Research Article

On Delay-Fractional-Dependent Stability Criteria for Takagi-Sugeno Fuzzy Systems with Interval Delay

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This paper investigates stability program of Takagi-Sugeno fuzzy systems with interval time-varying delay via a variable delay decomposition approach. By developing a delay decomposition approach, both lower and upper bound information of the delayed plant states can be taken into full consideration; two novel delay-fractional-dependent stability criteria are obtained based on the direct Lyapunov method allied with an appropriate and variable Lyapunov-Krasovskii functional choice and with two different bounding techniques to estimate some integral terms in the time-derivative of the Lyapunov-Krasovskii functional. The first stability criterion is derived by utilizing the suitable and generalized integral inequalities, while the second stability condition is obtained by employing a scalar inequality, without any direct approximation. Particularly, the proposed results differ from previous ones since the positiveness of the Lyapunov-Krasovskii functional is guaranteed by new relaxed conditions. When applying these two stability criteria to check the stability of a T-S fuzzy system, it is shown through some numerical examples that the first stability condition can provide a larger maximum allowable delay bound than the second stability criterion, and both stability criteria yield less conservative than the existing ones.

1. Introduction

Stability is a central issue in dynamical system and control theory. A dynamical system is called stable (in the sense of Lyapunov) if starting the system somewhere near its desired operating point implies that it will stay around that point ever after [1]. Fuzzy systems are nonlinear systems described by a set of “IF-THEN” rules which give a local linear representation for an underlying system. Consequent function-type fuzzy models with linear consequent functions are called Takagi-Sugeno (T-S) models; using such models, complex nonlinear systems can be represented by selection of membership functions and consequent functions, which can approximate a wide range of nonlinear systems [2, 3]. In practice, time delays are frequently encountered in many engineering systems, and they are usually regarded as a source of instability and poor performance [1, 3]. In

particular, the problem of checking stability of systems with an interval time-varying delay $\tau(t)$, that is, $\tau(t) \in [h_a, h_b]$, is more involving [4, 5] and may be found, for example, in networked control systems. As a result, stability analysis for T-S fuzzy systems with interval time-varying delay is of great significance both in theory and in practice. Much effort has been devoted to the analysis and synthesis of fuzzy systems with interval delay during the last decades; for example, see [5–18] and the references therein. Most recently, stability analyses for T-S fuzzy systems also have been extended to the fuzzy control and observer design, H_∞/H_∞ faults detection, and fault tolerant; for example, see [15, 19–23].

Recently, the development of the technologies for delay-dependent stability analysis has been focusing on effective reduction of the conservatism of the stability conditions. The main aim is to derive a maximum allowable delay bound (MADB) of the time delay such that the time delays'

system is asymptotically stable for any delay between the lower bound and the MADB. Via different types of LK functional together with the Jensen inequality approach, improved delay-dependent stability results were provided in [10, 12, 14, 16, 17, 24, 25]. An et al. [12, 14] improved the stability conditions for T-S fuzzy systems with norm-bounded-type uncertainties by constructing a new LKF and improving Jensen's inequality. Note that most of the aforementioned delay-dependent stability results were obtained by requiring all the involved symmetric matrices in a chosen LK functional to be positive definite. Such a requirement can cause conservatism in the criteria. Actually, this requirement is not necessary to guarantee an LK functional to be positive definite, which is similar to relaxed positiveness condition in [10], though the authors in [10] used a different LK functional. This observation has been overlooked when using the LK functional approach in the literature, which motivates the present study.

In order to reduce the conservatism introduced by model transformation and bounding techniques, a free-weighting matrices method is proposed in [17, 18, 26]. Z. Yang and Y.-P. Yang [18] studied the stability and synthesis problem of uncertain T-S fuzzy systems with time-varying delay by using LKF approach. Liu et al. [26] proposed new delay-dependent stability criteria for T-S fuzzy systems with time-varying delay. Recently, a delay partitioning approach has been proposed to further reduce their conservatism; see [4, 9, 11–14, 27–29] and references therein. However, the decomposition approach is complicated and the resulting stability condition is difficult to check in [27]. Liu [28] used a delay decomposition approach, and new stability results were derived in the sense of simpler complexity and less conservatism. Zhao et al. [29] proposed a delay partitioning approach to stability and stabilization of delayed T-S fuzzy systems. Most recently, Peng et al. [8, 9] proposed some less conservative results by using delay partitioning technique and free matrices based on Newton-Leibniz formula. Yet, there exists further room to investigate the upper bound of the time-derivative of the LK functional [5–14, 25, 26], which also motivates the present study.

This paper will focus on the delay-fractional-dependent stability problem of T-S fuzzy systems with interval time-varying delay. Inspired by the novel approach in Souza et al. [10], by some integral inequalities in [1, 24, 30], then by the works of Liu [28], Zhang et al. [24], and Briat [31], and by the aforementioned observation that the positive definiteness of a chosen LK functional does not necessarily require all the involved symmetric matrices to be positive definite, we firstly construct a novel delay-fractional-dependent LK functional by developing a variable delay decomposition technique. Secondly, we estimate the upper bound of its derivative less conservatively by suitably utilizing the integral inequalities and adopt the convex combination technique; then we obtain a less conservative stability criterion in terms of linear matrix inequalities (LMIs). On the other hand, by using a scalar inequality proposed in [32], another stability criterion is also proposed which is less conservative than the existing ones without any free matrices included; that is,

there has not any extra computation burden. Finally, three numerical examples are used to compare with some previous results and demonstrate the effectiveness of the proposed method.

2. Problem Formulation

Consider a nonlinear system with interval time-varying delay via T-S fuzzy model, which can be described as follows:

Rule j: IF $\theta_1(t)$ is N_{j1} and $\dots$ and $\theta_p(t)$ is N_{jp} , THEN

$$\begin{aligned}\dot{x}(t) &= A_j x(t) + A_{\tau j} x(t - \tau(t)), \\ x(t) &= \phi(t), \quad \forall t \in [-h_b, 0], \quad j = 1, 2, \dots, r,\end{aligned}\tag{1}$$

where $0 \leq h_a \leq \tau(t) \leq h_b$ and $x(t)$ is the state vector. $\phi(t)$ is the continuous initial vector function defined on $[-h_b, 0]$; $\theta_1(t), \theta_2(t), \dots, \theta_p(t)$ denote the premise variables while $N_{j1}, N_{j2}, \dots, N_{jp}$ represent the fuzzy sets; the system coefficient matrices are constant real matrices with appropriate dimensions, where $j = 1, 2, \dots, r$ and r is the number of IF-THEN rules. For the sake of convenience, we denote $\bar{\tau} = h_b - h_a$.

In this paper, the delay $\tau(t)$ is assumed to be time-varying delay as the following two cases.

Case I. $\tau(t)$ is a differentiable function, satisfying for all $t \geq 0$

$$0 < h_a \leq \tau(t) \leq h_b, \quad \dot{\tau}(t) \leq h_d.\tag{2}$$

Case II. $\tau(t)$ is not differentiable or the upper bound of the derivative of $\tau(t)$ is unknown, and $\tau(t)$ satisfies

$$0 < h_a \leq \tau(t) \leq h_b,\tag{3}$$

where h_a, h_b , and h_d are some given values.

