Research Article

Terminal-Dependent Statistical Inference for the FBSDEs Models

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The original stochastic differential equations (OSDEs) and forward-backward stochastic differential equations (FBSDEs) are often used to model complex dynamic process that arise in financial, ecological, and many other areas. The main difference between OSDEs and FBSDEs is that the latter is designed to depend on a terminal condition, which is a key factor in some financial and ecological circumstances. It is interesting but challenging to estimate FBSDE parameters from noisy data and the terminal condition. However, to the best of our knowledge, the terminal-dependent statistical inference for such a model has not been explored in the existing literature. We proposed a nonparametric terminal control variables estimation method to address this problem. The reason why we use the terminal control variables is that the newly proposed inference procedures inherit the terminal-dependent characteristic. Through this new proposed method, the estimators of the functional coefficients of the FBSDEs model are obtained. The asymptotic properties of the estimators are also discussed. Simulation studies show that the proposed method gives satisfying estimates for the FBSDE parameters from noisy data and the terminal condition. A simulation is performed to test the feasibility of our method.

1. Introduction

Since 1973, when the world's first options exchange opened in Chicago, a large number of new financial products have been introduced to meet the customer's demands from the derivative markets. In the same year, Black and Scholes [1] provided their celebrated formula for option pricing and Merton [2] proposed a general equilibrium model for security prices. Since then, modern finance has adopted stochastic differential equations as its basic instruments for portfolio management, asset pricing, risk management, and so on. Among these models, the backward stochastic differential equations (BSDEs for short) are a desirable choice for hedging and pricing an option. Its general form is as follows:

$$\begin{aligned} dY_s &= -g(s, Y_s, Z_s) ds + Z_s dB_s, \\ Y_T &= \xi, \quad s \in [t, T], \end{aligned} \quad (1)$$

where g is the generator, B_t is a Brownian motion, and ξ is a $\mathcal{R}$ -valued Borel function as the terminal condition. Usually

the terminal condition is designed as a random variable with given distribution. If g meets certain conditions, the BSDE has a unique solution.

In terms of the backward equation, within a complete market, it serves to characterize the dynamic value of replicating portfolio Y_s with a final wealth ξ and a special quantity Z_s that depends on the hedging portfolio. In particular, while the generator consists of diffusion process, the corresponding equation is proved to be a forward-backward stochastic differential equation (FBSDE), which can be expressed as

$$dY_s = -g(s, X_s, Y_s, Z_s) ds + Z_s dB_s, \quad Y_T = \xi, \quad (2)$$

where X_s satisfies the following ordinary stochastic differential equation (OSDE):

$$dX_s = \mu(s, X_s) dt + \sigma(s, X_s) dB_s, \quad s \in [t, T]. \quad (3)$$

Compared to the OSDE that contains an initial condition, the solution of the FBSDE is affected by the terminal condition $Y_T = \xi(X_T)$. As is well known, there exist a number

of parametric and nonparametric methods to deal with estimation and test for the OSDE. However, these methods cannot be directly employed to infer the BSDE and FBSDE because the two models are related to a terminal condition. Forward-backward stochastic differential equations are used in biology systems, mathematical finance, insurance, real estate, multiagent, and network control. See Antonelli [3], Wang et al. [4], Zhang and Li [5], and so on.

For the FBSDE defined above, the statistical inference was investigated initially by Su and Lin [6] and Chen and Lin [7]. Furthermore, by financial and ecological problems, a relevant statistical model was proposed by Lin et al. [8]. However, they did not take the terminal condition into account in the inference procedure. In the framework of the FBSDE mentioned above, the terminal condition is additional, which is not nested into the equation. Thus, there is an essential difficulty to use the terminal condition to refine the inference procedure.

As a result, their methods fail to cover the full problems given in the FBSDE. Zhang and Lin [9] proposed two terminal-dependent estimation methods via terminal control variable for the integral form models of FBSDE. However, they only considered the parametric form of the generator g in their paper.

This paper intends to explore the method to fulfill the terminal-dependent inference: quasi-instrumental variable methods. It is worth mentioning that the key point of our method is the use of the terminal condition information rather than neglecting it. This change leads to a completely new work among the existing researches. The key technique in our method is the use of quasi-instrumental variable which is similar but not the same as instrumental variable (IV). It is known that IV is widely employed in applied econometrics to achieve identification and carry out estimation and inference in the model containing endogenous explanatory variables or panel data; see Hsiao [10] for an overview of the relevant statistical inference and econometric interpretation and see Hall and Horowitz [11] for recent work on nonparametric instrumental variable estimation.

Through the backward equation (2) of FBSDE, we get a regression model. To use the terminal condition information, we put the terminal condition as a quasi-instrumental variable and introduce it into our model. However, when a constraint is appended artificially, the original model may change to be biased in the sense of $E(Z_s dB_s \mid X_s, \xi) \neq 0$, because the constraint condition influences the increase trend of wealth so that $Z_s dB_s$ may deviate from the original center zero; in other words, due to the constraint, the trajectory of Y_s may departure from the original expectation so that $Z_s dB_s$ cannot be regarded as error. Therefore, some problems arise naturally, including how to correct the bias of the model and how to construct the constraint-dependent estimation. To solve these problems, we will use remodeling method to draw terminal condition into differential equation, similar but not the same as IV, called quasi-instrumental variable methods; in other words, the terminal condition ξ enters into the equation as a control variable. This remodeling method takes advantage of the terminal information naturally, and the estimator performs quite well.

We use the nonparametric form of the generator g in model (2) because the correct FBSDEs model for a specific topic can neither be provided automatically by financial market nor be derived from theory of mathematical finance, and in lack of prior information about the structure of a model, nonparametric inference can provide a flexible as well as robust description of a data-generating process. Even in some cases when parametric models are available, nonparametric methods are still employed to avoid the model misspecification that may lead to large errors in option pricing and other problems from financial market. So we adopt the nonparametric form that can endow the model (2) with flexibility and robustness.

Note that Z_s is usually unobservable and g cannot be completely specified in the financial market. The problems of interest are therefore to give both proper estimations of the generator g and the process Z_s based on the observed data (X_s, Y_s) and the terminal expectation ξ .

The remainder of the paper is organized as follows. In Section 2, the FBSDE is rebuilt as a nonparametric model that contains the terminal condition as a quasi-instrumental variable. Consequently, a terminal-dependent estimation procedure is proposed. Next we discuss the asymptotic properties of the newly proposed estimations in Section 3. Simulation study is proposed in Section 4 to illustrate our methods. The proofs of the theorems are presented in Appendix.

2. Model and Method

In this section, we propose a nonparametric estimator with the help of quasi-instrumental variable.

2.1. Model and Its Statistical Version. We begin the following original model by combining (2)-(3):

$$\begin{aligned} dY_s &= -g(s, X_s, Y_s, Z_s) ds + Z_s dB_s, \quad Y_T = \xi, \\ dX_s &= \mu(s, X_s) dt + \sigma(s, X_s) dB_s, \quad s \in [t, T], \end{aligned} \quad (4)$$

where B_t is the standard Brownian motion and ξ is a $\mathcal{R}$ -valued Borel function. Here the generator g is a function of s, X_s, Y_s , and Z_s . For the FBSDEs model (4), only one of the backward components, Y_s , and the forward components, X_s , can be observed. Another backward component Z_s is totally unobservable. Furthermore, the adapted process Z_s and terminal condition could be indicated as a function of X_s .

