Research Article

On Certain Aspects of the Multidimensional S-Spectral Theory

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Received 10 August 2014; Accepted 1 December 2014; Published 28 December 2014

Academic Editor: Kishin Sadarangani

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This paper has two parts. The first one provides the preliminary notions introducing certain general concepts, in order to study, in the second part, the properties of some operator systems which admit spectral residual decompositions, S -decomposable, S -spectral, and A_S -scalar systems, and so forth. The results obtained by Frunză, 1975, are generalized, taking the results of Foias, 1963, as a model and adopting them.

1. Introduction

Across this paper we will try to generalize for operators systems some of the results obtained by Vasilescu in [1] for a single operator: residual single valued extension property, analytic residuum, the problem of local spectra, and so forth.

Most of the proofs are adaptations of the ones from [2, 3] with minor changes.

All operators with a reasonable spectral decomposition have S -decomposable restrictions and quotients. Subnormal, subscalar, and subdecomposable operators being restrictions and quotients of normal, scalar, and decomposable operators are thus S -decomposable (in fact S -normal, S -scalar, etc.). Eschmeier and Putinar [4] have shown that hyponormal operators are subscalar and therefore S -decomposable. Operators that admit scalar dilations (extensions) (C. Ionescu-Tulcea) or A -scalar (E. Stroescu) are S -decomposable. In fact, Eschmeier and Putinar [4] have shown that any operator is the quotient or restriction of a quotient of a decomposable operator and thus is S -decomposable or similar to an S -decomposable operator.

Colojară and Foiaş in [5, Chapter 6, Proposition 5b] formulated an open problem: any decomposable operator is strongly decomposable (meaning are that the restrictions and quotients in relation to the spectral maximal space decomposable?). A partial answer was given in [6]: the operators with the spectrum of dimension ≤ 1 thus situated on a curve are strongly decomposable, more specifically if

the spectrum is of dimension ≤ 1 and any of its subsets, in the relative topology, included, has a border ≤ 0 , sets of class C . There are subsets in the $\mathbb{C}$ plane which do not have this property. The result was strengthened by the example given by Albrecht and Vasilescu [7]. Some of the results obtained for one operator were generalized for systems of operators.

Let X be a Banach space and let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X .

Furthermore, let $\mathbf{S}(X)$ be the family of all closed linear subspaces of X , let $S \subset \mathbb{C}^n$ be a compact set, and let $\mathbf{F}_S$ be the family of all closed subsets $F \subset \mathbb{C}^n$ that have the property: either $F \cap S = \emptyset$ or $F \supset S$.

An S -spectral capacity is an application $\mathbf{E} : \mathbf{F}_S \rightarrow \mathbf{S}(X)$ verifying the following properties:

- (1) $\mathbf{E}(\emptyset) = \{0\}$, $\mathbf{E}(\mathbb{C}^n) = X$;
- (2) $\mathbf{E}(\bigcap_{i=1}^{\infty} F_i) = \bigcap_{i=1}^{\infty} \mathbf{E}(F_i)$, for any series $\{F_i\}_{i \in \mathbb{N}} \subset \mathbf{F}_S$;
- (3) for any open finite S -covering $\{G_S\} \cup \{G_j\}_{j=1}^m$ of $\mathbb{C}^n$, we have $X = \mathbf{E}(\overline{G_S}) + \sum_{j=1}^m \mathbf{E}(\overline{G_j})$.

A commuting system of operators $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ is said to be S -decomposable if there is an S -spectral capacity such that

- (4) $a_i \mathbf{E}(F) \subset \mathbf{E}(F)$, for any $F \in \mathbf{F}_S$ and any $i = 1, 2, \dots, n$;
- (5) $\sigma(a, \mathbf{E}(F)) \subset F$, for any $F \in \mathbf{F}_S$.

In case that $S = \emptyset$, the S -spectral capacity is said to be spectral capacity, and the system is decomposable [8].

2. Systems of Commuting Operators

This section provides the preliminary notions introducing certain general concepts necessary for the study of the S -decomposable systems in Section 3.