The fuzzy system (1) is supposed to have singleton fuzzifier, product inference, and centroid defuzzifier. The final output of the fuzzy system is inferred as follows:

$$\begin{aligned}\dot{x}(t) &= A(t) x(t) + A_{\tau}(t) x(t - \tau(t)), \\ x(t) &= \phi(t), \quad \forall t \in [-h_b, 0],\end{aligned}\tag{4}$$

where

$$\begin{aligned}A(t) &:= \sum_{j=1}^r h_j(\theta(t)) A_j, \\ A_{\tau}(t) &:= \sum_{j=1}^r h_j(\theta(t)) A_{\tau j},\end{aligned}\tag{5}$$

and $h_j(\theta(t)) = \mu_j(\theta(t)) / \sum_{i=1}^r \mu_i(\theta(t))$, $\mu_j(\theta(t)) = \prod_{k=1}^p N_{jk}(\theta_k(t))$, and $N_{jk}(\theta_k(t))$ is the membership function of $\theta_k(t)$ in N_{jk} . Here $\mu_j(\theta(t)) \geq 0$ and $\sum_{j=1}^r h_j(\theta(t)) = 1$.

The purpose of this paper is to find new stability criteria, which are less conservative than the existing results. Such a criterion may be used to compute the tolerable delay bound h_b for given h_a or vice versa.

To end this section, we introduce the following lemmas which are useful in stability analysis for the T-S fuzzy systems.

First, Lemma 1 induces some integral inequalities.

Lemma 1 (integral inequalities, Gu et al. [1], Zhang et al. [24], and Sun et al. [30]). *Let $x(t) \in \mathfrak{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, for any matrices $M, N \in \mathfrak{R}^{n \times n}$, $Z \in \mathfrak{R}^{2n \times 2n}$, $X = X^T \in \mathfrak{R}^{n \times n}$, and some given scalars $0 \leq \tau_1 < \tau_2$, the following integral inequality holds.*

Case 1. When $X > 0$ and τ_1, τ_2 are constant values,

$$\begin{aligned} & (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} x^T(s) X x(s) ds \\ & \geq \int_{t-\tau_2}^{t-\tau_1} x^T(s) ds X \int_{t-\tau_2}^{t-\tau_1} x(s) ds, \end{aligned} \quad (6)$$

$$\begin{aligned} & \left(\frac{\tau_2^2 - \tau_1^2}{2} \right) \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x^T(s) X x(s) ds d\theta \\ & \geq \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x^T(s) ds d\theta X \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t x(s) ds d\theta, \end{aligned} \quad (7)$$

$$\begin{aligned} & -(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) X \dot{x}(s) ds \\ & \leq \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix}^T \begin{bmatrix} -X & X \\ * & -X \end{bmatrix} \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix}. \end{aligned} \quad (8)$$

Case 2. When $X > 0$ and τ_1, τ_2 are time-varying, $h = \tau_2 - \tau_1 := h(t) \geq 0$,

$$\begin{aligned} & - \int_{t-\tau_2}^{t-\tau_1} x^T(s) X x(s) ds \\ & \leq \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix}^T \\ & \quad \times \left\{ \begin{bmatrix} M + M^T & -M + N^T \\ * & -N - N^T \end{bmatrix} + h \begin{bmatrix} M \\ N \end{bmatrix} X^{-1} \begin{bmatrix} M^T & N^T \end{bmatrix} \right\} \\ & \quad \times \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix}. \end{aligned} \quad (9)$$

Case 3. When τ_1, τ_2 are time-varying, $h = \tau_2 - \tau_1 := h(t) \geq 0$, and X is any symmetric matrix,

$$\begin{aligned} & - \int_{t-\tau_2}^{t-\tau_1} x^T(s) X x(s) ds \\ & \leq \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix}^T \left\{ \begin{bmatrix} M + M^T & -M + N^T \\ * & -N - N^T \end{bmatrix} + hZ \right\} \\ & \quad \times \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau_2) \end{bmatrix} \end{aligned} \quad (10)$$

with $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \geq 0$ and $Y = [M \ N]$.

Remark 2. Equations (6)–(10) are called some integral inequalities presented over the existing literatures. They play a key role in the derivation of a criterion for delay-fractional-dependent stability in this paper. To reduce conservatism of the proposed results, the relaxed matrix inequalities in (6)–(10) will be adopted in the present work. The relaxed matrix inequality is quite different from the ones in [1] and also can produce less conservative results than the descriptor model transformation method [18, 24, 26]. In [6–10], the Leibniz-Newton formula was employed and many free-weighting matrix variables were introduced. In fact, these free-weighting matrix variables are all redundant and can be eliminated, which was theoretically proven in [16]. Generally speaking, by using Jensen's inequality or equivalent Jensen inequality [31], the computational complexity of the obtained stability criterion may be lower than the ones in [6–10], and thus the integral inequality based results are carried out more efficiently as compared with the general free-weighting matrix based on the Newton-Leibniz formula, since fewer matrix variables are introduced for computing.

Second, we introduce the following scalar inequality which is essential in the proofs of our results without any free matrices.

Lemma 3 (scalar inequality, [32]). *For any scales $M \geq 0$, $N \geq 0$, and $h_2 \leq \tau(t) \leq h_3$, one has*

$$\begin{aligned} & - \frac{h_3 - h_2}{\tau(t) - h_2} M - \frac{h_3 - h_2}{h_3 - \tau(t)} N \\ & \leq \max \{-(M + 3N), -(N + 3M)\}. \end{aligned} \quad (11)$$

3. Main Results

This section aims to develop a novel delay decomposing approach for stability analysis of T-S fuzzy system (4).

Firstly, we divide the delay intervals $[0, h_a]$ and $[h_a, h_b]$ into four segments: $[h_{i-1}, h_i]$, $i = 1, 2, 3, 4$, where $h_0 = 0$, $h_1 = h_a/2$, $h_2 = h_a$, $h_3 = h_a + \alpha \bar{\tau}$, and $h_4 = h_b$, $0 < \alpha < 1$. For the sake of convenience, we denote $\tau_i = h_i - h_{i-1}$, $r_i = h_i^2 - h_{i-1}^2$, ($i = 1, 2, 3, 4$), and $\bar{\tau}_0 = h_a - 0$. For the T-S fuzzy system (4), based on the Lyapunov stability theorem, we will give a stability criterion by using a novel delay decomposition approach as follows.