In this section, we present the statistical structure of FBSDEs by taking advantage of quasi-instrumental variable and obtain the consistent asymptotically normal estimators of g and Z_s based on observed data $\{X_s, Y_s\}$ and the terminal condition ξ .

2.2. Remodeling for Model (4). To construct terminal-dependent estimation for the generator g and process Z_s , the key technique is how to plug the terminal condition into the equation. When ξ is plugged into the model, we call it the quasi-IV, similar but not the same as IV. Evidently, the property of Brownian motion shows that $E(Z_s dB_s \mid X_s) = 0$,

but $E(Z_s dB_s | X_s, \xi) \neq 0$, which means drawing the terminal control directly into the equation as the condition should not be encouraged at the cost of model bias. Rewriting the first equation of (4) enables us to construct an unbiased model:

$$dY_s = -g(s, X_s, Y_s, Z_s) ds + m(X_s, \xi) + U_s, \quad (5)$$

where $m(X_s, \xi) = E(Z_s dB_s | X_s, \xi)$, $U_s = Z_s dB_s - m(X_s, \xi)$, and $E(U_s | X_s, \xi) = 0$. The newly defined model (5), together with the second equation in (4), can be thought of as a quasi-IV FBSDE. Because the equation in (5) contains the terminal condition ξ , we can construct the terminal-dependent estimation. From the above definitions, we see that, by bias correction, the original model changes to be an additive nonparametric model with nonparametric components $-g(s, X_s, Y_s, Z_s)ds$ and $m(X_s, \xi)$. It shows that when terminal condition is regarded as a quasi-IV and then appended to the model, the result model is unbiased and changes to be nonparametric additive model.

2.3. Estimation for Z_s . Before estimating the model function $m(x_s, \xi)$ and the generator g , we need to estimate Z_s firstly because Z_s is unobservable and it will be seen that the estimators of the model function $m(x_s, \xi)$ and the generator g depend on Z_s . Since the distribution of ξ is supposed to be known, let $\{\xi_i, 1 \leq i \leq k\}$ for $k \geq 1/\Delta$ be a sample of ξ . Suppose that, for each terminal data ξ_j and equally spaced time points $\{s_i = s_1 + (i-1)\Delta, i = 1, \dots, n\} \subseteq [0, T]$, we record the observed time series data:

$$\begin{aligned} & \{X_{s_i, j}, Y_{s_i, j}, i = 1, \dots, n, j = 1, \dots, k\} \\ & = \{X_{i, j}, Y_{i, j}, i = 1, \dots, n, j = 1, \dots, k\}. \end{aligned} \quad (6)$$

At any time point $s \in [t, T]$, $Z_s^{t, x}$, denoting Z_s and satisfying the initial condition (t, x) , is a determined function of $X_s^{t, x}$. As was shown by Su and Lin [6] and Chen and Lin [7], we can adopt a difference-based method to approximate Z^2 as

$$(Z_s^{t, X_t})^2 = \frac{1}{\Delta} E(Y_{s+\Delta}^{t+\Delta, X_t+\Delta} - Y_s^{t, X_t} | X_t, t)^2 + O(\Delta). \quad (7)$$

It shows that the numerical approximation error to Z_t^2 converges to zero at rate of order $O_p(\Delta)$.

For each ξ_j , if Z_t depends on t only via variable X_t , by (7) and N-W kernel estimation method, we estimate Z_t^2 at x_0 by

$$\widehat{Z}_{x_0, j}^2 = \frac{\sum_{i=1}^{n-1} \Delta^{-1} (Y_{i+1, j} - Y_{i, j})^2 K_{h_X}(X_{i, j} - x_0)}{\sum_{i=1}^{n-1} K_{h_X}(X_{i, j} - x_0)}. \quad (8)$$

Otherwise, we estimate Z_t^2 at (x_0, t_0) by

$$\begin{aligned} & \widehat{Z}_{x_0, t_0, j}^2 \\ & = \frac{\sum_{i=1}^{n-1} \Delta^{-1} (Y_{i+1, j} - Y_{i, j})^2 K_{h_X}(X_{i, j} - x_0) K_{h_t}(t_i - t_0)}{\sum_{i=1}^{n-1} K_{h_X}(X_{i, j} - x_0) K_{h_t}(t_i - t_0)}, \end{aligned} \quad (9)$$

where $K_{h_X} = K(\cdot/h_X)/h_X$ and $K_{h_t} = K(\cdot/h_t)/h_t$, $K(\cdot)$ are regular kernel functions, with h_X and h_t being the corresponding bandwidths.

2.4. Estimation for $m(X_s, \xi)$. After plugging the estimator $\widehat{Z}_s$ into model (5), we still need to consider inference of $m(x_s, \xi)$. As we all know, the nonparametric function $m(X_s, \xi)$ in (5) can be acquired as $m(X_s, \xi) = E(dY_s + g(s, X_s, Y_s, Z_s)ds | X_s, \xi)$. We note that $g(s, X_s, Y_s, Z_s)ds$ is a higher order infinitesimal of $Z_s dB_s$ when Δ tends to zero. Under this situation, if $g(s, X_s, Y_s, Z_s)ds$ is ignored, then

$$m(X_s, \xi) \doteq E(dY_s | X_s, \xi). \quad (10)$$

It implies that we can use ordinary nonparametric method to estimate function m . For example, we use the N-W ordinary nonparametric method to estimate $m(X_s, \xi)$ valued at (x_0, ξ_0) :

$$\begin{aligned} & \widehat{m}(x_0, \xi_0) \\ & = \frac{\sum_{i=1}^{n-1} \sum_{j=1}^m (Y_{i+1, j} - Y_{i, j}) K_{h_X}(X_{i, j} - x_0) K_{h_\xi}(\xi_{i, j} - \xi_0)}{\sum_{i=1}^{n-1} \sum_{j=1}^m K_{h_X}(X_{i, j} - x_0) K_{h_\xi}(\xi_{i, j} - \xi_0)}, \end{aligned} \quad (11)$$

where $K_{h_X} = K(\cdot/h_X)/h_X$ and $K_{h_\xi} = K(\cdot/h_\xi)/h_\xi$ are regular kernel functions, with h_X and h_ξ being the corresponding bandwidths.