Definition 1. Let $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ be a commuting operators system and $S_a \subset \mathbb{C}^n$ a compact minimal set having the property that $H^{n-1}(\mathbb{C}^\infty(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open $G \subset \mathbb{C}^n$ with $G \cap S_a = \emptyset$ (minimal means that S_a is the intersection of all compact sets having the specified property). One will denote by $d(a, x)$ the union of all open sets $V \subset \mathbb{C}^n$ with the property that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V, X)]$ satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi$ meaning that

$$xs_1 \wedge s_2 \wedge \dots \wedge s_n = \left((z_1 - a_1)s_1 + \dots + (z_n - a_n)s_n + \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \dots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n \right) \wedge \psi(z) \quad (1)$$

(we recall that there exist sets V with this property, e.g., the solving set $r(a, X)$). We will also denote

$$\begin{aligned} g(a, x) &= \mathbb{C}^n \setminus d(a, x), \\ r(a, x) &= d(a, x) \cap (\mathbb{C}^n \setminus S_a), \\ sp(a, x) &= \mathbb{C}^n \setminus r(a, x) = g(a, x) \cup S_a. \end{aligned} \quad (2)$$

The set $r(a, x)$ will be said to be the *resolvent set* of x related to a , $sp(a, x)$ will be said to be the *spectrum* of x related to a , and S_a will be called the *residual spectrum* of a .

We will call *analytic resolvent set* of x related to a and we will denote by $\rho(a, x)$ the set $\rho(a, x) = \delta(a, x) \cap (\mathbb{C}^n \setminus S_a)$, where $\delta(a, x)$ is the set of $z \in \mathbb{C}^n$ for which there exists an open neighbourhood V of z and n analytic function on V taking values in X , $f_1, f_2, \dots, f_n$ satisfying the identity

$$x = (\zeta_1 - a_1)f_1(\zeta) + \dots + (\zeta_n - a_n)f_n(\zeta), \quad \zeta \in V. \quad (3)$$

We will understand through the *analytic spectrum* of x related to a the set

$$\sigma(a, x) = \mathbb{C}^n \setminus \rho(a, x) = \gamma(a, x) \cup S_a, \quad (4)$$

where

$$\gamma(a, x) = \mathbb{C}^n \setminus \delta(a, x). \quad (5)$$

We will prove that for an operators system that admits a spectral S -capacity we have

$$\begin{aligned} g(a, x) &= \gamma(a, x), \\ d(a, x) &= \delta(a, x), \\ \rho(a, x) &= r(a, x), \\ sp(a, x) &= \sigma(a, x). \end{aligned} \quad (6)$$

Proposition 2. For a commuting operators system $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ one has the following:

- (1) $x = 0$ implies $g(a, x) = \emptyset$, $sp(a, x) = S_a$;
- (2) $g(a, x+y) \subset g(a, x) \cup g(a, y)$, $sp(a, x+y) \subset sp(a, x) \cup sp(a, y)$ ($\forall x, y \in X$);
- (3) $g(a, by) \subset g(a, x)$, $sp(a, by) \subset sp(a, x)$ if $ba_i = a_i b$, $b \in B(X)$, $x \in X$;
- (4) $g(a, y) \subset sp(a, y) \subset \sigma(a, Y)$,

where Y is a (linear, closed) subspace of X invariant to all a_i and $y \in Y$.

Proof. (1°) follows from the fact that, for $x = 0$ and any neighbourhood $V \subset \mathbb{C}^n$, the form $\psi = 0 \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V, X)]$ verifies the relation $sx = (\alpha \oplus \bar{\partial})\psi$ meaning that

$$xs_1 \wedge s_2 \wedge \dots \wedge s_n = \left((z_1 - a_1)s_1 + \dots + (z_n - a_n)s_n + \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \dots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n \right) \wedge \psi(z). \quad (7)$$