Theorem 4. (i) *In Case I, for given scalars $0 < h_a \leq h_b$, $0 < \alpha < 1$, h_d , the T-S fuzzy system (4) is asymptotically stable if there exist real symmetry matrices $P, Q_i > 0$, $R_i, Q_\tau \geq 0$, R_τ , $\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} > 0$, $Z_j = \begin{bmatrix} Z_{j1} & Z_{j2} \\ * & Z_{j3} \end{bmatrix}$ ($j = 1, 2$), and any matrices*

$M_i, N_i, (i = 1, 2, 3, 4)$ with appropriate dimensions such that the LMIs in (12) and (13) are feasible. Consider

$$V_p = \begin{bmatrix} P + \sum_{i=1}^4 \frac{2\tau_i^3}{r_i} & -\frac{2\tau_1^2}{r_1} R_1 & -\frac{2\tau_2^2}{r_2} R_2 & -\frac{2\tau_3^2}{r_3} R_3 & -\frac{2\tau_4^2}{r_4} R_4 \\ * & \frac{1}{\tau_1} Q_1 + \frac{2\tau_1}{r_1} R_1 + \frac{2}{\bar{\tau}_0} S_{11} & \frac{2}{\bar{\tau}_0} S_{12} & 0 & 0 \\ * & * & \frac{1}{\tau_2} Q_2 + \frac{2\tau_2}{r_2} R_2 + \frac{2}{\bar{\tau}_0} S_{22} & 0 & 0 \\ * & * & * & \frac{1}{\tau_3} Q_3 + \frac{2\tau_3}{r_3} R_3 & 0 \\ * & * & * & * & \frac{1}{\tau_4} Q_4 + \frac{2\tau_4}{r_4} R_4 \end{bmatrix} > 0, \quad (12)$$

$$\Omega(i, k) := \Omega_0 + \Omega_1^k + \Omega_2^k + \Xi_i^k < 0, \quad (i = 1, 2; k = 3, 4) \quad (13)$$

$$\begin{bmatrix} \tau_4 R_4 + (1 - h_d) R_\tau & [M_4 \ N_4] \\ * & Z_2 \end{bmatrix} \geq 0, \quad (14)$$

with

$$\begin{bmatrix} \tau_3 R_3 + (1 - h_d) R_\tau & [M_1 \ N_1] \\ * & Z_1 \end{bmatrix} \geq 0,$$

where

$$\Omega_0 := \begin{bmatrix} PA + A^T P + Q_1 + S_{11} - R_1 & -\frac{PA_\tau}{(1 - h_d)} Q_\tau & S_{12} + R_1 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & Q_2 - Q_1 + S_{22} - S_{11} - R_1 - R_2 & -S_{12} + R_2 & 0 & 0 \\ * & * & * & Q_3 - Q_2 + Q_\tau - S_{22} - R_2 & 0 & 0 \\ * & * & * & * & Q_4 - Q_3 & 0 \\ * & * & * & * & * & -Q_4 \end{bmatrix},$$

$$\Omega_1^3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -N_1 - N_1^T + \tau_3 M_2 + \tau_3 M_2^T & 0 & -M_1^T + N_1 & -\tau_3 M_2 + \tau_3 N_2^T & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & M_1 + M_1^T & 0 & 0 \\ * & * & * & * & -R_4 - \tau_3 N_2 - \tau_3 N_2^T & R_4 \\ * & * & * & * & * & -R_4 \end{bmatrix},$$

$$\Omega_1^4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -N_4 - N_4^T + \tau_4 (M_3 + M_3^T) & 0 & 0 & -M_4^T + N_4 & -\tau_4 M_3 + \tau_4 N_3^T \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & M_1 + M_1^T + \tau_3 Z_{11} & -M_1 + N_1^T + \tau_3 Z_{12} & 0 \\ * & * & * & * & -N_1 - N_1^T + M_4 + M_4^T + \tau_3 Z_{13} & 0 \\ * & * & * & * & * & -\tau_4 (N_3 + N_3^T) \end{bmatrix},$$

$$\Omega_2^3 := \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \tau_3 R_\tau \right) \Gamma, \quad \Omega_2^4 := \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \bar{\tau} R_\tau \right) \Gamma, \quad \Gamma := [A \ A_\tau \ 0 \ 0 \ 0 \ 0],$$

$$\Xi_1^k := \tau_k Z^k, \quad Z^3 := [Z_{ij}^3]_{6 \times 6}, \quad Z_{ij}^3 \triangleq \begin{cases} Z_{13} & i=2, j=2 \\ Z_{12}^T & i=2, j=4 \\ Z_{11} & i=4, j=4 \\ Z_{12} & i=4, j=2 \\ 0 & \text{other,} \end{cases} \quad Z^4 := [Z_{ij}^4]_{6 \times 6}, \quad Z_{ij}^4 \triangleq \begin{cases} Z_{23} & i=2, j=2 \\ Z_{22}^T & i=2, j=5 \\ Z_{21} & i=5, j=5 \\ Z_{22} & i=5, j=2 \\ 0 & \text{other,} \end{cases}$$

$$\Xi_2^k := \tau_k^2 \Phi^k R_k^{-1} (\Phi^k)^T, \quad \Phi^3 := \text{col} \{0, M_2, 0, 0, N_2, 0\}, \quad \Phi^4 := \text{col} \{0, M_3, 0, 0, 0, N_3\}.$$

(15)

(ii) In Case II, for given scalars $0 < h_a \leq h_b$ and $0 < \alpha < 1$, the T-S fuzzy system (4) is asymptotically stable if there exist real symmetry matrices $P, Q_i > 0, R_i, \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} > 0, Z_j = \begin{bmatrix} Z_{j1} & Z_{j2} \\ * & Z_{j3} \end{bmatrix} (j = 1, 2)$, and any matrices $M_i, N_i (i = 1, 2, 3, 4)$, with appropriate dimensions such that the LMIs in (12) and (13) with $Q_\tau = 0, R_\tau = 0$ are feasible.

Proof. In Case I, choose the following delay-dependent LK functional:

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t) + V_3(t, x_t) + V_4(t, x_t), \quad (16)$$

where x_t denotes the function $x(s)$ defined on the interval $[t - h_b, t]$ and

$$\begin{aligned} V_1(t, x_t) &= x^T(t) P x(t), \\ V_2(t, x_t) &= \sum_{i=1}^4 \int_{t-h_i}^{t-h_{i-1}} x^T(s) Q_i x(s) ds \\ &\quad + \int_{t-\tau(t)}^{t-h_2} x^T(s) Q_\tau x(s) ds, \\ V_3(t, x_t) &= \int_{t-h_1}^t \left[x \left(s - \frac{\bar{\tau}_0}{2} \right) \right]^T \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \left[x \left(s - \frac{\bar{\tau}_0}{2} \right) \right] ds, \\ V_4(t, x_t) &= \sum_{i=1}^4 \tau_i \int_{-h_i}^{-h_{i-1}} \int_{t+\theta}^t \dot{x}^T(s) R_i \dot{x}(s) ds d\theta \\ &\quad + \int_{-\tau(t)}^{-h_2} \int_{t+\theta}^t \dot{x}^T(s) R_\tau \dot{x}(s) ds d\theta. \end{aligned} \quad (17)$$

with $P, Q_i > 0 (i = 1, 2, 3, 4), Q_\tau \geq 0, \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} > 0, R_i (i = 1, 2, 3, 4), R_\tau$ being real symmetry matrices.