2.5. Estimation for Generator g . As was shown in the nonparametric instrumental variables estimator of Hall and Horowitz [11] (hereinafter HH), we can adopt a nonparametric quasi-instrumental variables estimation to estimate the generator g . So in the section we summarize the HH estimator of g in the model:

$$E[-dY_t - m(X_t, \xi) | X_t, \xi] = E[g(t, X_t, Y_t, Z_t) dt | X_t, \xi]. \quad (12)$$

Since $\widehat{m}(x_0, \xi_0)$ and $\widehat{Z}_{x_0, j}^2$ are the consistent estimator of $m(x_0, \xi_0)$ and $Z_{x_0, j}^2$, respectively, we use them instead of $m(X_s, \xi)$ and Z_s in the above model and we get

$$\begin{aligned} & E[-dY_s - \widehat{m}(X_s, \xi) | X_s, \xi] \\ & = E[g(s, X_s, Y_s, \widehat{Z}_s) dt | X_s, \xi]. \end{aligned} \quad (13)$$

Because $\widehat{Z}_s$ is function of X_s and Y_s , for simplicity of presentation, we denote $g(s, X_s, Y_s, \widehat{Z}_s) = g(X_s, Y_s)$. Thus, the model becomes

$$E[-dY_s - \widehat{m}(X_s, \xi) | X_s, \xi] = E[g(X_s, Y_s) dt | X_s, \xi]. \quad (14)$$

Let $\mathbb{Y}_i = ((Y_{i+\Delta} - Y_i) - \widehat{m}(X_i, \xi))/\Delta$, $\mathbb{X}_i = X_i$, $\mathbb{Z}_i = Y_i$, $\mathbb{W} = \xi$, and $\mathbb{U}_i = V_i/\sqrt{\Delta}$; the model becomes

$$\mathbb{Y}_i = g(\mathbb{X}_i, \mathbb{Z}_i) + \mathbb{U}_i, \quad E(\mathbb{U}_i | \mathbb{X}_i, \mathbb{W}_i) = 0. \quad (15)$$

It is assumed that the support of $(\mathbb{X}, \mathbb{Z}, \mathbb{W})$ is contained in $[0, 1]^3$. This assumption can always be satisfied by, if necessary, carrying out monotone increasing transformations of $\mathbb{X}$, $\mathbb{Z}$, and $\mathbb{W}$. For example, one can replace $\mathbb{X}$, $\mathbb{Z}$, and $\mathbb{W}$ by

$\Phi(\mathbb{X})$, $\Phi(\mathbb{Z})$, and $\Phi(\mathbb{W})$, where Φ is the normal distribution function. We take $(\mathbb{Y}, \mathbb{X}, \mathbb{Z}, \mathbb{W}, \mathbb{U})$ to be a vector, where $\mathbb{Y}$ and $\mathbb{U}$ are scalars, $\mathbb{X}$ and $\mathbb{W}$ are supported on $[0, 1]$, and $\mathbb{Z}$ is supported on $[0, 1]$. The model is

$$\mathbb{Y}_i = g(\mathbb{X}_i, \mathbb{Z}_i) + \mathbb{U}_i, \quad E(\mathbb{U}_i | \mathbb{Z}_i, \mathbb{W}_i) = 0, \quad (16)$$

where $(\mathbb{Y}_i, \mathbb{X}_i, \mathbb{Z}_i, \mathbb{W}_i, \mathbb{U}_i)$, for $i \geq 1$, are independent and identically distributed as $(\mathbb{Y}, \mathbb{X}, \mathbb{Z}, \mathbb{W}, \mathbb{U})$. Thus, $\mathbb{X}$ and $\mathbb{Z}$ are endogenous and exogenous explanatory variables, respectively. Data $(\mathbb{Y}_i, \mathbb{X}_i, \mathbb{Z}_i, \mathbb{W}_i, \mathbb{U}_i)$, for $1 \leq i \leq n$, are observed.

Let $f_{\mathbb{X}\mathbb{Z}\mathbb{W}}$ denote the density of $(\mathbb{X}, \mathbb{Z}, \mathbb{W})$, write $f_{\mathbb{Z}}$ for the density of $\mathbb{Z}$, and, for each $x_1, x_2 \in [0, 1]^P$, and put

$$t_z(x_1, x_2) = \int f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x_1, z, w) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x_2, z, w) dw. \quad (17)$$

Define the operator T_z on $L_2[0, 1]^P$ by

$$(T_z \psi)(x) = \int t_z(\xi, x) \psi(\xi) d\xi. \quad (18)$$

It may be proved that, for each z for which T_z^{-1} exists,

$$\begin{aligned} g(x, z) &= f_{\mathbb{Z}}(z) E_{\mathbb{W}|\mathbb{Z}} \\ &\times \left\{ E(\mathbb{Y} | \mathbb{Z} = z, \mathbb{W}) (T_z^{-1} f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, \mathbb{W}) | \mathbb{Z} = z \right\}, \end{aligned} \quad (19)$$

where $E_{\mathbb{W}|\mathbb{Z}}$ denotes the expectation with respect to the distribution of $\mathbb{W}$ conditional on $\mathbb{Z}$. In this formulation, $(T_z^{-1} f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, \mathbb{W})$ denoted the result of applying T_z^{-1} to the function $f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\cdot, z, \mathbb{W})$ and evaluating the resulting function at x .

To construct an estimator of $g(x, z)$, given $h > 0$ and $x = x^{(1)}$ and $\xi = \xi^{(1)}$, let $K_h(x, \xi) = K_h(x^{(j)}, \xi^{(j)})$, put $K_h(z, \xi)$ analogously for z and ξ , let $h_x, h_z > 0$, and define

$$\begin{aligned} \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) &= \frac{1}{nh_x^2 h_z} \sum_{i=1}^n K_{h_x}(x - \mathbb{X}_i, x) K_{h_z}(z - \mathbb{Z}_i, z) K_{h_x}(w - \mathbb{W}_i, w), \\ \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{-i}(x, z, w) &= \frac{1}{(n-1)h_x^2 h_z} \sum_{1 \leq j \leq n: j \neq i} K_{h_x}(x - \mathbb{X}_j, x) \\ &\quad \times K_{h_z}(z - \mathbb{Z}_j, z) K_{h_x}(w - \mathbb{W}_j, w), \\ \hat{t}_z(x_1, x_2) &= \int \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x_1, z, w) \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{-i}(x_2, z, w) dw, \\ (\hat{T}_z \psi)(x, z, w) &= \int \hat{t}_z(\xi, x) \psi(\xi, z, w) d\xi, \end{aligned} \quad (20)$$

where ψ is a function from R^3 to a real line. Then the estimator of $g(x, z)$ is

$$\hat{g}(x, z) = \frac{1}{n} \sum_{i=1}^n (\hat{T}_z^{-1} \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{-i})(x, z, \mathbb{W}_i) Y_i K_{h_z}(z - \mathbb{Z}_i, z). \quad (21)$$

3. Asymptotic Results

In this section, we study the asymptotic properties of our proposed estimators. All proofs are presented in Appendix.

3.1. Asymptotic results of $\hat{Z}_s$. To give the asymptotic results of $\hat{Z}_s$, we need the following conditions.

- (a) $X_1, \dots, X_n$ are ρ -mixing dependent; namely, the ρ -mixing coefficients $\rho(l) \rightarrow 0$ as $l \rightarrow \infty$, where

$$\rho(l) = \sup_{E(X_{i+l}X_i) - E(X_{i+l})E(X_i) \neq 0} \frac{|E(X_{i+l}X_i) - E(X_{i+l})E(X_i)|}{\sqrt{\text{Var}(X_{i+l})\text{Var}(X_i)}} \quad (22)$$

with $X_i = X(t_i)$.

- (b) $|Z_i| \leq C$ (a. s.) uniformly for $i = 1, \dots, n$, where C is a positive constant and $Z_i = Z(t_i)$.

- (c) The continuous kernel function $K(\cdot)$ is symmetric about 0, with a support of interval $[-1, 1]$, and

$$\begin{aligned} \int_{-1}^1 K(u) du &= 1, \quad \sigma_K^2 = \int_{-1}^1 u^2 K(u) du \neq 0, \\ \int_{-1}^1 |u|^j K^k(u) du &< \infty \quad \text{for } j \leq k = 1, 2. \end{aligned} \quad (23)$$

Condition (a) is commonly used for weakly dependent process; see, for example, Kolmogorov and Rozanov [12], Bradley and Bryc [13], Lu and Lin [14], and Su and Lin [6]. Condition (b) is also reasonable because, as is shown by (10), Z_t can be regarded as the deviation between the adjacent two observations. Condition (c) is standard for nonparametric kernel function.