Let $z \in d(a, x) \cap d(a, y)$ and $z \in V_i$ such that there exist the forms

$$\psi_i \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_i, X)], \quad (i = 1, 2) \quad (8)$$

verifying the equalities $sx = (\alpha \oplus \bar{\partial})\psi_1$, $sy = (\alpha \oplus \bar{\partial})\psi_2$. Then the form

$$\psi_1 + \psi_2 \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cup V_2, X)] \quad (9)$$

verifies the equality

$$s(x+y) = (\alpha \oplus \bar{\partial})(\psi_1 + \psi_2), \quad (10)$$

and hence (2°) is verified. The inclusions from (3°) result from the fact that, by considering the form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V, X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$ and by applying operator b to the coefficients of ψ , its commuting with each a_i ($i = 1, 2, \dots, n$) implies

$$(\alpha \oplus \bar{\partial})b\psi = b(\alpha \oplus \bar{\partial})\psi = bxs \quad (11)$$

(admitting the equality on components). The inclusion (4°) and the remark on the resolvent set $r(a, Y)$ lead to the equality $xs = (\alpha \oplus \bar{\partial})\psi$. $\square$

Lemma 3. Let V_1, V_2 be two open sets in $\mathbb{C}^n$ such that $V_1 \cap V_2 \neq \emptyset$.

Then for any $f \in \mathbb{C}^\infty(V_1 \cap V_2, X)$ there exist $f_i \in \mathbb{C}^\infty(V_i, X)$ ($i = 1, 2$) such that $f = f_1 - f_2$ on $V_1 \cap V_2$ ([8], 1.2.1.).

Lemma 4. Let $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ be an operators system with the residual spectrum S_a and V_i ($i = 1, 2$) two open

sets in $\mathbb{C}^n \setminus S_a$ such that there exist the forms $\psi_i \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_i, X)]$ with the property that $sx = (\alpha \oplus \bar{\partial})\psi_i$ on V_i . Then there exists a form

$$\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cup V_2, X)] \quad (12)$$

having the following property: $sx = (\alpha \oplus \bar{\partial})\psi$ on $V_1 \cup V_2$.

Proof. When $V_1 \cap V_2 = \emptyset$ we can consider $\psi(z) = \psi_i(z)$ for $z \in V_i$ ($i = 1, 2$) and we have $sx = (\alpha \oplus \bar{\partial})\psi$ on $V_1 \cup V_2$. If $V_1 \cap V_2 \neq \emptyset$ we have $(\alpha \oplus \bar{\partial})(\psi_2 - \psi_1) = 0$ on $V_1 \cap V_2$. Since $V_1 \cap V_2 = G \subset \mathbb{C}^n \setminus S_a$, it results that there exists a form $\varphi \in \Lambda^{n-2}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cap V_2, X)]$ such that

$$\psi_2 - \psi_1 = (\alpha \oplus \bar{\partial})\varphi. \quad (13)$$

Indeed, the nucleus of the cofrontier operator $\alpha \oplus \bar{\partial}$ is:

$$\begin{aligned} & \text{Ker}(\alpha \oplus \bar{\partial} : \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cap V_2, X)]) \\ & \longrightarrow \Lambda^n[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cap V_2, X)] \\ & = \text{Im}(\alpha \oplus \bar{\partial} : \Lambda^{n-2}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cap V_2, X)]) \\ & \longrightarrow \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_1 \cap V_2, X)]. \end{aligned} \quad (14)$$

By applying the preceding lemma to the coefficients of φ there follows $\varphi = \varphi_1 - \varphi_2$ where $\varphi_i \in \Lambda^{n-2}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(V_i, X)]$ ($i = 1, 2$).