First, we show that the condition $V(t, x_t) \geq \varepsilon_1 \|x(t)\|, (\varepsilon_1 > 0)$ is satisfied. Assume that $Q_i > 0$ and $R_i > 0 (i = 1, 2, 3, 4)$; then, suitably applying Lemma 1 yields

$$\begin{aligned} V(t, x_t) &\geq x^T(t) P x(t) + \int_{t-\tau(t)}^{t-h_2} x^T(s) Q_\tau x(s) ds \\ &\quad + \int_{-\tau(t)}^{-h_2} \int_{t+\theta}^t \dot{x}^T(s) R_\tau \dot{x}(s) ds d\theta \\ &\quad + \sum_{i=1}^4 \frac{1}{\tau_i} \int_{t-h_i}^{t-h_{i-1}} x^T(s) ds Q_i \int_{t-h_i}^{t-h_{i-1}} x(s) ds \\ &\quad + \int_{t-h_1}^t \left[x \left(s - \frac{\bar{\tau}_0}{2} \right) \right]^T ds \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \\ &\quad \times \int_{t-h_1}^t \left[x \left(s - \frac{\bar{\tau}_0}{2} \right) \right] ds \\ &\quad + \sum_{i=1}^4 \frac{2\tau_i}{r_i} \int_{-h_i}^{-h_{i-1}} \int_{t+\theta}^t \dot{x}^T(s) ds d\theta \\ &\quad \times R_i \int_{-h_i}^{-h_{i-1}} \int_{t+\theta}^t \dot{x}(s) ds d\theta. \end{aligned} \quad (18)$$

We denote $\zeta(t) := \text{col} \left\{ x(t) \int_{t-h_1}^t x(s) ds \int_{t-h_2}^{t-h_1} x(s) ds \int_{t-h_3}^{t-h_2} x(s) ds \int_{t-h_4}^{t-h_3} x(s) ds \right\}$; after the simple arrangement, we have

$$\begin{aligned} V(t, x_t) &\geq \zeta^T(t) V_p \zeta(t) + \int_{t-\tau(t)}^{t-h_2} x^T(s) Q_\tau x(s) ds \\ &\quad + \int_{-\tau(t)}^{-h_2} \int_{t+\theta}^t \dot{x}^T(s) R_\tau \dot{x}(s) ds d\theta, \end{aligned} \quad (19)$$

where

$$V_p \triangleq \begin{bmatrix} P + \sum_{i=1}^4 \frac{2\tau_i^3}{r_i} & -\frac{2\tau_1^2}{r_1} & -\frac{2\tau_2^2}{r_2} & -\frac{2\tau_3^2}{r_3} & -\frac{2\tau_4^2}{r_4} \\ * & \frac{1}{\tau_1} Q_1 + \frac{2\tau_1}{r_1} R_1 + \frac{2}{\bar{\tau}_0} S_{11} & \frac{2}{\bar{\tau}_0} S_{12} & 0 & 0 \\ * & * & \frac{1}{\tau_2} Q_2 + \frac{2\tau_2}{r_2} R_2 + \frac{2}{\bar{\tau}_0} S_{22} & 0 & 0 \\ * & * & * & \frac{1}{\tau_3} Q_3 + \frac{2\tau_3}{r_3} R_3 & 0 \\ * & * & * & * & \frac{1}{\tau_4} Q_4 + \frac{2\tau_4}{r_4} R_4 \end{bmatrix}. \quad (20)$$

Therefore, if the right side of (19) is positive, it yields a sufficient condition to guarantee that $V(t, x_t) \geq \varepsilon_1 \|x(t)\|, (\varepsilon_1 > 0)$. Moreover, the LMI in (12) implies that $R_i > 0 (i = 1, 2, 3, 4)$

and $R_\tau > 0$. Consequently, if the LMIs in (7) are satisfied, $Q_i > 0 (i = 1, 2, 3, 4)$, and $Q_\tau \geq 0$, then $V(t, x_t) \geq \varepsilon_1 \|x(t)\|, (\varepsilon_1 > 0)$.

Secondly, we show that the condition $\dot{V}(t, x_t) \leq -\varepsilon_2 \|x(t)\|$ ($\varepsilon_2 > 0$) is also guaranteed if the LMIs in (13) and $Q_\tau \geq 0, Q_i > 0$ ($i = 1, 2, 3, 4$), $\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} > 0$ hold. Taking the derivative of (16) with respect to t along the trajectory of system (4), we have

$$\begin{aligned} \dot{V}_1(t, x_t) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\ &= [A x(t) + A_\tau x(t - \tau(t))]^T P x(t) \\ &\quad + x^T(t) P [A x(t) + A_\tau x(t - \tau(t))], \end{aligned} \quad (21)$$

$$\begin{aligned} \dot{V}_2(t, x_t) &= \sum_{i=1}^4 [x^T(t - h_{i-1}) Q_i x(t - h_{i-1}) \\ &\quad - x^T(t - h_i) Q_i x(t - h_i)] \\ &\quad + x^T(t - h_2) Q_\tau x(t - h_2) - (1 - \dot{\tau}(t)) x^T \\ &\quad \times (t - \tau(t)) Q_\tau x(t - \tau(t)) \\ &\leq \sum_{i=1}^4 [x^T(t - h_{i-1}) Q_i x(t - h_{i-1}) \\ &\quad - x^T(t - h_i) Q_i x(t - h_i)] \\ &\quad + x^T(t - h_2) Q_\tau x(t - h_2) \\ &\quad - (1 - h_d) x^T(t - \tau(t)) Q_\tau x(t - \tau(t)), \end{aligned} \quad (22)$$

$$\begin{aligned} \dot{V}_3(t, x_t) &= \begin{bmatrix} x(t) \\ x(t - h_1) \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h_1) \end{bmatrix} \\ &\quad - \begin{bmatrix} x(t - h_1) \\ x(t - h_2) \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \begin{bmatrix} x(t - h_1) \\ x(t - h_2) \end{bmatrix}, \end{aligned} \quad (23)$$

$$\begin{aligned} \dot{V}_4(t, x_t) &= \sum_{i=1}^4 \tau_i^2 \dot{x}^T(t) R_i \dot{x}(t) \\ &\quad + (\tau(t) - h_2) \dot{x}^T(t) R_\tau \dot{x}(t) \\ &\quad - \sum_{i=1}^4 \tau_i \int_{t-h_i}^{t-h_{i-1}} \dot{x}^T(s) R_i \dot{x}(s) ds \\ &\quad - (1 - \dot{\tau}(t)) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds. \end{aligned} \quad (24)$$

For any $t \geq 0$, it is a fact that $h_a \leq \tau(t) \leq h_a + \alpha\tau$ or $h_a + \alpha\tau \leq \tau(t) \leq h_b$, ($0 < \alpha < 1$). In the case of $h_a \leq \tau(t) \leq h_a + \alpha\tau$, that is, $\tau(t) \in [h_2, h_3]$, $k = 3$, suitably using the integral inequalities in Lemma 1, the following inequalities are true:

$$\begin{aligned} &(\tau(t) - h_2) \dot{x}^T(t) R_\tau \dot{x}(t) \\ &\leq \alpha \bar{\tau} \dot{x}^T(t) R_\tau \dot{x}(t) = \tau_3 \dot{x}^T(t) R_\tau \dot{x}(t), \end{aligned} \quad (25)$$

$$\begin{aligned} & - \tau_i \int_{t-h_i}^{t-h_{i-1}} \dot{x}^T(s) R_i \dot{x}(s) ds \\ &\leq \begin{bmatrix} x(t - h_{i-1}) \\ x(t - h_i) \end{bmatrix}^T \begin{bmatrix} -R_i & R_i \\ * & -R_i \end{bmatrix} \begin{bmatrix} x(t - h_{i-1}) \\ x(t - h_i) \end{bmatrix}, \end{aligned} \quad (26)$$