Theorem 1. Besides conditions (a), (b), and (c), let $\{X_1, \dots, X_n\}$ be an observation sequence on a stationary ρ -mixing Markov process with the ρ -mixing coefficients satisfying $\rho(l) = \rho^l$ for $0 < \rho < 1$. Furthermore, $X_1, \dots, X_n$ have a common and probability density $p(x)$, and for each interior point x_0 in the support of $p(\cdot)$, $p(x_0) > 0$, $Z^2(x_0) > 0$, the functions $p(x)$ and $Z(x)$ have continuous two derivatives in neighborhood of x_0 . As $n \rightarrow \infty$, such that $nh \rightarrow \infty$, $nh^5 \rightarrow 0$, and $nh\Delta^2 \rightarrow 0$, then

$$\sqrt{nh}(\hat{Z}^2(x_0) - Z^2(x_0)) \xrightarrow{d} \left(0, \frac{Z^4(x_0) J_K}{p(x_0)}\right), \quad (24)$$

where $J_K = \int_{-1}^1 K^2(u) du < \infty$.

The asymptotic result in Theorem 1 is standard for nonparametric kernel estimator and here undersmoothing is used to eliminate asymptotic bias.

3.2. Asymptotic results of $\hat{g}(x, z)$. This section gives conditions under which the HH estimator of the generator

g is asymptotically distributed as $N(0, I)$. The following additional notations are used.

Define $\mathbb{U}_i = \mathbb{Y}_i - g(\mathbb{X}_i, \mathbb{Z}_i)$, $S_{n1}(x, z) = n^{-1} \sum_{i=1}^n \mathbb{U}_i \hat{T}^+ \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)}(x, z, \mathbb{W}_i) K_{q, h_z}(z - \mathbb{Z}_i, z)$, and $S_{n2}(x, z) = n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, \mathbb{Z}_i) \hat{T}^+ \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)}(x, z, \mathbb{W}_i) K_{q, h_z}(z - \mathbb{Z}_i, z)$. Then, $\hat{g}(x, z) = S_{n1}(x, z) + S_{n2}(x, z)$. Define $T^+ = (T + a_n I)^{-1}$. Write

$$\begin{aligned} S_{n1}(x, z) &= n^{-1} \sum_{i=1}^n \mathbb{U}_i (T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, \mathbb{W}_i) K_{q, h_z}(z - \mathbb{Z}_i, z) \\ &\quad + n^{-1} \sum_{i=1}^n \mathbb{U}_i (\hat{T}^+ \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)} - T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}}) \\ &\quad \times (x, z, \mathbb{W}_i) K_{q, h_z}(z - \mathbb{Z}_i, z) \\ &= S_{n11}(x, z) + S_{n12}(x, z). \end{aligned} \quad (25)$$

Define $V_n(x, z) = n^{-1} \text{Var}[\mathbb{U}(T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, \mathbb{W})]$. It follows from a triangular array version of the Lindeberg-Levy central limit theorem that $S_{n11}(x, z)/\sqrt{V_n(x, z)} \rightarrow^d N(0, 1)$ as $n \rightarrow \infty$. Therefore, $[\hat{g}(x, z) - g(x, z)]/\sqrt{V_n(x, z)} \rightarrow^d N(0, 1)$ if $[S_{n12}(x, z) + S_{n2}(x, z) - g(x, z)]/\sqrt{V_n(x, z)} = o_p(1)$.

Assumption 2. The data $\mathbb{Y}_i, \mathbb{X}_i, \mathbb{Z}_i, \mathbb{W}_i$ are independently and identically distributed as $(\mathbb{Y}, \mathbb{X}, \mathbb{Z}, \mathbb{W})$, where $(\mathbb{X}, \mathbb{Z}, \mathbb{W})$ is supported on $[0, 1]^3$ and $E[\mathbb{Y} - g(\mathbb{X}, \mathbb{Z}) | \mathbb{W}, \mathbb{Z}] = 0$.

Assumption 3. The distribution of $(\mathbb{X}, \mathbb{Z}, \mathbb{W})$ has a density $f_{\mathbb{X}\mathbb{Z}\mathbb{W}}$ with respect to Lebesgue measure. Moreover, $f_{\mathbb{X}\mathbb{Z}\mathbb{W}}$ is r times differentiable with respect to any combination of its arguments, where derivatives at the boundary of $[0, 1]^3$ are defined as one sided derivatives. The derivatives are bounded in absolute value by C . In addition, g is r times differentiable on $[0, 1]^2$ with derivatives at 0 and 1 defined as one sided. The derivatives of g are bounded in absolute value by C . In addition, $E[\mathbb{Y}^2 | \mathbb{X}, \mathbb{Z}, \mathbb{W}] \leq C$ and $E[\mathbb{Y}^2 | \mathbb{X}, \mathbb{Z}, \mathbb{W}] \leq C$, and $E[\mathbb{U}^2 | \mathbb{Z}, \mathbb{W}] \geq C_U$ for some finite constant C_U .

Assumption 4. The constants α and β satisfy $\alpha > 1$, $\beta > 1/2$, and $\beta - 1/2 \leq \alpha < 2\beta$. Moreover, $b_j \leq Cj^{-\beta}$, $j^{-\alpha} \leq C\lambda_j$, and $\sum_{k=1}^{\infty} |d_{zjk}| \leq Cj^{-\alpha/2}$ for all $j \geq 1$. In addition, there are finite strictly positive constants, $C_{\lambda 1}$ and $C_{\lambda 2}$, such that $C_{\lambda 1} \leq \lambda_j \leq C_{\lambda 2} j^{-\alpha}$ for all $j \geq 1$.

Assumption 5. The tuning parameters a_n and h satisfy $a_n \asymp n^{-(\rho\alpha)/(2\beta+\alpha)}$ and $h \asymp n^{-1}$, where $r \in [A'_2, A'_3]$.

Assumption 6. K_h denotes a generalized kernel function, with the properties $K_h(u, t) = 0$ if $u > t$ or $u < t - 1$, for all $t \in [0, 1]h^{-(j+1)} \int_{t-1}^t tu^j K_h(u, t) du = 1$ if $j = 0$, else 0 if $1 \leq j \leq r - 1$. For each $\xi \in [0, 1]$, $K_h(h, \xi)$ is supported on $[(\xi - 1)/h, \xi/h] \cap \kappa$, where κ is a compact interval not depending on ξ . Moreover,

$$\sup_{h>0, \xi \in [0, 1], u \in \kappa} K_h(hu, \xi) < \infty. \quad (26)$$

Assumption 7. Consider $E_{\mathbb{W}}[T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, \mathbb{W})]^2 \asymp E_{\mathbb{W}}[T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\cdot, \cdot, \mathbb{W})]^2$ and $E_{\mathbb{W}}[T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\cdot, \cdot, \mathbb{W})]^2 \asymp \int_0^1 \|T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\cdot, \cdot, \mathbb{W})\|^2 dw$.

