end{aligned} \quad (10)$$

where $H_t, H_s, H_v, H_\lambda, H_{ss}, H_{\lambda\lambda}, H_{vv}, H_{sv}, H_{v\lambda}$, and $H_{s\lambda}$ denote partial derivatives of first and second orders with respect to time, stock price, wealth, and consumption.

Differentiating (10) with respect to π , we have

$$\begin{aligned} \pi v^2 k^2 s^{2\beta} H_{vv} + v (\mu - r) H_v \\ + v k^2 s^{2\beta+1} H_{sv} + v a \rho k s^\beta H_{v\lambda} = 0. \end{aligned} \quad (11)$$

The optimal strategy π_t^* is

$$\pi_t^* = - \frac{(\mu - r) H_v + k^2 s^{2\beta+1} H_{sv} + a \rho k s^\beta H_{v\lambda}}{v k^2 s^{2\beta} H_{vv}}. \quad (12)$$

Putting (12) into (10), we obtain a partial differential equation for the value function H

$$\begin{aligned} H_t + \mu s H_s + (rv - \lambda) H_v + b H_\lambda + \frac{1}{2} k^2 s^{2\beta+2} H_{ss} \\ + \frac{1}{2} a^2 H_{\lambda\lambda} + a \rho k s^{\beta+1} H_{s\lambda} \\ - \frac{[(\mu - r) H_v + k^2 s^{2\beta+1} H_{sv} + a \rho k s^\beta H_{v\lambda}]^2}{2k^2 s^{2\beta} H_{vv}} = 0. \end{aligned} \quad (13)$$

4. The Legendre Transform and Dual Theory

Definition 1. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. For $z > 0$, we write the Legendre transform

$$L(z) = \max_x \{f(x) - zx\}. \quad (14)$$

The function $L(z)$ is called the Legendre dual of the function $f(x)$ (see [16]).

If $f(x)$ is strictly convex, the maximum in the above equation will be attained at just one point, which is denoted by x_0 . It arrives at the unique solution to the first-order condition, namely,

$$\frac{df(x)}{dx} - z = 0. \quad (15)$$

Therefore, we write

$$L(z) = f(x_0) - zx_0. \quad (16)$$

According to Definition 1, we take advantage of the convexity of value function $H(t, s, \lambda, v)$ to define the Legendre transform

$$\begin{aligned} \widehat{H}(t, s, \lambda, z) := \sup_{v>0} \{H(t, s, \lambda, v) - zv \mid 0 < v < \infty\}, \\ (0 < t < T), \end{aligned} \quad (17)$$

where $z > 0$ denotes the dual variable to v . The value of v where this optimum is attained is denoted by $g(t, s, \lambda, z)$. Namely,

$$\begin{aligned} g(t, s, \lambda, z) := \inf_{v>0} \{v \mid H(t, s, \lambda, v) \geq zv + \widehat{H}(t, s, \lambda, z)\}, \\ (0 < t < T). \end{aligned} \quad (18)$$

Using (16) and (17), we have

$$z = H_v. \quad (19)$$

From (18) and (4), the function $\widehat{H}$ related to g is given by $g = -\widehat{H}_z$. Therefore, we take one of the two functions g and $\widehat{H}$ as the dual of H . Here, we work mainly with the function g , as it is easy to be computed numerically for the purpose of computing optimal strategies.

Differentiating (18) and (4) with respect to t, s , and z , we know that the transformation rules for the derivatives of

the value function H and the dual function $\widehat{H}$ are given by (see [13, 16])