Consequently,

$$(\alpha \oplus \bar{\partial})(\varphi_1 - \varphi_2) = (\alpha \oplus \bar{\partial})\varphi = \varphi_2 - \varphi_1, \quad (15)$$

where $\psi_1 + (\alpha \oplus \bar{\partial})\psi_1 = \psi_2 + (\alpha \oplus \bar{\partial})\psi_2$ on $V_1 \cap V_2$. By putting $\psi'_i = \psi_i + (\alpha \oplus \bar{\partial})\varphi_i$ ($i = 1, 2$) we will obtain $sx = (\alpha \oplus \bar{\partial})\psi'_i$ on V_i ($i = 1, 2$) and $\psi'_1 = \psi'_2$ on $V_1 \cap V_2$. Hence by defining $\psi(z) = \psi'_i(z)$ for $z \in V_i$ ($i = 1, 2$) one obtains a form as the one required in the text of the lemma. The lemma is proved. $\square$

Corollary 5. Let $\{V_i\}_{i=1}^m$ be a finite family of open sets from $\mathbb{C}^n \setminus S_a$ such that the equation $sx = (\alpha \oplus \bar{\partial})\psi$ has a solution ψ on each of them. If $K \subset r(a, x)$ is a compact set, there exists an open neighbourhood V of K ($V \subset r(a, x)$) on which the equation $sx = (\alpha \oplus \bar{\partial})\psi$ has a solution.

Proof. Let $\{K_\nu\}_{\nu=1}^\infty$ be a growing sequence of compact sets such that $r(a, x) = \bigcup_{\nu=1}^\infty K_\nu$. We will prove that there exists a corresponding sequence of forms $\psi_\nu \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(r(a, x), X)]$ that verify the equality $sx = (\alpha \oplus \bar{\partial})\psi_\nu$ on a neighbourhood of K_ν . Then $\psi = \lim_{\nu \rightarrow \infty} \psi_\nu$ exists and it is a global solution. We will start with K_1 (see [9, 10]).

By Corollary 5 there exists a form ψ_1 defined in an open neighbourhood of K_1 and satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi_1^*$ on this neighbourhood. Since the space $\mathbb{C}^\infty(r(a, x), X)$ is invariant to multiplication by scalar functions of a $\mathbb{C}^\infty$ class ([3], 2.16.1) we can assume, without limiting the generality, that ψ_1^* is defined on $r(a, x)$; indeed, by multiplying ψ_1^* by

a suitable scalar function, the new form can be extended to $r(a, x)$ and we will obtain a form ψ_1 on $r(a, x)$ verifying the equality $sx = (\alpha \oplus \bar{\partial})\psi_1$ on a neighbourhood of K_1 . We will now suppose that the forms $\psi_1, \psi_2, \dots, \psi_i$ from the desired sequences were already determined and let us determine ψ_{i+1} .

According to the preceding corollary there exists a neighbourhood V_{i+1} of the set K_{i+1} and a form ψ_{i+1} defined on this neighbourhood satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi_{i+1}^*$, and we are allowed to suppose moreover that ψ_{i+1}^* is defined on the whole $r(a, x)$. But $sx = (\alpha \oplus \bar{\partial})\psi_i$ on a vicinity V_i of K_i , and hence by subtraction we obtain $(\alpha \oplus \bar{\partial})(\psi_{i+1}^* - \psi_i) = 0$ on $V_i \cap V_{i+1}$; since $V_i \cap V_{i+1} \subset \mathbb{C}^n \setminus S_a$, it will result that there exists a form φ' such that $\psi_{i+1}^* - \psi_i = (\alpha \oplus \bar{\partial})\varphi'$ on $V_i \cap V_{i+1}$, and we may suppose that φ' is defined on $r(a, x)$. We will put $\psi_{i+1} = \psi_{i+1}^* - (\alpha \oplus \bar{\partial})\varphi'$ and obtain a form defined on $r(a, x)$ equal to ψ_i on $V_i \cap V_{i+1}$ and satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi_{i+1}$ on the neighbourhood V_{i+1} of K_{i+1} . By this the demonstration ends. $\square$

Remark 6. A local version of the Cauchy-Weil formula ([8, 1.2.4]) can be established on the same way as in [8, formula 1.5.1].

Let $a = (a_1, a_2, \dots, a_n) \in B(X)$ be a commuting operators system with the residual spectrum S_a and U an open neighbourhood of $\text{sp}(a, x)$; obviously $U \supset S_a$.