Theorem 8. Let Assumptions 2–7 hold. Then

$$\frac{\hat{g}(x, z) - g(x, z)}{\sqrt{V_n(x, z)}} \rightarrow^d N(0, I) \quad (27)$$

holds except, possibly, on a set of (x, z) values whose Lebesgue is 0.

Corollary 9. Let Assumptions 2–7 hold. And if $V_n(x, z)$ is replaced with the consistent estimator,

$$\hat{V}_n(x, z) = n^{-1} \sum_{i=1}^n \hat{\mathbb{U}}_i^2 [\hat{T}^+ \hat{f}_{xw}^{(-i)}(z, \mathbb{W}_i) K_{q, h_z}(z - \mathbb{Z}_i, z)]^2, \quad (28)$$

where $\hat{\mathbb{U}}_i = \mathbb{Y}_i - \hat{g}(\mathbb{X}_i, \mathbb{Z}_i)$. This yields the Studentized statistic $[\hat{g}(x, z) - g(x, z)]/\sqrt{\hat{V}_n(x, z)}$. Then

$$\frac{\hat{g}(x, z) - g(x, z)}{\sqrt{\hat{V}_n(x, z)}} \rightarrow^d N(0, I) \quad (29)$$

holds except, possibly, on a set of (x, z) values whose Lebesgue is 0.

As was shown in the remark given in the previous section, even the conditional mean of error of the model is nonzero, and the newly proposed estimation is consistent because of the mixing dependency; for details see the proof of Theorem 8. Furthermore, because of the terminal condition, the asymptotic variance is larger than that without the use of the terminal condition.

4. Simulation Studies

In this section, we investigate the finite-sample behaviors by simulation.

Example 10. We consider a simple FBSDE as

$$\begin{aligned} dY_t &= \left(\frac{\mu - r}{\sigma} Z_t + rY_t \right) dt + Z_t dB_t \\ &\triangleq (bY_t + cZ_t) + Z_t dB_t; \quad Y_T = \xi, \end{aligned} \quad (30)$$

where X_t is Geometric Brownian motion for modeling stock price satisfying

$$dX_t = \mu X_t dt + \sigma X_t dB_t; \quad X_0 = x, \quad (31)$$

while the riskless asset is the same as formula (31); $dP_t = rP_0 dt$.

Firstly, let $\mu = 0.1$, $\sigma = 0.01$, $\Delta = 0.12$, $n = 300$, $T = 36.6$, and $n_0 = n_1 = 10$. Obviously, $Z_t = n_1 \sigma X_t$. We adopt Epanechnikov kernel defined by $K(u) = 3/4(1 - u^2)I(|u| \leq 1)$,

FIGURE 1: The real lines are the true curves of Z_t and function $g(t)$, respectively, and the dashed ones are estimated curves for them in Example 10.

where $I(\cdot)$ is the indicator function. For bandwidth selection, various data-driven techniques have been developed, such as cross-validation, the plug-in method, and the empirical bias method. However, these useful tools require additional computation intensiveness. In our simulation, we simply apply the rule of thumb bandwidth selector. For bandwidth selection, bandwidth $h = \text{std}(x)n^{-1/5}$. The values of the tuning parameters are $a_n = 0.05$, $\alpha = 1.2$, $\beta = 1$. Figure 1 presents the estimated curves for diffusion Z_t and drift g by one simulation.

Example 11. According to the theory of mathematical finance, we represent a European call option by the following FBSDEs model:

$$\begin{aligned} dX_s &= bX_s ds + \sigma X_s dW_s, \\ dY_s &= [rY_s + (b-r)\sigma^{-1}Z_s] ds + Z_s dW_s, \end{aligned} \quad (32)$$

$$X_0 = x, \quad Y_T = (X_T - K)^+, \quad s \in [0, T].$$

Here $\{X_s\}_{0 \leq s \leq T}$ and $\{Y_s\}_{0 \leq s \leq T}$ are the price processes of the stock and the option, respectively, and K is the striking price at the expiration date T . $\{X_s\}_{0 \leq s \leq T}$ follows the geometric Brownian motion as

$$\begin{aligned} dX_s &= bX_s ds + \sigma X_s dW_s, \\ X_0 &= x, \quad s \in [0, T]. \end{aligned} \quad (33)$$

We use the Euler scheme to generate the price series of the stock as

$$X_{i+1} - X_i = bX_i \Delta + \sigma X_i \Delta^{1/2} \epsilon_i, \quad i = 0, \dots, n-1, \quad (34)$$

where $\{\epsilon_i\}_{i=0}^{n-1}$ is an i.i.d. series with standard normality.

The price series by Black Scholes formula is part of the solution of the FBSDEs above at discrete time points; that is,

$$Y_i = X_i N(d_+^i) - e^{-r(n-i)\Delta} K N(d_-^i), \quad (35)$$

which, together with

$$Z_i = \sigma X_i N(d_+^i), \quad (36)$$

gives us data generating formulae, where

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx \quad (37)$$

is a cumulative normal function, and

$$d_{\pm}^i = \frac{\ln(X_i/K) + (r \pm \sigma^2/2)((n-i)\Delta)}{\sigma \sqrt{(n-i)\Delta}}. \quad (38)$$

We produce the true curve of the drift coefficient by

$$g_i = -rY_i - (b-r)\sigma^{-1}Z_i. \quad (39)$$

We first apply formulas (21) and (11) to estimate g_i and Z_i^2 , respectively. We adopt Epanechnikov kernel defined by $K(u) = 3/4(1-u^2)I(|u| \leq 1)$, where $I(\cdot)$ is the indicator function. For bandwidth selection, we simply apply the rule of thumb bandwidth selector:

$$h = \text{constant} \times \text{std}(Y_0, \dots, Y_{n-1}) n^{-1/5} \quad (40)$$

to implement the estimation.

Let $K = 110$, $X_0 = 100$, $b = 0.1$, $\sigma = 0.18$, $r = 0.08$, $T = 60$, and $\Delta = 1/100$. The bandwidth parameters $h = 6.06$ and $h = 0.67$ are used for estimation of g_s and Z_s , respectively. The values of the tuning parameters are $a_n = 0.05$, $\alpha = 1.2$, and $\beta = 1$. To see the performance of our estimation method, the simulated and the estimated curves of the two coefficients of the backward equation are displayed in Figures 2 and 3.

FIGURE 2: The simulated curve and the estimated curves of g_s in Example 11.

FIGURE 3: The simulated curve and the estimated curves of Z_s in Example 11.