$$\begin{aligned} \widehat{H}(t, s, \lambda, z) &= H(t, s, \lambda, g) - zg, \quad g(t, s, \lambda, z) = v, \quad (20) \\ H_v &= z, \quad H_t = \widehat{H}_t, \quad H_s = \widehat{H}_s, \quad H_\lambda = \widehat{H}_\lambda, \\ g &= -\widehat{H}_z, \quad H_{vv} = -\frac{1}{\widehat{H}_{zz}}, \quad H_{ss} = \widehat{H}_{ss} - \frac{\widehat{H}_{sz}^2}{\widehat{H}_{zz}}, \\ H_{\lambda\lambda} &= \widehat{H}_{\lambda\lambda} - \frac{\widehat{H}_{z\lambda}^2}{\widehat{H}_{zz}}, \quad H_{vs} = -\frac{\widehat{H}_{sz}}{\widehat{H}_{zz}}, \quad H_{v\lambda} = -\frac{\widehat{H}_{\lambda z}}{\widehat{H}_{zz}}, \\ H_{s\lambda} &= \widehat{H}_{s\lambda} - \frac{\widehat{H}_{sz}\widehat{H}_{z\lambda}}{\widehat{H}_{zz}}. \end{aligned} \quad (21)$$

At the terminal time T , we define

$$\widehat{U}(z) = \sup_{v>0} \{U(v) - zv\}, \quad (22)$$

$$G(z) = \inf_{v>0} \{v \mid U(v) \geq zv + \widehat{U}(z)\}.$$

Kramkov and Schachermayer [18] and Cox and Huang [19] have shown that the functions $\widehat{U}(z)$ and $U(v)$ can be obtained from each other by using the Legendre transform as follows:

$$\widehat{U}(z) = \sup_{v>0} \{U(v) - zv\}, \quad U(v) = \inf_{z>0} \{\widehat{U}(z) + zv\}. \quad (23)$$

Thus, the primary problem is turned into a dual problem.

Putting (21) into (13) yields

$$\begin{aligned} \widehat{H}_t + \mu s \widehat{H}_s + (rv - \lambda) z + b \widehat{H}_\lambda + \frac{1}{2} k^2 s^{2\beta+2} \widehat{H}_{ss} + \frac{1}{2} a^2 \widehat{H}_{\lambda\lambda} \\ - \frac{1}{2} a^2 (1 - \rho^2) \frac{\widehat{H}_{\lambda z}^2}{\widehat{H}_{zz}} + a \rho k s^{\beta+1} \widehat{H}_{s\lambda} + \frac{(\mu - r)^2 a \rho z^2}{2k^2 s^{2\beta}} \widehat{H}_{zz} \\ - (\mu - r) s z \widehat{H}_{sz} - \frac{(\mu - r) a \rho z}{k s^\beta} \widehat{H}_{\lambda z} = 0. \end{aligned} \quad (24)$$

Differentiating $g(t, s, \lambda, z) = -\widehat{H}_z$ on both sides with respect to t, s, λ , and z , we have the following first order and second order partial derivatives:

$$\begin{aligned} \widehat{H}_{tz} &= -g_t, \quad \widehat{H}_{sz} = -g_s, \quad \widehat{H}_{ssz} = -g_{ss}, \\ \widehat{H}_{zz} &= -g_z, \quad \widehat{H}_{s\lambda z} = -g_{s\lambda}, \quad \widehat{H}_{zzz} = -g_{zz}, \\ \widehat{H}_{ssz} &= -g_{sz}, \quad \widehat{H}_{\lambda zz} = -g_{\lambda z}, \quad \widehat{H}_{\lambda z} = -g_\lambda, \\ \widehat{H}_{\lambda\lambda z} &= -g_{\lambda\lambda}, \quad \frac{\widehat{H}_{\lambda z}^2}{\widehat{H}_{zz}} = -\frac{g_\lambda^2}{g_z}. \end{aligned} \quad (25)$$

Therefore, differentiating (24) for $\widehat{H}$ with respect to z and using the above derivatives, we derive

$$\begin{aligned} g_t + rsg_s - rg + \lambda + \left[\frac{(\mu - r)^2}{k^2 s^{2\beta}} - r \right] zg_z + bg_\lambda \\ + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} + \frac{1}{2} a^2 g_{\lambda\lambda} + apks^{\beta+1} g_{s\lambda} \\ - (\mu - r) zsg_{sz} - \frac{(\mu - r) a\rho}{ks^\beta} [g_\lambda + zg_{\lambda z}] \\ + \frac{1}{2} a^2 \frac{g_\lambda^2 g_{zz} - 2g_\lambda g_z g_{\lambda z}}{g_z^2} = 0. \end{aligned} \quad (26)$$

Thus, the optimal strategy (12) denoted by g is rewritten as

$$\pi^* = \frac{k^2 s^{2\beta+1} g_s - (\mu - r) zg_z + apks^\beta g_\lambda}{gk^2 s^{2\beta}}. \quad (27)$$

We solve (26) to obtain the dual variable g . Then, replacing g into (27) we obtain the optimal strategy.