We will prove that there exists a form $\chi \in \Lambda^n[\sigma \cup d\bar{z}, \mathbb{C}_0^\infty(\mathbb{C}^n, X)]$ in the same cohomology class related to $\alpha \oplus \bar{\partial}$ as sx such that $\text{support}(\chi) \in U$.

It follows that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathbb{C}^\infty(r(a, x), X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$. Let U_1 and U_2 be two open neighbourhoods relatively compact of $\text{sp}(a, x)$, such that $\text{sp}(a, x) \subset U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U} \subset U$, and let us consider scalar $\mathbb{C}^\infty$ -function h on $\mathbb{C}^n$, $h = 1$ outside U_2 , and $h = 0$ on U_1 . By using h let us define the form $\tilde{\psi}$ by $\tilde{\psi} = h\psi$ on $r(a, x)$ and $\tilde{\psi} = 0$ on U_1 .

This form has the coefficients in $\mathbb{C}^\infty(\mathbb{C}^n, X)$ and satisfies the condition $sx = (\alpha \oplus \bar{\partial})\tilde{\psi}$ outside the relatively compact set U_2 . Hence by setting $\chi = sx - (\alpha \oplus \bar{\partial})\tilde{\psi}$ we will obtain a form defined on $\mathbb{C}^n$ with $\text{support}(\chi) \subset \bar{U}_2 \subset U$ that is precisely the form having the specified properties. Considering formula [8, 1.2.4.] and using form χ above we can write

$$x = \frac{1}{(2\pi i)^n} \int_U (-1)^n \pi_\chi \wedge dz_1 \wedge \dots \wedge dz_n \quad (16)$$

which will yield the local version of Cauchy-Weil formula.

3. Some Properties of S-Decomposable Systems

For the Banach space X and for an arbitrary open set $U \subset \mathbb{C}^n$, we denote by $\mathbf{U}(U, X)$ the space of all X -valued analytic functions on U .

Proposition 7. Let $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ be an S-decomposable system, let D be an open polydisk with $D \cap S = \emptyset$,

let p be an integer, $0 \leq p \leq n-1$, and let $\psi \in \Lambda^p[\sigma, U(D, X)]$ such that $\sigma\psi = 0$ where α is defined by

$$(\alpha\psi)(z) = ((z_1 - a_1)s_1 + (z_2 - a_2)s_2 + \cdots + (z_n - a_n)s_n) \wedge \psi(z). \quad (17)$$

Then for any polydisk $D' \subset D$ with $\overline{D'} \subset D$ there exists a form $\psi \in \Lambda^{p-1}[\sigma, U(D', X)]$ such that $\psi = \alpha\varphi$ on D' .

The proof of Proposition 2.1.3. presented in [8] is also true in this case, with a single comment that D is not anymore any polydisk of $\mathbb{C}^n$, but a polydisk that does not cross S .

Theorem 8. If $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ is S -decomposable, then $S \supset S_a$.

Proof. With minor differences, the proof is identical with the one for the decomposable systems ($S = \emptyset$) ([8, Proposition 2.1.4.]) where $S_a = \emptyset$ is called property (L). It will have to show that for any polydisk $U \subset \mathbb{C}^n$ such that $U \cap S = \emptyset$ we have $H^i(U(U, X), \alpha) = 0$ ($0 \leq i \leq n-1$).

We note that $H^i(U(U, X), \alpha) = 0$ implies $H^i(\mathbb{C}^\infty(G, X), \alpha \oplus \bar{\partial}) = 0$ ($0 \leq i \leq n-1$) where U is any open polydisk from $\mathbb{C}^n$ and G is any open set $G \subset \mathbb{C}^n$ such that $U \cap S = \emptyset$, $G \cap S = \emptyset$; the proof is given in [8, Theorem 1.5.16.] for any $U, G \subset \mathbb{C}^n$.