Appendix

A. Proofs

Proof of Theorem 1. Denote $\mathcal{E} = \{X_1, \dots, X_n, \dots\}$. By the Taylor expansion and formula (8), we have

$$\begin{aligned}
 E(\hat{Z}^2(x_0) | \mathcal{E}) &= \frac{\sum_{i=1}^{n-1} \Delta^{-1} K_h(X_i - x_0) E((Y_{i+1} - Y_i)^2 | \mathcal{E})}{\sum_{i=1}^{n-1} K_h(X_i - x_0)} \\
 &= \frac{\sum_{i=1}^{n-1} K_h(X_i - x_0) (Z_i^2 + O(\Delta))}{\sum_{i=1}^{n-1} K_h(X_i - x_0)} \\
 &= \frac{\int K_h(X_i - x_0) (Z^2(x) + O(\Delta)) p(x) dx (1 + O_p(nh)^{-1/2})}{\int K_h(X_i - x_0) p(x) dx (1 + O_p(nh)^{-1/2})}
 \end{aligned}$$

$$\begin{aligned}
 &= \left((Z^2(x_0) + O(\Delta)) \right. \\
 &\quad \times (p(x_0) + (1/2) h^2 p^{(2)}(x_0) \sigma_K^2 + o(h^2)) \\
 &\quad \times (1 + O_p(nh)^{-1/2}) \Big) \\
 &\quad \times \left((p(x_0) + (1/2) h^2 p^{(2)}(x_0) \sigma_K^2 + o(h^2)) \right. \\
 &\quad \times (1 + O_p(nh)^{-1/2}) \Big)^{-1} \\
 &= Z^2(x_0) + \frac{p^{(2)}(x_0)}{2p(x_0)} h^2 Z^2(x_0) \sigma_K^2 + o(h^2) + O(\Delta).
 \end{aligned} \tag{A.1}$$

Furthermore,

$$\begin{aligned}
 \text{Var}(\hat{Z}^2(x_0) | \mathcal{E}) &= \frac{1}{\sum_{i=1}^{n-1} K_h^2(X_i - x_0)} \\
 &\quad \times \left\{ \sum_{i=1}^{n-1} \Delta^{-2} K_h^2(X_i - x_0) \text{Var}((Y_{i+1} - Y_i)^2 | \mathcal{E}) \right. \\
 &\quad \left. + \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \Delta^{-2} \text{cov}(K_h(X_i - x_0)(Y_{i+1} - Y_i), \right. \\
 &\quad \left. K_h(X_{i+k} - x_0)(Y_{i+k+1} - Y_{i+k}) | \mathcal{E}) \right\}.
 \end{aligned} \tag{A.2}$$

From the conditions of Markov process and ρ -mixing coefficient,

$$\begin{aligned}
 &\left| \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \Delta^{-2} \text{cov} \left(K_h(X_i - x_0)(Y_{i+1} - Y_i), \right. \right. \\
 &\quad \left. \left. K_h(X_{i+k} - x_0)(Y_{i+k+1} - Y_{i+k}) \right) \right| \\
 &= \frac{1}{(n-1)^2} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \left| E \left((\Delta)^{-2} (Y_{i+1} - Y_i)^2 (Y_{i+k+1} - Y_{i+k})^2 \right. \right. \\
 &\quad \times (K_h(X_i - x_0) - E(K_h(X_i - x_0))) \\
 &\quad \times (K_h(X_{i+k} - x_0) - E(K_h(X_{i+k} - x_0))) \Big) \Big| \\
 &= \frac{1}{(n-1)^2} \left| E \left(Z_i^2 Z_{i+i}^2 (K_h(X_i - x_0) - E(K_h(X_i - x_0))) \right. \right. \\
 &\quad \times (K_h(X_{i+k} - x_0) - E(K_h(X_{i+k} - x_0))) \Big) \Big| \\
 &\quad + O(\Delta) \\
 &\leq \frac{C}{(n-1)^2 h} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \rho^k = O\left(\frac{1}{nh}\right) = o(1).
 \end{aligned} \tag{A.3}$$

Note that $(Y_{i+1} - Y_i)/\sqrt{\Delta} = g(t_i, Y_i, Z_i)\sqrt{\Delta} + Z_i\eta_i$, where $E(\eta_i) = 0$, $\text{Var}(\eta_i) = 1$. Thus $\text{Var}((Y_{i+1} - Y_i)/\sqrt{\Delta}) = Z_i^4 + O(\Delta)$ and furthermore

$$\begin{aligned} \text{Var}(\hat{Z}^2(x_0) | \mathcal{C}) &= \frac{\sum_{i=1}^{n-1} \Delta^{-2} K_h^2(X_i - x_0) \text{Var}((Y_{i+1} - Y_i)^2 | \mathcal{C})}{\sum_{i=1}^{n-1} K_h^2(X_i - x_0)} + O_p(1) \\ &= \frac{\sum_{i=1}^{n-1} K_h^2(X_i - x_0) (Z_i^4(x_0) + O(\sqrt{\Delta}))}{\sum_{i=1}^{n-1} K_h^2(X_i - x_0)} + O_p(1) \\ &= \frac{Z^4(x_0) J_K + O(\sqrt{\Delta})}{nhp(x_0)} (1 + O_p(nh)^{-1/2}). \end{aligned} \quad (\text{A.4})$$

To our interest, both the conditional expectation and variance are independent on $\mathcal{C}$, so the condition could be erased.

From Lemma A.1 of Politis and Romano [15] and the relation between the α -mixing condition and the ρ -mixing condition (e.g., Theorem 1.1.1 of Lu and Lin [14]), we can ensure that $\{(Y_{i+1} - Y_i)^2, i = 1, \dots, n-1\}$ is a ρ -mixing dependent process and the mixing coefficient, denoted by $\rho_Y(l)$, satisfies

$$\sum_{k=1}^{\infty} \rho_Y(2^k) \leq C \sum_{k=1}^{\infty} \rho(2^k) = \sum_{k=1}^{\infty} \rho^{2^k} < \infty, \quad (\text{A.5})$$

where C is a positive constant. Finally, we use the central limit theorems for ρ -mixing dependent process (e.g., Theorem 4.0.1 of Lu and Lin [14]) to complete this proof. $\square$

Proof of Theorem 8. Theorem 8 follows from proving that $S_{n1}(x, z)/\sqrt{V_n(x, z)} \rightarrow^d N(0, I_2)$ and $[S_{n2}(x, z) - g(x, z)]/\sqrt{V_n(x, z)} = o_p(1)$ except, possibly, if (x, z) belongs to a set of Lebesgue measure 0. The first result is established in Lemma A.1, and the second is established in Lemma A.2. Throughout this Appendix, “for almost every (x, z) ” means “for every $(x, z) \in [0, 1]^2$ except, possibly, a set of Lebesgue measure 0.” We make repeated use of the fact that if $E\|\psi\|^2 = O(n^{-s})$ for some $s > 0$, then $\psi(x, z) = o_p(n^{-s})$ for almost every (x, z) . $\square$

Lemma A.1 (asymptotic normality of $S_{n1}(x, z)/\sqrt{V_n(x, z)}$). *Let Assumptions 2–7 hold. Then $S_{n1}(x, z)/\sqrt{V_n(x, z)} \rightarrow^d N(0, I_2)$ for almost every (x, z) .*

Proof. Define $S_{n11}(x, z) = n^{-1} \sum_{i=1}^n \mathbb{U}_i(T^{++} f_{\mathbb{XZW}})(x, z, \mathbb{W}_i)$,

$$\begin{aligned} A_{n2}(x, z) &= n^{-1} \sum_{i=1}^n \mathbb{U}_i \left[T^{++} \left(\hat{f}_{\mathbb{XZW}}^{(-i)} - f_{\mathbb{XZW}} \right) \right] (x, z, \mathbb{W}_i), \end{aligned}$$

$$\begin{aligned} A_{n3}(x, z) &= n^{-1} \sum_{i=1}^n \mathbb{U}_i \left[(\hat{T}^{++} - T^{++}) f_{\mathbb{XZW}} \right] (x, z, \mathbb{W}_i), \end{aligned}$$

$$\begin{aligned} A_{n4}(x, z) &= n^{-1} \sum_{i=1}^n \mathbb{U}_i \left[(\hat{T}^{++} - T^{++}) \left(\hat{f}_{\mathbb{XZW}}^{(-i)} - f_{\mathbb{XZW}} \right) \right] (x, z, \mathbb{W}_i). \end{aligned} \quad (\text{A.6})$$

Then $S_{n1}(x, z) = S_{n11}(x, z) + A_{n2}(x, z) + A_{n3}(x, z) + A_{n4}(x, z)$. $S_{n11}(x, z)/\sqrt{V_n(x, z)} \rightarrow^d N(0, I_2)$ by a triangular array version of the Lindeberg-Levy central limit theorem. The proof of the triangular-array version of the theorem is identical to the proof of the ordinary Lindeberg-Levy theorem. The lemma follows if we can prove that $A_{nj}(x, z)/\sqrt{V_n(x, z)} = o_p(1)$ for $j = 2, 3, 4$ and almost every $(x, z) \in [0, 1]^2$.