($i = 1, 2, 4$),

$$\begin{aligned} & - \tau_3 \int_{t-h_3}^{t-h_2} \dot{x}^T(s) R_3 \dot{x}(s) ds \\ &\quad - (1 - \dot{\tau}(t)) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds \\ &\leq - \tau_3 \int_{t-h_3}^{t-h_2} \dot{x}^T(s) R_3 \dot{x}(s) ds \\ &\quad - (1 - h_d) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds \\ &= - \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) (\tau_3 R_3 + (1 - h_d) R_\tau) \dot{x}(s) ds \\ &\quad - \tau_3 \int_{t-h_3}^{t-\tau(t)} \dot{x}^T(s) R_3 \dot{x}(s) ds \\ &\leq \begin{bmatrix} x(t - h_2) \\ x(t - \tau(t)) \end{bmatrix}^T \left\{ \begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T \\ * & -N_1 - N_1^T \end{bmatrix} + \rho \cdot \alpha \bar{\tau} \cdot Z_1 \right\} \\ &\quad \times \begin{bmatrix} x(t - h_2) \\ x(t - \tau(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} x(t - \tau(t)) \\ x(t - h_3) \end{bmatrix}^T \left\{ \tau_3 \begin{bmatrix} M_2 + M_2^T & -M_2 + N_2^T \\ * & -N_2 - N_2^T \end{bmatrix} \right. \\ &\quad \left. + (1 - \rho) \cdot \alpha \bar{\tau} \cdot \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R_3^{-1} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T \right\} \\ &\quad \times \begin{bmatrix} x(t - \tau(t)) \\ x(t - h_3) \end{bmatrix} \end{aligned} \quad (27)$$

with $\begin{bmatrix} \tau_3 R_3 + (1 - h_d) R_\tau & [M_1 \ N_1] \\ * & Z_1 \end{bmatrix} \geq 0$, and $\rho = ((\tau(t) - h_2)/\alpha \bar{\tau})$, $0 \leq \rho \leq 1$.

It follows from (21)–(27) that

$$\begin{aligned} &\dot{V}(t, x_t) \\ &\leq \rho \xi^T(t) \left[\Omega_0 + \Omega_1^3 + \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \alpha \bar{\tau} R_\tau \right) \Gamma + \alpha \bar{\tau} Z^3 \right] \\ &\quad \times \xi(t) + (1 - \rho) \xi^T(t) \\ &\quad \times \left[\Omega_0 + \Omega_1^3 + \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \alpha \bar{\tau} R_\tau \right) \Gamma \right. \\ &\quad \left. + (\alpha \bar{\tau})^2 \Phi^3 R_3^{-1} (\Phi^3)^T \right] \xi(t) \end{aligned} \quad (28)$$

with $\begin{bmatrix} \tau_3 R_3 + (1-h_d) R_r & [M_1 \ N_1] \\ * & Z_1 \end{bmatrix} \geq 0$ where $\Omega_0, \Omega_1^3, \Gamma, Z^3$, and Φ^3 are defined in (15), and $\xi(t) := \text{col}\{x(t) \ x(t-\tau(t)) \ x(t-h_1) \ x(t-h_2) \ x(t-h_3) \ x(t-h_4)\}$.

Since $0 \leq \rho \leq 1$, applying the convex combination method, we conclude that if the following LMIs

$$\begin{aligned} \Omega_0 + \Omega_1^3 + \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \alpha \bar{\tau} R_r \right) \Gamma + \alpha \bar{\tau} Z^3 &< 0, \\ \Omega_0 + \Omega_1^3 + \Gamma^T \left(\sum_{i=1}^4 \tau_i^2 R_i + \alpha \bar{\tau} R_r \right) \Gamma & \\ + (\alpha \bar{\tau})^2 \Phi^3 R_3^{-1} (\Phi^3)^T &< 0 \end{aligned} \quad (29)$$

are simultaneously feasible, then $\dot{V}(t, x_t) < -\varepsilon_2 \|x(t)\|$ is true for a sufficiently small $\varepsilon_2 > 0$.

Meanwhile, if $h_a + \alpha \tau \leq \tau(t) \leq h_b$, that is, $\tau(t) \in [h_3, h_4]$, $k = 4$, similar to the above deduction process, we also can obtain similar stability conditions.

In Case II, a Lyapunov functional can be chosen as (16) with $Q_r = 0, R_r = 0$. Similar to the above analysis, one can get that $\dot{V}(t, x_t) < -\varepsilon_2 \|x(t)\|$ ($\varepsilon_2 > 0$).

Thus, this completes the proof. $\square$

Remark 5. It may be noted that, in the above, no approximation of the delay term is involved excepting exploiting a convex combination of the uncertain terms involved. In fact, Lemma 1 plays a key effect on the present results, which is different from the common Jensen's inequality, although their similarity can be established following the equivalency results in [31]. However, if $h = \tau_2 - \tau_1 := h(t)$ is uncertain and is required to be approximated with its lower or upper bound then use of (9) or (10) would be beneficial since the free variables M, N are introduced. Such a feature leads to less

conservative results compared to the existing ones as is shown in the next section using numerical examples.

When the lower bound of the delay is 0, that is, $h_a = 0$, and the interval term $[0, h_a]$ is missing, we may set $Q_1 = 0, Q_2 = 0, S_{ij} = 0, R_1 = 0$, and $R_2 = 0$ in (16), according to Theorem 4, and then obtain Corollary 6.

Corollary 6. (i) In Case 1, for given scalars $0 = h_a \leq h_b, 0 < \alpha < 1, h_d$, the T-S fuzzy systems (4) are asymptotically stable if there exist real symmetric matrices $P, Q_i > 0, R_i$ ($i = 3, 4$), $Q_r \geq 0, R_r, Z_i = \begin{bmatrix} Z_{i1} & Z_{i2} \\ * & Z_{i3} \end{bmatrix}$ ($i = 1, 2$), and any matrices M_i, N_i ($i = 1, 2, 3, 4$) with appropriate dimensions such that the LMIs in (30) are feasible as follows:

$$\begin{aligned} \widehat{V}_P &= \begin{bmatrix} P + \sum_{i=3}^4 \frac{2\tau_i^3}{r_i} R_i & -\frac{2\tau_3^2}{r_3} R_3 & -\frac{2\tau_4^2}{r_4} R_4 \\ * & \frac{1}{\tau_3} Q_3 + \frac{2\tau_3}{r_3} R_3 & 0 \\ * & * & \frac{1}{\tau_4} Q_4 + \frac{2\tau_4}{r_4} R_4 \end{bmatrix} \\ &> 0, \\ \widehat{\Omega}(i, k) &:= \widehat{\Omega}_0 + \widehat{\Omega}_1^k + \widehat{\Omega}_2^k + \widehat{\Xi}_i^k < 0, \quad (i = 1, 2; k = 3, 4) \end{aligned} \quad (30)$$

with

$$\begin{aligned} \begin{bmatrix} \tau_3 R_3 + (1-h_d) R_r & [M_1 \ N_1] \\ * & Z_1 \end{bmatrix} &\geq 0, \\ \begin{bmatrix} \tau_4 R_4 + (1-h_d) R_r & [M_4 \ N_4] \\ * & Z_2 \end{bmatrix} &\geq 0, \end{aligned} \quad (31)$$