One motivates this through induction on i , beginning with $i = 0$. Let $f \in U(U, X)$ such that $\alpha f = 0$; according to the preceding proposition we will have $f = 0$ on any polydisk D' with $D' \subset U$ and $f = 0$ on U . Suppose that for any open polydisk $D \subset \mathbb{C}^n$ with $D \cap S = \emptyset$ we have $H^{i-1}(U(D, X), \alpha) = 0$ with i fixed, $0 \leq i \leq n-1$, and let us prove that $H^i(U(U, X), \alpha) = 0$.

Let $\{D_\nu\}$ be a sequence of polydisks, $D_\nu \cap S = \emptyset$, such that $D_\nu \subset \overline{D_{\nu+1}}$ for any ν with $\bigcup_{\nu=1}^\infty D_\nu = U$ and $\psi \in \Lambda^i[\sigma, U(U, X)]$ such that $\alpha\psi = 0$. By applying the preceding proposition for D_2 , we infer that there exists a form $\varphi_1 \in \Lambda^{i-1}[\sigma, U(D_2, X)]$ such that $\psi = \alpha\varphi_1$ on D_2 ; analogously we can find a form φ'_2 on D_3 with $\psi = \alpha\varphi'_2$ on D_3 . One obtains $\alpha(\varphi_1 - \varphi'_2) = 0$ on D_2 where, by applying the inductive hypothesis, we infer that there exists a form $\chi \in \Lambda^{i-2}[\sigma, U(D_2, X)]$, such that $\varphi_1 - \varphi'_2 = \alpha\chi$. We will retain from Taylor's decomposition of χ on D_2 a sufficient number of terms, such that χ' (the retained part) verifies $\|\alpha\chi - \alpha\chi'\| < 1/2$ on $\overline{D_1}$. Thinking analogously, we can define a sequence of forms $\varphi_\nu, \varphi'_\nu \in \Lambda^{i-1}[\sigma, U(D_{\nu+1}, X)]$, enjoying the properties

$$\begin{aligned} \psi &= \alpha\varphi_\nu \quad \text{on } D_{\nu+1}, \\ \|\varphi_{\nu+1} - \varphi'_\nu\| &< \frac{1}{2^{\nu+1}} \quad \text{on } D. \end{aligned} \quad (18)$$

The sequence φ_ν obviously converges to a form having the analytic coefficients on U and satisfying $\psi = \alpha\varphi$ on U . $\square$

The uniqueness of the spectral S -capacities for S -decomposable operators systems can be proved. We will

now prove this on other ways, emphasizing the connection between the spectral S -capacity related to an operator and certain linear subspaces, described using the local spectrum, which is most useful.

Let a be a commuting system of operators on the space X , $a \in B(X)$, with the residual spectrum S_a . If H is an arbitrary set from $\mathbb{C}^n$ such that $H \supset S_a$, we will put $X_{[a]}(H) = \{x, x \in X, \text{sp}(a, x) \subset H\}$ and $X_a(H) = \{x, x \in X, \sigma(a, x) \subset H\}$; $X_{[a]}(H)$ and $X_a(H)$ are linear subspaces of X and $X_a(H) \subset X_{[a]}(H)$.

Theorem 9. If $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ is an S -decomposable system and $\mathbf{E}$ is its S -spectral capacity, then

$$\mathbf{E}(F) = X_{[a]}(F), \quad (19)$$

for any closed set $F \subset \mathbb{C}^n$, $F \supset S$.

Proof. According to Theorem 8, $S \supset S_a$; hence $F \supset S_a$ and $X_{[a]}(F)$ makes sense. The inclusion $\mathbf{E}(F) \subset X_{[a]}(F)$ follows by the fact that $\text{sp}(a, x) \subset \sigma(a, \mathbf{E}(F))$ (Proposition 2 (4°)). In the same manner as for [8, Theorem 2.2.1], one proves the inverse inclusion, with the observation that F is not an arbitrary subset of $\mathbb{C}^n$, but $F \supset S$. $\square$

Corollary 10. Let a be an S -decomposable system. Then for any closed $F \supset S$, the subspace $X_{[a]}(F)$ is spectral maximal space of a ; more precisely, for any subspace Z invariant to a such that $\sigma(a, Z) \subset F$, one has $Z \subset X_{[a]}(F)$; moreover $\sigma(a, X_{[a]}(F)) \subset F$.

