Research Article

Strong Convergence Theorems for Two Total Quasi- ϕ -Asymptotically Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is to establish some strong convergence theorems for a common fixed point of two total quasi- ϕ -asymptotically nonexpansive mappings in Banach space by means of the hybrid method in mathematical programming. The results presented in this paper extend and improve on the corresponding ones announced by Martinez-Yanes and Xu (2006), Plubtieng and Ungchittrakool (2007), Qin et al. (2009), and many others.

1. Introduction

Let E be a smooth Banach space and let E^* be the topological dual of E . Let the function $\phi : E \times E \rightarrow \mathbb{R}^+$ be defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E, \quad (1)$$

where J is the normalized duality mapping from E to E^* . Let K be a nonempty subset of E and $T : K \rightarrow K$ a nonlinear mapping. Recall that a mapping T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$. A point $x \in K$ is said to be a *fixed point* of T [1] if $Tx = x$. Let $F(T) := \{x \in K : Tx = x\}$ be the set of fixed points of T . A point $p \in K$ is said to be an *asymptotic fixed point* of T [1] if K contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Recall that a mapping $T : K \rightarrow K$ is called (i) *relatively nonexpansive* [2] if $\hat{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$, $\forall x \in K, \forall p \in F(T)$; (ii) *closed* if for any sequence $\{x_n\} \subset K$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, $Tx = y$; (iii) *quasi- ϕ -nonexpansive* [3] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$, $\forall x \in K, \forall p \in F(T)$; (iv) *quasi- ϕ -asymptotically nonexpansive* [3] if $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} \mu_n = 0$ such that $\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x)$, $\forall x \in$

$K, \forall p \in F(T), \forall n \in \mathbb{N}$; (v) *total quasi- ϕ -asymptotically nonexpansive* [4–6] if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}, \{\nu_n\}$ with $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n = 0$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(p, T^n x) \leq \phi(p, x) + \mu_n \zeta(\phi(p, x)) + \nu_n$, $\forall x \in K, \forall p \in F(T), \forall n \in \mathbb{N}$; (vi) *uniformly L -Lipschitz continuous* [4] if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall x, y \in K, \forall n \geq 1$.

It is well known that total quasi- ϕ -asymptotically nonexpansive mappings contain relatively nonexpansive mappings, quasi- ϕ -nonexpansive mappings, and quasi- ϕ -asymptotically nonexpansive mappings as its special cases; see [4, 6, 7] for more details.

Iterative approximation of fixed points for nonexpansive mappings has been considered by many papers for either the Mann iteration [8] or the Ishikawa iteration [9]; see, for example, [9–12] and the references therein.

In 1967, Halpern [11] considered the following explicit iteration which was referred to as Halpern iteration:

$$\begin{aligned} x_0 &\in K \text{ arbitrarily chosen,} \\ x_{n+1} &= \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \end{aligned} \quad (2)$$

where T is nonexpansive. He proves the strong convergence of $\{x_n\}$ to a point of T provided that $\alpha_n = n^{-\theta}$, where $\theta \in (0, 1)$.

Recently, many authors improved and generalized the results of Halpern [11] by means of different methods; see, for example, [2, 4, 10, 12–17] and the references therein. In general, there are two ways as follows.

- (i) One of the methods is to combine Halpern iteration with Mann iteration; see, for example, C.E. Chidume and C.O. Chidume [10] and Hu [12].
- (ii) Another method is to use the hybrid projection algorithm method; see, for example, Martinez-Yanes and Xu [15], Plubtieng and Ungchittrakool [16], and Qin et al. [17].

Motivated and inspired by [2, 4, 10, 12–17], the purpose of this paper is to modify Halpern iteration (2) by means of both methods above for two total quasi- ϕ -asymptotically nonexpansive mappings and then to prove the strong convergence in the framework of Banach spaces. The results presented in this paper extend and improve the corresponding results of Martinez-Yanes and Xu [15], Plubtieng and Ungchittrakool [16], Qin et al. [17], and others.

Throughout the paper, we denote $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.

2. Preliminaries

Let E be a real Banach space endowed with the norm $\|\cdot\|$ and let E^* be the topological dual of E . For all $x \in E$ and $x^* \in E^*$, the value of f at x is denoted by $\langle x, f \rangle$ and is called the duality pairing. Then, the normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E. \quad (3)$$

It is well known that the operator J is well defined and J is the identity mapping if and only if E is a Hilbert space. But in general, J is nonlinear and multiple-valued.