5. An Approximation Solution for CARA Utility

Consider the CARA utility

$$U(v) = -\frac{1}{q} e^{-qv}, \quad q > 0, \quad (28)$$

where q is the coefficient of absolute risk aversions. We have

$$g(T, s, \lambda, z) = -\frac{1}{q} \ln z. \quad (29)$$

We try to find a solution for (26) in the following form:

$$\begin{aligned} g(t, s, \lambda, z) = -\frac{1}{q} [m(t) (\ln z + h(t, y, \lambda))] + n(t), \\ y = s^{-2\beta}. \end{aligned} \quad (30)$$

Then

$$\begin{aligned} g_t &= -\frac{1}{q} [m_t (\ln z + h(t, y, \lambda)) + m(t) h_t] + n_t, \\ g_s &= \frac{2\beta}{q} s^{-2\beta-1} m(t) h_y, \\ g_z &= -\frac{m(t)}{qz}, \quad g_\lambda = -\frac{m(t) h_\lambda}{q}, \\ g_{\lambda\lambda} &= -\frac{m(t) h_{\lambda\lambda}}{q}, \quad g_{zz} = \frac{m(t)}{qz^2}, \\ g_{ss} &= -\frac{m(t)}{q} [4\beta^2 s^{-4\beta-2} h_{yy} + 2\beta(2\beta+1) s^{-2\beta-2} h_y], \\ g_{sz} &= g_{\lambda z} = 0, \\ g_{s\lambda} &= \frac{2\beta}{q} s^{-2\beta-1} m(t) h_{y\lambda}. \end{aligned} \quad (31)$$

Substituting the above derivatives into (26), we obtain

$$\begin{aligned} [m_t - rm] \ln z + \left[rn - n_t - \frac{rm}{q} \right] q \\ + l_0 h + l_1 h + l_2 h \\ - \frac{1}{2} a^2 (1 - \rho^2) m h_\lambda^2 = 0, \end{aligned} \quad (32)$$

where

$$\begin{aligned} l_0 h &= -bq + bm h_\lambda + \frac{1}{2m} a^2 (1 - \rho^2) q^2 \\ &\quad + a^2 (1 - \rho^2) q h_\lambda, \\ l_1 h &= m_t h + m h_t - 2\beta \mu y m h_y - rm h \\ &\quad + k^2 m (2\beta^2 y h_{yy} + \beta(2\beta+1) h_y) \\ &\quad + 2\beta(\mu - r) y m h_y, \\ l_2 h &= -2apm\beta k \sqrt{y} h_{y\lambda} \\ &\quad + \frac{1}{k} a\rho(\mu - r) \sqrt{y} q \\ &\quad - \frac{1}{k} a\rho(\mu - r) m \sqrt{y} h. \end{aligned} \quad (33)$$

We decompose (32) into the three equations

$$m_t - rm = 0, \quad (34)$$

$$n_t - rn - \frac{rm}{q}, \quad (35)$$

$$(l_0 + l_1 + l_2) h - \frac{1}{2} a^2 (1 - \rho^2) m h_\lambda^2 = 0. \quad (36)$$

We state that m and n are found from (34) and (35). We know that (36) is a nonlinear partial differential equation, which has not a general solution.