Assumption 7 and arguments like those leading to (6.2) of HH [11] show that

$$\int_0^1 \int_0^1 V_n(x, z) dx dz \asymp n^{-[2\beta+\alpha-\rho(\alpha+1)]/(2\beta+\alpha)}. \quad (\text{A.7})$$

It follows from the Cauchy-Schwartz inequality, $E(\hat{f}_{\mathbb{XZW}}^{(-i)} - f_{\mathbb{XZW}}) = O(h')$, and $\text{Var}(\hat{f}_{\mathbb{XZW}}^{(-i)}) = O[1/(nh^2)]$ that

$$E\|A_{n2}\|^2 = O\left(\frac{1}{n^2 h^2 a_n^2} + \frac{h^{2r}}{n a_n^2}\right). \quad (\text{A.8})$$

Therefore, it follows from Assumptions 5 and 7 that $A_{n2}(x, z)/\sqrt{V_n(x, z)} = o_p(1)$ for almost every (x, z) . Now consider $A_{n3}(x, z)$. Define the operator $\Delta = \hat{T} - T$. Then

$$A_{n3}(x, z) = -(\hat{T} + a_n I) \Delta A_{n1}(x, z). \quad (\text{A.9})$$

Therefore, the Cauchy-Schwartz inequality gives

$$\begin{aligned} E\|A_{n2}\|^2 &\leq E\|(\hat{T} + a_n I) \Delta\|^2 E\|S_{n11}\|^2 \\ &= E\|(\hat{T} + a_n I) \Delta\|^2 \int_0^1 \int_0^1 V_n(x, z) dx dz. \end{aligned} \quad (\text{A.10})$$

HH show that

$$E\|(\hat{T} + a_n I) \Delta\|^2 = O\left(\frac{1}{n h a_n^2} + \frac{h^{2r}}{a_n^2}\right). \quad (\text{A.11})$$

Therefore, it follows from Assumptions 5 and 7 that $A_{n3}(x, z)/\sqrt{V_n(x, z)} = o_p(1)$ for almost every (x, z) . Finally, some algebra shows that

$$A_{n4}(x, z) = -(\hat{T} + a_n I)^{-1} \Delta A_{n2}(x, z). \quad (\text{A.12})$$

Therefore, $A_{n4}(x, z)/\sqrt{V_n(x, z)} = o_p(1)$ for almost every (x, z) follows from (A.11) and $A_{n2}(x, z)/\sqrt{V_n(x, z)} = o_p(1)$. $\square$

Lemma A.2 (asymptotic negligibility of $S_{n2}(x, z) - g(x, z)$). *Let Assumptions 2–7 hold. Then $S_{n2}(x, z) - g(x, z)/\sqrt{V_n(x, z)} = o_p(1)$ for almost every (x, z) .*

Proof. Define

$$\begin{aligned} D_n(x, z) &= \iiint_0^1 g(\theta, \eta) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\theta, \eta, w) T^+ \\ &\quad \times (\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}} - f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, w) d\theta d\eta dw, \\ A_{n1}(x, z) &= n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, Z_i) (T^+ f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, \mathbb{W}_i). \end{aligned} \quad (\text{A.13})$$

Redefine

$$\begin{aligned} A_{n2}(x, z) &= n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, Z_i) [T^+ (\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)} - f_{\mathbb{X}\mathbb{Z}\mathbb{W}})](x, z, \mathbb{W}_i) \\ &\quad - D_n(x, z), \\ A_{n3}(x, z) &= n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, Z_i) [(\hat{T}^+ - T^+) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}](x, z, \mathbb{W}_i) + D_n(x, z), \\ A_{n4}(x, z) &= n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, Z_i) [(\hat{T}^+ - T^+) (\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)} - f_{\mathbb{X}\mathbb{Z}\mathbb{W}})] \\ &\quad \times (x, z, \mathbb{W}_i). \end{aligned} \quad (\text{A.14})$$

Then $S_{n2}(x, z) = \sum_{j=1}^4 A_{nj}(x, z)$. Arguments identical to those used to derive (6.2) and (6.3) of HH [11] show that $\|EA_{n1} - g\|^2 = O[n^{-\rho(2\beta_1)/(2\beta+\alpha)}]$ and

$$\iint_0^1 \text{Var}[A_{n1}(x, z)] dx dz = O n^{-[2\beta+\alpha-\rho(\alpha+1)]/(2\beta+\alpha)}. \quad (\text{A.15})$$

Therefore, it follows from Assumptions 5 and 7 that

$$\frac{[EA_{n1}(x, z) - g(x, z)]}{\sqrt{V_n(x, z)}} = o_p(1), \quad (\text{A.16})$$

$$V_n^{-1}(x, z) \iint_0^1 \text{Var}[A_{n1}(x, z)] dx dz = O(1) \quad (\text{A.17})$$

for almost every (x, z) .

Now consider $A_{n2}(x, z)$. Define

$$\begin{aligned} D_{ni}(x, z) &= \iiint_0^1 g(\theta, \eta) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(\theta, \eta, w) T^+ \\ &\quad \times (\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)} - f_{\mathbb{X}\mathbb{Z}\mathbb{W}})(x, z, w) d\theta d\eta dw, \\ A_{n21}(x, z) &= n^{-1} \sum_{i=1}^n g(\mathbb{X}_i, Z_i) [T^+ (\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}^{(-i)} - f_{\mathbb{X}\mathbb{Z}\mathbb{W}})] \\ &\quad \times (x, z, \mathbb{W}_i) - D_{ni}(x, z), \end{aligned} \quad (\text{A.18})$$

and $A_{n22}(x, z) = n^{-1} \sum_{i=1}^n [D_{ni}(x, z) - D_n(x, z)]$. HH show that $\|EA_{n21}\|^2 = O((h^{2r}/na_n^2) + (1/n^2 h^2 a_n^2))$ and $\|EA_{n22}\|^2 = O(1/n^2 a_n^2)$. Therefore, it follows from Assumptions 5 and 7 that

$$\frac{A_{n2}(x, z)}{\sqrt{V_n(x, z)}} = o_p(1) \quad (\text{A.19})$$

for almost every (x, z) . Now consider $A_{n3}(x, z)$. Write

$$A_{n3}(x, z) = A_{n31}(x, z) + A_{n32}(x, z), \quad (\text{A.20})$$

where $A_{n31}(x, z) = -(I + T^+ \Delta)^{-1} T^+ \Delta g(x, z) + D_n(x, z)$ and $A_{n32}(x, z) = -(\hat{T}^+ + a_n I)^{-1} \Delta (A_{n1} - g)(x, z)$. It follows from (A.11)-(A.16) and (A.20) that

$$\frac{A_{n32}(x, z)}{\sqrt{V_n(x, z)}} = o_p(1) \quad (\text{A.21})$$

for almost every (x, z) .