where

$$\begin{aligned} \widehat{\Omega}_0 &:= \begin{bmatrix} PA + A^T P + Q_r + Q_3 & P A_{\tau} & 0 & 0 \\ * & (1-h_d) Q_r & 0 & 0 \\ * & * & Q_4 - Q_3 & 0 \\ * & * & * & -Q_4 \end{bmatrix}, \\ \widehat{\Omega}_1^3 &:= \begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T & 0 & 0 \\ * & -N_1 - N_1^T + \tau_3 M_2 + \tau_3 M_2^T & -\tau_3 M_2 + \tau_3 N_2^T & 0 \\ * & * & R_4 - \tau_3 N_2 - \tau_3 N_2^T & R_4 \\ * & * & * & -R_4 \end{bmatrix}, \\ \widehat{\Omega}_1^4 &:= \begin{bmatrix} M_1 + M_1^T + \tau_3 Z_{11} & 0 & -M_1 + N_1^T + \tau_3 Z_{12} & 0 \\ * & -N_4 - N_4^T + \tau_4 (M_3 + M_3^T) & -M_4^T + N_4 & -\tau_4 M_3 + \tau_4 N_3^T \\ * & * & -N_1 - N_1^T + M_4 + M_4^T + \tau_3 Z_{13} & 0 \\ * & * & * & -\tau_4 (N_3 + N_3^T) \end{bmatrix}, \\ \widehat{\Omega}_2^3 &:= \widehat{\Gamma}^T \left(\sum_{i=3}^4 \tau_i^2 R_i + \tau_3 R_r \right) \widehat{\Gamma}, \quad \widehat{\Omega}_2^4 := \widehat{\Gamma}^T \left(\sum_{i=3}^4 \tau_i^2 R_i + \bar{\tau} R_r \right) \widehat{\Gamma}, \quad \widehat{\Gamma} := [A \ A_r \ 0 \ 0], \end{aligned}$$

$$\begin{aligned}
\widehat{\Xi}_1^k &:= \tau_k \widehat{Z}^k, \quad (k = 3, 4), \quad \widehat{Z}^3 := [\widehat{Z}_{ij}^3]_{4 \times 4}, \quad \widehat{Z}_{ij}^3 \triangleq \begin{cases} Z_{11} & i = 1, j = 1 \\ Z_{12} & i = 1, j = 2 \\ Z_{12}^T & i = 2, j = 1 \\ Z_{13} & i = 2, j = 2 \\ 0 & \text{other,} \end{cases} \\
\widehat{Z}^4 &:= [\widehat{Z}_{ij}^4]_{4 \times 4}, \quad \widehat{Z}_{ij}^4 \triangleq \begin{cases} Z_{23} & i = 2, j = 2 \\ Z_{22}^T & i = 2, j = 3 \\ Z_{21} & i = 3, j = 3 \\ Z_{22} & i = 3, j = 2 \\ 0 & \text{other,} \end{cases} \\
\widehat{\Xi}_2^k &:= \tau_k^2 \widehat{\Phi}^k R_k^{-1} (\widehat{\Phi}^k)^T, \quad (k = 3, 4), \quad \widehat{\Phi}^3 := \text{col} \{0, M_2, N_2, 0\}, \quad \widehat{\Phi}^4 := \text{col} \{0, M_3, 0, N_3\}.
\end{aligned} \tag{32}$$

(ii) In Case 2, if the LMIs in (30) with $Q_\tau = 0$ and $R_\tau = 0$ are feasible, the aforementioned system is asymptotically stable for fast-varying delay.

Notice that Theorem 4 and Corollary 6 still have some free-weighting matrices. In the following we present the second stability criterion in which no free-weighting matrix is involved.

Similar to the line of derivation process of Theorem 4, for (27), we have

$$\begin{aligned}
& -\tau_3 \int_{t-h_3}^{t-h_2} \dot{x}^T(s) R_3 \dot{x}(s) ds \\
& - (1 - \dot{\tau}(t)) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds \\
& \leq -\tau_3 \int_{t-h_3}^{t-h_2} \dot{x}^T(s) R_3 \dot{x}(s) ds \\
& - (1 - h_d) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds \\
& = - \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) [\tau_3 R_3 + (1 - h_d) R_\tau] \dot{x}(s) ds \\
& - \tau_3 \int_{t-h_3}^{t-\tau(t)} \dot{x}^T(s) R_3 \dot{x}(s) ds \\
& \leq - \frac{h_3 - h_2}{\tau(t) - h_2} (x^T(t - h_2) - x^T(t - \tau(t))) \\
& \times \left(R_3 + \frac{(1 - h_d)}{\tau_3} R_\tau \right) (x(t - h_2) - x(t - \tau(t))) \\
& - \frac{h_3 - h_2}{h_3 - \tau(t)} (x^T(t - \tau(t)) - x^T(t - h_3)) \\
& \times (R_3) (x(t - \tau(t)) - x(t - h_3)).
\end{aligned} \tag{33}$$

Based on Lemma 3, if we denote

$$\begin{aligned}
M &\triangleq (x^T(t - h_2) - x^T(t - \tau(t))) \\
&\times \left(R_3 + \frac{(1 - h_d)}{\tau_3} R_\tau \right) (x(t - h_2) - x(t - \tau(t))), \\
N &\triangleq (x^T(t - \tau(t)) - x^T(t - h_3)) (R_3) \\
&\times (x(t - \tau(t)) - x(t - h_3)),
\end{aligned} \tag{34}$$

then we have $-\tau_3 \int_{t-h_3}^{t-h_2} \dot{x}^T(s) R_3 \dot{x}(s) ds - (1 - \dot{\tau}(t)) \int_{t-\tau(t)}^{t-h_2} \dot{x}^T(s) R_\tau \dot{x}(s) ds \leq \max\{-(M + 3N), -(N + 3M)\}$. After rearrangement of (21)–(26) and (33), we can deduce a similar formula as (28)–(29) and then obtain the following result for the T-S fuzzy system (4).

Theorem 7. (i) In Case 1, for given scalars $0 < h_a \leq h_b$, $0 < \alpha < 1$, h_d , the T-S fuzzy systems (4) are asymptotically stable if there exist real symmetric matrices $P, Q_i > 0, R_i (i = 1, 2, 3, 4)$, $Q_\tau \geq 0, R_\tau, \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} > 0$ such that the LMIs in (12) and (35) are feasible. Consider

$$\widetilde{\Omega}(i, k) := \Omega_0 + \Omega_2^k + \widetilde{\Xi}_i^k < 0 \quad (i = 1, 2; k = 3, 4) \tag{35}$$

with $\tau_3 R_3 + (1 - h_d) R_\tau > 0$ and $\tau_4 R_4 + (1 - h_d) R_\tau > 0$, where the terms are defined in (15) except that $\widetilde{\Xi}_i^k$ are described as follows:

$$\begin{aligned}
\tilde{\mathcal{W}}_1^3 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -4R_3 - \frac{1-h_d}{\tau_3}R_\tau & 0 & R_3 + \frac{1-h_d}{\tau_3}R_\tau & 3R_3 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -R_3 - \frac{1-h_d}{\tau_3}R_\tau & 0 & 0 \\ * & * & * & * & -R_4 - 3R_3 & R_4 \\ * & * & * & * & * & -R_4 \end{bmatrix}, \\
\tilde{\mathcal{W}}_2^3 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -4R_3 - \frac{3(1-h_d)}{\tau_3}R_\tau & 0 & 3R_3 + \frac{3(1-h_d)}{\tau_3}R_\tau & R_3 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -3R_3 - \frac{3(1-h_d)}{\tau_3}R_\tau & 0 & 0 \\ * & * & * & * & -R_4 - R_3 & R_4 \\ * & * & * & * & * & -R_4 \end{bmatrix}, \\
\tilde{\mathcal{W}}_1^4 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -4R_4 - \frac{1-h_d}{\tau_4}R_\tau & 0 & 0 & R_4 + \frac{1-h_d}{\tau_4}R_\tau & 3R_4 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -R_3 - \frac{1-h_d}{\tau_3}R_\tau & R_3 + \frac{1-h_d}{\tau_3}R_\tau & 0 \\ * & * & * & * & -R_3 - \frac{1-h_d}{\tau_3}R_\tau - R_4 - \frac{1-h_d}{\tau_4}R_\tau & 0 \\ * & * & * & * & * & -3R_4 \end{bmatrix}, \\
\tilde{\mathcal{W}}_2^4 &:= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & -4R_4 - \frac{3(1-h_d)}{\tau_4}R_\tau & 0 & 0 & 3R_4 + \frac{3(1-h_d)}{\tau_4}R_\tau & R_4 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -R_3 - \frac{1-h_d}{\tau_3}R_\tau & R_3 + \frac{1-h_d}{\tau_3}R_\tau & 0 \\ * & * & * & * & -R_3 - \frac{1-h_d}{\tau_3}R_\tau - 3R_4 - \frac{3(1-h_d)}{\tau_4}R_\tau & 0 \\ * & * & * & * & * & -R_4 \end{bmatrix}.
\end{aligned} \tag{36}$$

(ii) In Case 2, if the LMIs in (12) and (35) with $Q_\tau = 0$ and $R_\tau = 0$ are feasible, the aforementioned system is asymptotically stable for fast-varying delay.

We have proposed two techniques to investigate the stability of T-S fuzzy system (4) and obtain two stability results, Theorems 4 and 7, both less conservative with smaller numerical complexity than the existing ones. The two techniques are different mainly in the way of estimating the cross terms. The first estimation has no conservatism but involves seven free-weighting matrices, which are introduced to make use of the constraints M_i, N_i ($i = 1, 2, 3, 4$) and

Z_1, Z_2 . By contrast, the second estimation involves no free-weighting matrices though a little more conservative. This ends in that Theorem 7 has smaller numerical complexity than Theorem 4.

Remark 8. By constructing new augmented LKF (16) and using useful inequalities in Lemmas 1 and 3, Theorems 4 and 7 are obtained with and without using any free matrices, respectively, which are dependent on τ_i, r_i ($i = 1, 2, 3, 4$) with relation to a tuning parameter. Moreover, the proposed results can be extended to robust H_-/H_∞ control and fault detection/fault tolerant tracking control for T-S systems with

unknown inputs or input constraints and with unmeasurable premise variables as [19–22], robust observer and output feedback control of the above systems [15, 23], NCSs, and multidelay systems, and the corresponding results will appear in the near future.

4. Numerical Examples

This section gives three examples to demonstrate the effectiveness and reduced conservatism of the proposed approach. For comparisons, we study T-S fuzzy system (4) with fuzzy rules investigated in recent publications [6–10, 12–14, 26].

Example 1 (Example 1 of [7]). Consider a two-rule system:

$$\dot{x}(t) = \sum_{i=1}^2 h_i [A_i x(t) + A_{\tau i} x(t - \tau(t))], \quad (37)$$

where $h_1 = 1/(1 + \exp(-2x_1(t)))$, $h_2 = 1 - h_1$, and

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, & A_{\tau 1} &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -1.5 & 1 \\ 0 & -0.75 \end{bmatrix}, & A_{\tau 2} &= \begin{bmatrix} -1 & 0 \\ 1 & -0.85 \end{bmatrix}. \end{aligned} \quad (38)$$

For this T-S fuzzy system, it is easy to notice, from Table 1, that the proposed result asserts the system's stability with the larger MADB to the time-varying delay with unknown time-derivative. If the additional information $h_d = 0.3$ is given, larger upper bounds of the delay can be computed by Theorems 4 and 7 for various tuning delay-fractional parameters α , respectively, which are shown in Table 2. Moreover, when $h_d = 0.5$, the maximum allowable delay bound (MADB) h_b obtained from the criterion [28] are 1.3451 and 1.4016 with $h_a = 0.1$ and $h_a = 0.3$, while Theorem 4 ($\alpha = 0.65$) in this paper can compute the MADB as 1.5503 and 1.4874 with $h_a = 0.1$ and $h_a = 0.3$, respectively.

It is worth mentioning that Theorems 4 and 7 can provide larger upper bound than the existing ones. At the same time, Theorem 7 has some larger conservative than Theorem 4 but has less numerical complexity than some existing ones since Theorem 7 does not involve any free matrices. Next, we will give another example to illustrate the merit of our method, which is reduced conservative than the existing results.

Example 2 (Example 1 of [26]). Consider a two-rule system:

$$\dot{x}(t) = \sum_{i=1}^2 h_i [A_i x(t) + A_{\tau i} x(t - \tau(t))], \quad (39)$$

where $h_1 = 1/(1 + \exp(-2x_1(t)))$, $h_2 = 1 - h_1$, and

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, & A_2 &= \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \\ A_{\tau 1} &= \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}, & A_{\tau 2} &= \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}. \end{aligned} \quad (40)$$

TABLE 1: Comparison of MADB h_b for various h_a with unknown delay derivative h_d .

Method\ h_a	0.2	0.4	0.6	0.8
Lien et al. [6]	0.7945	0.8487	0.9316	1.0480
Peng et al. [8]	0.9119	0.9793	1.0639	1.1662
Tian et al. [7]	1.1410	1.1500	1.1720	1.2090
Souza et al. [10]	1.1639	1.1734	1.1994	1.2532
Theorem 7 ($\alpha = 0.55$)	1.1640	1.1844	1.2100	1.2567
Theorem 4 ($\alpha = 0.6$)	1.4618	1.4202	1.4005	1.4095

TABLE 2: Comparison of MADB h_b for various h_a with $h_d = 0.3$.

Method\ h_a	0.2	0.4	0.6	0.8
Theorem 7 ($\alpha = 0.5$)	1.2872	1.3018	1.3155	1.3230
Theorem 7 ($\alpha = 0.55$)	1.2842	1.3018	1.3165	1.3234
Theorem 4 ($\alpha = 0.6$)	1.5934	1.5535	1.4999	1.4510
Theorem 4 ($\alpha = 0.65$)	1.6013	1.5485	1.4882	1.4381

TABLE 3: MADB on h_b for various delay derivatives h_d and h_a .

$h_d \setminus h_a$	0	1	2	3
0.1	3.288	3.874	3.774	3.480
0.2	2.819	3.229	3.013	3.252
0.3	2.494	2.754	2.566	3.252
1.1	1.779	2.031	2.563	3.252
h_d unknown	1.779	2.031	2.563	3.252

Choosing h_d and h_a in Table 3 and applying Theorem 4 ($\alpha = 0.7$) and Corollary 6 ($\alpha = 0.8$), the larger MADB on h_b of the delay, which guarantees the asymptotic stability of the fuzzy system, is listed in Table 3. Moreover, when h_d are unknown, that is, fast-varying delay case, by setting $Q_\tau = 0$ and $R_\tau = 0$, the obtained h_b are computed as shown in the last column of Table 3. It is worthwhile to point out that, when $h_d \geq 1$, the Q_τ, R_τ in LK functional authentically will no longer be helpful to improve the stability criterion.