In order to find solutions of (36), we let λ vary slowly and a small parameter ε ($0 < \varepsilon \ll 1$) satisfy

$$d\lambda_t^\varepsilon = \varepsilon b(\lambda) + \sqrt{\varepsilon} a(\lambda) dz(t). \quad (37)$$

Substituting (37) into (36), namely, replacing b and a with εb and $\sqrt{\varepsilon} a$, respectively in (36), we know that the corresponding h^ε in (36) is rewritten in the form

$$(\varepsilon l_0 + l_1 + \sqrt{\varepsilon} l_2) h^\varepsilon - \frac{1}{2} \varepsilon a^2 (1 - \rho^2) m(h_\lambda^\varepsilon)^2 = 0. \quad (38)$$

Theorem 2. *If a solution to (38) takes in the form*

$$\begin{aligned} h^\varepsilon(t, y, \lambda) &= h^{(0)}(t, y, \lambda) + \sqrt{\varepsilon} h^{(1)}(t, y, \lambda) \\ &\quad + \varepsilon h^{(2)}(t, y, \lambda) + \dots + \varepsilon^{n/2} h^{(n)}(t, y, \lambda), \end{aligned} \quad (39)$$

then the approximation solutions to (36) in the slow-fluctuating regime are $h^{(0)}(t, y, \lambda)$, $h^{(1)}(t, y, \lambda)$, and $h^{(2)}(t, y, \lambda)$. Moreover,

$$h^{(0)}(t, y, \lambda) = e^{-\int p_0 dt} \left[c_0 + \int \left(q_0 e^{-\int p_0 dt} \right) dt \right], \quad (40)$$

where $p_0 = (m_t - rm)/m$, $q_0 = k^2 \beta(2\beta + 1)e^{-((\mu-r)+\mu)/km\beta)y}$;

$$h^{(1)}(t, y, \lambda) = e^{-\int p_1 dt} \left[c_1 + \int \left(q_1 e^{-\int p_1 dt} \right) dt \right], \quad (41)$$

where $p_1 = (m_t - rm)/m$, $q_1 = k^2 \beta(2\beta + 1)e^{-((\mu-r)m+\mu)/km\beta)y} + (\mu - r)apq\sqrt{y}/km$;

$$h^{(2)}(t, y, \lambda) = e^{-\int p_2 dt} \left[c_2 + \int \left(q_2 e^{-\int p_2 dt} \right) dt \right], \quad (42)$$

where $p_2 = (m_t - rm)/m$, $q_2 = k^2 \beta(2\beta + 1)e^{((\mu-m(\mu-r))/mk\beta)y} + (\mu - r)apq\sqrt{y}/mk - bq/m - (1/2m^2)a^2(1 - \rho^2)q^2$.

Proof. Substituting (39) into (38), we derive

$$\begin{aligned} & l_1 h^{(0)} + \sqrt{\varepsilon} (l_1 h^{(1)} + l_2 h^{(0)}) \\ & + \varepsilon \left[l_0 h^{(0)} + l_1 h^{(2)} + l_2 h^{(1)} \right. \\ & \left. - \frac{1}{2} a^2 (1 - \rho^2) m (h_\lambda^{(0)})^2 \right] = 0. \end{aligned} \quad (43)$$

Collecting the same order of the terms, we obtain the following three equations.

Zero-order term:

$$l_1 h^{(0)} = 0, \quad \text{with } h^{(0)}(T, y, \lambda) = 0; \quad (44)$$

term of order $\sqrt{\varepsilon}$:

$$l_1 h^{(1)} + l_2 h^{(0)} = 0, \quad \text{with } h^{(1)}(T, y, \lambda) = 0; \quad (45)$$

term of order ε

$$\begin{aligned} l_0 h^{(0)} + l_1 h^{(2)} + l_2 h^{(1)} - \frac{1}{2} a^2 (1 - \rho^2) m (h_\lambda^{(0)})^2 &= 0, \\ \text{with } h^{(2)}(T, y, \lambda) &= 0. \end{aligned} \quad (46)$$

Considering the boundary condition, we will obtain solutions to (44)–(46).