Proof. The inclusion $\sigma(a, X_{[a]}(F)) \subset F$ follows by the preceding theorem, since $\sigma(a, \mathbf{E}(F)) \subset F$.

If Z is invariant to a with $\sigma(a, Z) \subset F$ then any $z \in Z$ $\text{sp}(a, z) \subset \sigma(a, Z) \subset F$; hence $z \in X_{[a]}(F)$, meaning $Z \subset X_{[a]}(F)$. $\square$

Proposition 11. If a is S -decomposable then, for any $x \in X$, one has $\text{sp}(a, x) = \sigma(a, x)$.

Proof. Let us prove first that $\text{sp}(a, x) \subset \sigma(a, x)$ or its equivalent $\sigma(a, x) \subset r(a, x)$. Let $z \in \delta(a, x)$ and according to [8, Theorem 1.1.3.] let us consider an open neighbourhood V of z and n analytic functions defined on V taking values in X , $f_1, f_2, \dots, f_n$ that verify the equality

$$x = (\zeta_1 - a_1)f_1(\zeta) + \cdots + (\zeta_n - a_n)f_n(\zeta), \quad (20)$$

$\zeta \in V$. We consider the $n-1$ degree form defined on V ,

$$\psi(\zeta) = \sum_{i=1}^n (-1)^{i-1} f_i(\zeta) s_1 \wedge s_2 \wedge \cdots \wedge s_i \wedge \cdots \wedge s_n. \quad (21)$$

This form can be considered as an element of $\Lambda^{n-1}[\sigma \oplus \bar{\partial}, \mathbb{C}^\infty(V, X)]$ and it easily verifies the equality $sx = (\alpha \oplus \bar{\partial})\psi$ on V taking into account the analyticity of the functions f_i ($\delta f_i = 0$); hence it results that $V \subset d(a, x)$; that is, $\delta(a, x) \subset d(a, x)$ or $g(a, x) \subset \gamma(a, x)$ hence $g(a, x) \cup S_a = \text{sp}(a, x) \subset \gamma(a, x) \cup S_a = \sigma(a, x)$. For the inverse inclusion $\sigma(a, x) \subset \text{sp}(a, x)$, let $z \in r(a, x)$ and let D be an open polydisk with its centre in z such that $D \subset r(a, x)$.

Since $x \in X_{[a]}(\text{sp}(a, x) = \text{E}(\text{sp}(a, x)))$, hence by [8, Theorem 1.1.3.] there exist the analytic functions $f_1, f_2, \dots, f_n$ defined on D and taking values in X , such that

$$x = \sum_{i=1}^n (\zeta_i - a_i) f_i(\zeta), \quad \zeta \in D. \quad (22)$$

That means that $z \in \rho(a, x)$; hence $r(a, x) \subset \rho(a, x)$ hence $\sigma(a, x) \subset \text{sp}(a, x)$. $\square$

Corollary 12. *If a is an S -decomposable system, then for any $H \subset \mathbb{C}^n$ with $H \supset S$ one has $X_{[a]}(H) = X_a(H)$.*

Proof. It easily follows by the preceding proposition. $\square$

Proposition 13. *If a is an arbitrary system of operators, then*

$$\sigma(a, X) = \bigcup_{x \in X} \text{sp}(a, x). \quad (23)$$

Proof. The inclusion $\bigcup_{x \in X} \text{sp}(a, x) \subset \sigma(a, X)$ results from Proposition 2 (4°), $\text{sp}(a, x) \subset \sigma(a, X)$.

Conversely, if $z \in \bigcap_{x \in X} r(a, x)$, then $H^n(\mathbf{U}(z, X), \alpha) = 0$; since $z \in S_a$, there exists an open polydisk D , with $D \cap S_a = \emptyset$, for which, according to Theorem 8, we have $H^i(\mathbf{U}(D, X), \alpha) = 0$ ($i = 0, 1, \dots, n-1$).