To analyze $A_{n31}(x, z)$, define

$$\begin{aligned} B_{n1}(x, z) &= \iiint_0^1 [\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw, \\ B_{n2}(x, z) &= \iiint_0^1 [\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, x, w) g(x, x) dx dz dw, \\ B_{n3}(x, z) &= \iiint_0^1 [\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w), \\ &\quad \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times g(x, z) dx dz dw, \\ B_{n11}(x, z) &= \iiint_0^1 [E \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw, \\ B_{n12}(x, z) &= \iiint_0^1 [\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - E \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw, \\ B_{n21}(x, z) &= \iiint_0^1 [E \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw, \\ B_{n22}(x, z) &= \iiint_0^1 [\hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) - E \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w)] \\ &\quad \times f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw. \end{aligned} \quad (\text{A.22})$$

Define $\delta = h^{2r} + (nh)^{-1}$. HH show that

$$\begin{aligned} A_{n31}(x, z) = & -(I + T^+ \Delta)^{-1} T^+ (B_{n11} + B_{n12} + B_{n3})(x, z) \\ & + (I + T^+ \Delta)^{-1} T^+ \Delta T^+ (B_{n21} + B_{n22})(x, z). \end{aligned} \quad (\text{A.23})$$

Define

$$\begin{aligned} \tilde{A}_{n31}(x, z) = & -(I + T^+ \Delta)^{-1} T^+ (B_{n11} + B_{n12} + B_{n3})(x, z) \\ & + (I + T^+ \Delta)^{-1} T^+ \Delta T^+ B_{n21}. \end{aligned} \quad (\text{A.24})$$

Then

$$\begin{aligned} E\|A_{n31}\|^2 & \leq \text{const.} \left[E\|\tilde{A}_{n31}\|^2 + E\|(I + T\Delta)^{-1} T^+ \Delta T^+ B_{n22}\|^2 \right], \end{aligned} \quad (\text{A.25})$$

$$\begin{aligned} E\|\tilde{A}_{n31}\|^2 & \leq \text{const.} \left(\|T^+ B_{n11}\|^4 + E\|T^+ B_{n12}\|^4 \right. \\ & \quad \left. + E\|T^+ \Delta T^+ B_{n21}\|^4 + E\|T^+ B_{n3}\|^4 \right)^{1/2}. \end{aligned} \quad (\text{A.26})$$

HH show that

$$\|T^+ B_{n11}\| = O\left(\frac{h^r}{a_n}\right), \quad (\text{A.27})$$

$$\left(E\|T^+ \Delta T^+ B_{n21}\|^4 \right)^{1/2} = O\left(\frac{\delta h^{2r}}{a_n}\right), \quad (\text{A.28})$$

$$\left(E\|T^+ B_{n3}\|^4 \right)^{1/2} = O\left(\frac{\delta^2}{a_n^2}\right). \quad (\text{A.29})$$

See (6.11), (6.13), (6.14), and (6.15) of HH [11]. Moreover,

$$\begin{aligned} E\|(I + T\Delta)^{-1} T^+ \Delta T^+ B_{n22}\|^2 & = O\left(\frac{h^{2r-1}}{na_n^{2+(\alpha+1)/\alpha}} + \frac{1}{n^3 h^5 a_n^4} + \frac{h^{4r}}{nha_n^2} \right). \end{aligned} \quad (\text{A.30})$$

See the arguments leading to (6.24) in HH [11] and the analogous result for their equation (6.24) in HH [11] and the analogous result for their quantity $E\|H_{n2}\|^2$. Combining (A.25)–(A.30) with Assumptions 5 and 7 yields the result that

$$\frac{A_{n4}(x, z)}{\sqrt{V_n(x, z)}} = \frac{-(I + T^+ \Delta)^{-1} T^+ B_{n12}}{\sqrt{V_n(x, z)}} + o_p(1). \quad (\text{A.31})$$

Now consider $-(I + T^+ \Delta)^{-1} T^+ B_{n12}$. Standard calculations for kernel estimators show that

$$\begin{aligned} & \iiint_0^1 \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw \\ & = n^{-1} \sum_{i=1}^n f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, \mathbb{W}_i) g(\mathbb{X}_i, \mathbb{Z}_i) + O(h^r). \end{aligned} \quad (\text{A.32})$$

Therefore,

$$\begin{aligned} & T^+ \iiint_0^1 \hat{f}_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) f_{\mathbb{X}\mathbb{Z}\mathbb{W}}(x, z, w) g(x, z) dx dz dw \\ & = A_{n1}(x, z) + o\left(\frac{h^r}{a_n}\right), \end{aligned} \quad (\text{A.33})$$

$$T^+ B_{n12}(x, z) = A_{n1}(x, z) - EA_{n1}(x, z) + o\left(\frac{h^r}{a_n}\right). \quad (\text{A.34})$$

But

$$\begin{aligned} & (I + T^+ \Delta)^{-1} T^+ B_{n12}(x, z) \\ & = T^+ B_{n12} + \left[(I + T^+ \Delta)^{-1} - I \right] T^+ B_{n12} \\ & = T^+ B_{n12} + (\hat{T} + a_n I)^{-1} \Delta T^+ B_{n12}. \end{aligned} \quad (\text{A.35})$$

Therefore, it follows, by combining Assumption 7 and equations (A.11), (A.17), and (A.34), that

$$(I + T^+ \Delta)^{-1} T^+ B_{n12}(x, z) = A_{n1}(z) - EA_{n1}(x, z) + r_n, \quad (\text{A.36})$$

where $E\|r_n\|^2 / \sqrt{V_n(x, z)} = o(1)$ for almost every (x, z) . Combining this result with (A.21) and (A.31) gives

$$\frac{A_{n3}(x, z)}{\sqrt{V_n(x, z)}} = \frac{-[A_{n1}(x, z) - EA_{n1}(x, z)]}{\sqrt{V_n(x, z)}} + o_p(1) \quad (\text{A.37})$$

for almost every (x, z) .

Now consider $A_{n4}(x, z)$. HH show that

$$A_{n4}(x, z) = -(I + T^+ \Delta)^{-1} T^+ \Delta (A_{n2} - T^+ B_{n2})(x, z). \quad (\text{A.38})$$

Therefore, it follows from (A.19) and (A.30) that

$$\frac{A_{n4}(x, z)}{\sqrt{V_n(x, z)}} = o_p(1) \quad (\text{A.39})$$

for almost every (x, z) .

Now combine (A.19), (A.37), and (A.39) to obtain

$$\frac{S_{n2}(x, z)}{\sqrt{V_n(x, z)}} = \frac{\sum_{j=1}^4 A_{nj}(x, z)}{\sqrt{V_n(x, z)}} = \frac{EA_{n1}(x, z)}{\sqrt{V_n(x, z)}} + o_p(1) \quad (\text{A.40})$$

for almost every (x, z) . The lemma follows by combining this result with (A.16).

This completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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