(i) From (44), we have

$$\begin{aligned} & m_t h^{(0)} + m h_t^{(0)} - 2\beta \mu y h_y^{(0)} - r m h^{(0)} \\ & + m k^2 \left[2\beta^2 y h_{yy}^{(0)} + \beta (2\beta + 1) h_y^{(0)} \right] \\ & + 2\beta m y (\mu - r) h_y^{(0)} = 0. \end{aligned} \quad (47)$$

We decompose (47) into two equations

$$y \left[-2\beta \mu h_y^{(0)} + 2m k^2 \beta^2 h_{yy}^{(0)} + 2\beta m (\mu - r) h_y^{(0)} \right] = 0, \quad (48)$$

$$m_t h^{(0)} + m h_t^{(0)} - r m h^{(0)} + m k^2 \beta (2\beta + 1) h_y^{(0)} = 0. \quad (49)$$

From (48), we obtain $h_y^{(0)}$. It is

$$\begin{aligned} & \left[-2\beta \mu h_y^{(0)} + 2\beta m (\mu - r) h_y^{(0)} \right] = -2m k^2 \beta^2 h_{yy}^{(0)}, \\ & \frac{\mu h_y^{(0)} - m (\mu - r) h_y^{(0)}}{m k^2 \beta h_{yy}^{(0)}} = \frac{d h_y^{(0)}}{h_y^{(0)}} = d (\ln h_y^{(0)}), \end{aligned} \quad (50)$$

$$h_y^{(0)} = e^{((\mu-m(\mu-r))/mk^2\beta)y}.$$

Putting (50) into (49), we obtain

$$\begin{aligned} & m_t h^{(0)} + m h_t^{(0)} - r m h^{(0)} \\ & + m k^2 \beta (2\beta + 1) e^{((\mu-m(\mu-r))/mk^2\beta)y} = 0, \end{aligned} \quad (51)$$

$$h^{(0)}(t, y, \lambda) = e^{-\int p_0 dt} \left[c_0 + \int \left(q_0 e^{-\int p_0 dt} \right) dt \right],$$

where $p_0 = (m_t - rm)/m$, $q_0 = k^2 \beta(2\beta + 1)e^{-((\mu-r)+\mu)/km\beta)y}$.

(ii) From (45), we have

$$\begin{aligned} & m_t h^{(1)} + m h_t^{(1)} - 2\beta \mu y m h_y^{(0)} - r m h^{(1)} \\ & + m k^2 \left[2\beta^2 y h_{yy}^{(1)} + \beta (2\beta + 1) h_y^{(1)} \right] \\ & + 2\beta m y (\mu - r) h_y^{(1)} + \frac{1}{k} (\mu - r) a p q \sqrt{y} = 0. \end{aligned} \quad (52)$$

We decompose (52) into two equations

$$y \left[-2\beta \mu h_y^{(1)} + 2m k^2 \beta^2 h_{yy}^{(1)} + 2\beta m (\mu - r) h_y^{(1)} \right] = 0, \quad (53)$$

$$\begin{aligned} & m_t h^{(1)} + m h_t^{(1)} - r m h^{(1)} + m k^2 \beta (2\beta + 1) h_y^{(1)} \\ & + \frac{1}{k} (\mu - r) a p q \sqrt{y} = 0. \end{aligned} \quad (54)$$

From (53), we obtain $h_y^{(1)}$. It is

$$h_y^{(1)} = e^{((\mu-m(\mu-r))/mk\beta)y}. \quad (55)$$

Putting $h_y^{(1)}$ into (54), we obtain the solution

$$h^{(1)}(t, y, \lambda) = e^{-\int p_1 dt} \left[c_1 + \int \left(q_1 e^{-\int p_1 dt} \right) dt \right], \quad (56)$$

where $p_1 = (m_t - rm)/m$, $q_1 = k^2 \beta(2\beta + 1)e^{-((\mu-r)m+\mu)/km\beta)y} + (\mu - r)apq\sqrt{y}/km$.

(iii) From (46), we have

$$\begin{aligned} & y \left[2\beta \mu m h_y^{(2)} + 2m k^2 \beta^2 h_{yy}^{(2)} + 2\beta m (\mu - r) h_y^{(2)} \right] \\ & + m h_t^{(2)} + (m_t - r m) h^{(2)} + m k^2 \beta (2\beta + 1) h_y^{(2)} \\ & + \frac{1}{k} (\mu - r) a p q \sqrt{y} - \frac{1}{2m} a^2 (1 - \rho^2) q^2 = 0. \end{aligned} \quad (57)$$