To compare with the existing results in [14, 25, 26], we assume that $h_a = 0$. Using Corollary 6 yields larger MADB on h_b that guarantees the stability of fuzzy system for various h_d , which are listed in Table 4. From Table 4, it can be seen that when the delay is time-invariant, that is, $h_d = 0$, the obtained results in Corollary 6 are better than those in [14, 25, 26]. Furthermore, when the delay is time-varying, the authors in [25] fail to verify that the fuzzy system is stable, while Corollary 6 of this paper can obtain the better upper bounds than those in [14, 26]. Moreover, it should be noted that $h_d = 0$ does not mean a constant delay (i.e., $\tau(t) \equiv 0$), but it means a time-varying delay with $\dot{\tau}(t) \leq 0, \forall t \geq 0$. Also, our result gives lesser conservativeness for a relatively large value or unknown case of h_d .

Furthermore, we will give another example to show the effectiveness and merit of nonlinear system via T-S fuzzy models, and the proposed approach also yields less conservative than the existing results.

TABLE 4: MADB on h_b for various h_d ($h_a = 0$).

Method	$h_d = 0$	$h_d = 0.1$	$h_d = 0.5$	h_d unknown
Chen et al. [25]	3.150	—	—	—
Liu et al. [26]	3.300	2.650	1.500	0.790
An et al. [14]	3.704	3.011	1.657	1.197
Corollary 6 ($\alpha = 0.7$)	3.942	3.111	2.054	1.895
Corollary 6 ($\alpha = 0.8$)	4.062	3.288	2.076	1.779

Example 3 (Example 1 of [9]). Consider the following time-delayed nonlinear system:

$$\begin{aligned}
 \dot{x}_1(t) &= 0.5(1 - \sin^2(\theta(t)))x_2(t) - x_1(t - \tau(t)) \\
 &\quad - (1 + \sin^2(\theta(t)))x_1(t), \\
 \dot{x}_2(t) &= \operatorname{sgn}\left(|\theta(t)| - \frac{\pi}{2}\right)(0.9 \cos^2(\theta(t)) - 1)x_1(t - \tau(t)) \\
 &\quad - x_2(t - \tau(t)) - (0.9 + 0.1 \cos^2(\theta(t)))x_2(t)
 \end{aligned} \quad (41)$$

which can be exactly expressed as a T-S delayed system (4) with the following rules [8–10, 13]:

$$\begin{aligned}
 R^1 : & \text{ if } \theta(t) \text{ is } \pm \frac{\pi}{2}, \text{ then} \\
 & \dot{x}(t) = A_1 x(t) + A_{\tau 1} x(t - \tau(t)), \\
 R^2 : & \text{ if } \theta(t) \text{ is } 0, \text{ then} \\
 & \dot{x}(t) = A_2 x(t) + A_{\tau 2} x(t - \tau(t)).
 \end{aligned} \quad (42)$$

The membership functions for above rules 1 and 2 are

$$h_1(x_1(t)) = \sin^2(x_1(t)), \quad h_2(x_1(t)) = \cos^2(x_1(t)) \quad (43)$$

with the following system parameters:

$$\begin{aligned}
 A_1 &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, & A_{\tau 1} &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, & A_{\tau 2} &= \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.
 \end{aligned} \quad (44)$$

To compare with the existing results, we assume that h_d is unknown. The improvement of this paper is shown in Table 5. If the delay is fast time-varying case, the LMIs in Theorem 4 are feasible with $1.2 \leq \tau(t) \leq 1.634$. If the additional information $h_d = 0.1$ is given, larger upper bounds of the delay can be computed by Theorem 4, which is shown at the last row of Table 5.

TABLE 5: MADB on h_b for varying h_a (h_d unknown).

Method \ h_a	0.4	0.8	1.0	1.2
Lien et al. [6]	0.8829	1.0677	1.1874	1.3181
Peng et al. [8]	1.0183	1.1817	1.2776	1.3816
Li et al. [5]	1.038	1.158	1.252	1.359
Peng and Han [13]	1.180	1.310	1.370	1.430
Tian et al. [7]	1.2647	1.3032	1.3528	1.4214
An and Wen [12]	1.277	1.311	1.358	1.419
Peng and Fei [9]	1.320	1.320	1.380	1.420
Souza et al. [10]	1.2836	1.3394	1.4009	1.4815
Theorem 4 ($\alpha = 0.7$)	1.537	1.556	1.586	1.634
Theorem 4 ($\alpha = 0.7$ and $h_d = 0.1$)	2.079	2.058	2.029	1.986

As an example, when $h_a = 1.2$, $h_b = 1.634$, and h_d is unknown, by applying Theorem 4 in this paper, we can obtain the following solutions:

$$\begin{aligned}
 P &= \begin{bmatrix} 0.2142 & 0.0715 \\ 0.0715 & 0.0842 \end{bmatrix}, & S_{11} &= \begin{bmatrix} 0.1455 & 0.0346 \\ 0.0346 & 0.0330 \end{bmatrix}, \\
 S_{12} &= \begin{bmatrix} -0.0090 & -0.0220 \\ -0.0084 & -0.0274 \end{bmatrix}, & S_{22} &= \begin{bmatrix} 0.1534 & 0.0460 \\ 0.0460 & 0.0641 \end{bmatrix}, \\
 R_1 &= \begin{bmatrix} 0.0475 & 0.0486 \\ 0.0486 & 0.0822 \end{bmatrix}, & R_2 &= \begin{bmatrix} 0.0317 & 0.0217 \\ 0.0217 & 0.0305 \end{bmatrix}, \\
 R_3 &= \begin{bmatrix} 0.4544 & 0.1034 \\ 0.1034 & 0.1084 \end{bmatrix}, & R_4 &= \begin{bmatrix} 0.0779 & 0.0365 \\ 0.0365 & 0.0421 \end{bmatrix}, \\
 Q_1 &= \begin{bmatrix} 0.1375 & 0.0246 \\ 0.0246 & 0.0202 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 0.1513 & 0.0390 \\ 0.0390 & 0.0409 \end{bmatrix}, \\
 Q_3 &= \begin{bmatrix} 0.1738 & 0.0541 \\ 0.0541 & 0.0345 \end{bmatrix}, \\
 Q_4 &= 10^{-4} \begin{bmatrix} 0.9310 & -0.0495 \\ -0.0495 & 0.4243 \end{bmatrix}.
 \end{aligned} \quad (45)$$

Furthermore, with initial state condition $\phi(t) = [1, -1]^T$, Figure 1 depicts the state responses of the system (41) with $1.2 \leq \tau(t) \leq 1.634$ listed in Table 5.

5. Conclusion

Two delay-interval-dependent stability criteria have been proposed for T-S fuzzy systems with interval time-varying delays through constructing a novel delay-dependent LK functional via variable delay partitioned approach, utilizing suitably some integral inequalities, and employing convex combination technique and a scalar inequality. The proposed criteria provide better results since approximation of the uncertain delay factors in the time-derivative of LK functional has been avoided, while the positiveness of LK functional also must not be guaranteed by imposing the positiveness of all the terms composing the LK functional separately. The superiority of the present result has been validated through numerical examples. Moreover, the proposed

FIGURE 1: State responses of the system (41) and (42) for Example 3.

approach is simple and may easily be extended to robust stabilization and H_∞ control problems for uncertain time-delayed T-S fuzzy systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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