Then by [8, Corollary 1.4.3.] it follows that $z \in r(a, x)$; hence $\bigcup_{x \in X} \text{sp}(a, x) \supset \sigma(a, X)$. $\square$

Definition 14. *The support of an S -spectral capacity $\mathbf{E}$ is the set*

$$\text{supp } \mathbf{E} = \cap \{F; F \text{ closed, } \mathbf{E}(F) = X\}. \quad (24)$$

Proposition 15. *If $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ is an S -decomposable system and $\mathbf{E}$ is its S -spectral capacity, then $\text{supp } \mathbf{E} = \sigma(a, X)$.*

Proof. The inclusion $\sigma(a, X) \subset \text{supp } \mathbf{E}$ results from the fact that, for any closed F such that $\mathbf{E}(F) = X$, we have $\sigma(a, X) = \sigma(a, \mathbf{E}(F)) \subset F$, where $\sigma(a, X) \subset \cap \{F; F \text{ closed, } \mathbf{E}(F) = X\} = \text{supp } \mathbf{E}$. For the inverse inclusion, let $z_0 \in r(a, X)$ and let us prove that $z_0 \notin \text{supp } \mathbf{E}$. Let V be an open neighborhood of z_0 such that $\bar{V} \subset r(a, X)$ and let F be a closed set under the conditions $z_0 \notin F$, $F \supset S$ and $X = \mathbf{E}(F) + \mathbf{E}(\bar{V})$; these conditions are possible because $z_0 \in S$ ($S \subset \sigma(a, X)$).

Let $x \in \mathbf{E}(\bar{V})$; since $V \subset r(a, X)$ it results that $sx = (\alpha \oplus \bar{\partial})\psi$ in a neighborhood $(\mathbb{C}^n \setminus V)$ of the spectrum $\text{sp}(a, X)$; hence by applying [8, formula 1.2.4], we find $x = 0$. Therefore $\mathbf{E}(\bar{V}) = \{0\}$; that is, $\mathbf{E}(F) = X$; hence from $z_0 \notin F$ it follows that $z_0 \notin \text{supp } \mathbf{E}$. $\square$

Corollary 16. *If $a = (a_1, a_2, \dots, a_n) \in \mathbf{B}(X)$ is an S -decomposable system, then one has $\sigma(a, \mathbf{E}(F)) \subset \sigma(a, X)$, for any closed set $F \subset \mathbb{C}^n$.*

Proof. We have

$$\begin{aligned} \mathbf{E}(F \cap \sigma(a, X)) &= \mathbf{E}(F) \cap \mathbf{E}(\sigma(a, X)) \\ &= \mathbf{E}(F) \cap X = \mathbf{E}(F), \end{aligned} \quad (25)$$

hence

$$\begin{aligned} \sigma(a, \mathbf{E}(F)) &= \sigma(a, \mathbf{E}(F \cap \sigma(a, X))) \subset F \cap \sigma(a, X) \\ &\subset \sigma(a, X). \end{aligned} \quad (26)$$

$\square$

4. Conclusions

The originality consists in the adaptation of the results obtained for the spectral capacities by Frunză to the decomposable systems of operators to the S -spectral capacities which are appropriate to the S -decomposable systems of operators.

These theorems remain valid also for a special class of S -decomposable systems, namely, S -spectral systems. An important result was obtained in the particular case when the topological dimension of S is zero and the S -decomposable (S -spectral) systems become decomposable (spectral systems).

We will underline the relevance, importance, and necessity of studying the S -decomposable operators showing the consistence of this class, in the sense of how many and how substantial its subfamilies are. Relations with this same subject can also be found in the “regularities” of Müller [11]. The S -decomposable operators appear firstly as restrictions and quotients of decomposable operators, particularly restrictions and quotients of spectral operators, and therefore A_S -scalar or S -spectral operators (generalized S -spectral, generalized S -scalar, etc.).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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