We decompose (57) into the following two equations

$$y \left[2\beta \mu m h_y^{(2)} + 2m k^2 \beta^2 h_{yy}^{(2)} + 2\beta m (\mu - r) h_y^{(2)} \right] = 0, \quad (58)$$

$$\begin{aligned} & m h_t^{(2)} + (m_t - r m) h^{(2)} + m k^2 \beta (2\beta + 1) h_y^{(2)} \\ & + \frac{1}{k} (\mu - r) a p q \sqrt{y} - \frac{1}{2m} a^2 (1 - \rho^2) q^2 = 0. \end{aligned} \quad (59)$$

From (58), we obtain $h_y^{(2)}$. It is

$$h_y^{(2)} = e^{((\mu-m(\mu-r))/mk)y}. \quad (60)$$

Putting $h_y^{(2)}$ into (59), we get the solution

$$h^{(2)}(t, y, \lambda) = e^{-\int p_2 dt} \left[c_2 + \int (q_2 e^{-\int p_2 dt}) dt \right], \quad (61)$$

where $p_2 = (m_t - rm)/m$, $q_2 = k^2 \beta (2\beta + 1) e^{((\mu-m(\mu-r))/mk)y} + (\mu - r)apq\sqrt{y}/mk - bq/m - (1/2m^2)a^2(1 - \rho^2)q^2$. Therefore, (51), (56), and (61) are the solutions to (44)–(46) which are the solutions to (36). $\square$

Considering

$$\begin{aligned} g &= -\frac{\ln z}{q}, & zg_z &= -\frac{m}{q}, \\ g_s &= \frac{2}{q} m \beta s^{-2\beta-1} h_y, & g_\lambda &= -\frac{m}{q} h_\lambda, \\ m_t &= rm, & g &= v, \end{aligned} \quad (62)$$

from (27), (51), (56), and (61), we have

$$\begin{aligned} \pi_t^* &= \frac{m(\mu - r) + k^2 s^{2\beta+1} 2\beta s^{-2\beta-1} m h_y - k s^\beta \rho a m h_\lambda}{q v k^2 s^{2\beta}} \\ &= \frac{m}{q v k^2 s^{2\beta}} \{ (\mu - r) + 2\beta k^2 h_y - \rho k s^\beta h_\lambda \} \\ &= \frac{m}{q v k^2 s^{2\beta}} \{ (\mu - r) + 2\beta k^2 [h_y^{(0)} + \sqrt{\varepsilon} h_y^{(1)} + \varepsilon h_y^{(2)}] \\ &\quad - \rho k s^\beta [h_\lambda^{(0)} + \sqrt{\varepsilon} h_\lambda^{(1)} + \varepsilon h_\lambda^{(2)}] \\ &\quad + o(\varepsilon^{3/2}) \} \\ &= \frac{m}{q v k^2 s^{2\beta}} \{ (\mu - r) + 2\beta k^2 h_y^{(0)} - \rho k s^\beta h_\lambda^{(0)} \\ &\quad + \sqrt{\varepsilon} (2\beta k^2 h_y^{(1)} - \rho k s^\beta h_\lambda^{(1)}) \\ &\quad + \varepsilon (2\beta k^2 h_y^{(2)} - \rho k s^\beta h_\lambda^{(2)}) + o(\varepsilon^{3/2}) \}. \end{aligned} \quad (63)$$

When $\varepsilon \rightarrow 0$ and according to (50) and (51), we have

$$\pi_t^* = \frac{m}{q v k^2 s^{2\beta}} \{ (\mu - r) + 2\beta k^2 e^{((\mu-m(\mu-r))/mk)y} \}. \quad (64)$$

6. Conclusion

In the optimal investment strategies, we consider personal wealth with consumption invested in risk-free asset and risky asset. We apply the CEV model to the optimal personal portfolio and obtain the dual solution by using the Legendre transform and dual theory. Maximizing the expected utility and using the corresponding Hamilton-Jacobi-Bellman

(HJB) equation, we obtain an approximation solution and the optimal investment strategies for the exponential utility function. The results obtained in this paper are only applicable for the exponential utility function. For other cases, it would extend this result for further research in our forthcoming paper.

Conflict of Interests

The authors declare that they have no conflict of interests.

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