The following basic properties for a Banach space E can be found in [18].

- (i) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E .
- (ii) If E is reflexive and strictly convex, then J^{-1} is norm-weak-continuous.
- (iii) If E is a smooth, strictly convex, and reflexive Banach space, then J is single-valued, one-to-one, and onto.
- (iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
- (v) Each uniformly convex Banach space E has the *Kadec-Klee property*; that is, for any sequence $\{x_n\} \in E$, if $x_n \rightarrow x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$.
- (vi) Every uniformly smooth Banach space is reflexive.

Moreover, we have the Lyapunov functional ϕ defined by (1). It is obvious from the definition of the function ϕ that we have the following property.

Property 1 (see [19]). If the function ϕ is defined by (1), then we have

- (i) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \forall x, y \in E$;
- (ii) $\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \forall x, y, z \in E$;
- (iii) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \forall x, y, z \in E$;
- (iv) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|Jy\|, \forall x, y \in E$.

It follows Albert [20] that the generalized projection $\Pi_K : E \rightarrow K$ is defined by $\Pi_K(x) = \arg \inf_{y \in K} \phi(y, x), \forall x \in E$. So

we have the following lemmas.

Lemma 1 (see [20]). *Let E be a smooth, strictly convex, and reflexive Banach space and let K be a nonempty closed convex subset of E . Then the following conclusions hold:*

- (i) $\phi(x, \Pi_K y) + \phi(\Pi_K y, y) \leq \phi(x, y), \forall x \in K, \forall y \in E$;
- (ii) if $x \in K$ and $z \in K$, then $z = \Pi_K x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in K$;
- (iii) for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Remark 2. If H is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_K = P_K$ (the metric projection of H onto a closed convex subset K).

Lemma 3 (see [20]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ (as $n \rightarrow \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \rightarrow 0$ (as $n \rightarrow \infty$).*

Lemma 4 (see [4]). *Let E be a smooth, strictly convex, and reflexive Banach space with Kadec-Klee property, and let K be a nonempty closed convex subset of E . Let $T : K \rightarrow K$ be a closed and totally quasi- ϕ -asymptotically nonexpansive mapping with nonnegative real sequences $\{\mu_n\}, \{\nu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n = 0$ and $\zeta(0) = 0$. If $v_1 = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of K .*

Lemma 5 (see [21]). *Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E . Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \quad (4)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

3. Main Results

To prove the main results, we need the following hypotheses for the sequences $\{\alpha_n\}$ and $\{\beta_n^{(j)}\}$ ($j = 1, 2, 3$):

- (i) $0 < \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$,
(ii) $0 \leq \beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \leq 1$, $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$ and $\liminf_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

The main results of this paper are stated as follows.

Theorem 6. Let E be a real strictly convex and uniformly smooth Banach space with Kadec-Klee property, and let K be a nonempty closed convex subset of E . Let $S, T : K \rightarrow K$ be two closed, uniformly L -Lipschitz continuous and totally quasi- ϕ -asymptotically nonexpansive mappings with nonnegative real sequences $\{\mu_n^{(j)}\}, \{\nu_n^{(j)}\}, j = 1, 2$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{n \rightarrow \infty} \mu_n^{(j)} = \lim_{n \rightarrow \infty} \nu_n^{(j)} = 0, j = 1, 2$ and $\zeta(0) = 0$. Let $\{x_n\}$ be a sequence defined by

$$\begin{aligned} x_0 &\in C_1 = K \text{ arbitrarily chosen,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) Jz_n), \\ z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_0) \\ &\quad + (1 - \alpha_n) \phi(z, x_n) + (1 - \alpha_n) \xi_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \quad (5)$$

where $\xi_n = (\beta_n^{(2)} \mu_n^{(2)} + \beta_n^{(3)} \mu_n^{(1)}) \sup_{p \in \mathfrak{F}} \zeta(\phi(p, x_n)) + (\beta_n^{(2)} \nu_n^{(2)} + \beta_n^{(3)} \nu_n^{(1)})$. Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n^{(j)}\}$ ($j = 1, 2, 3$) satisfy hypotheses (i) and (ii). If $\mathfrak{F} := F(S) \cap F(T) \neq \emptyset$ is bounded and $\nu_1 = 0$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathfrak{F}} x_0$.

Proof. We divide the proof of Theorem 6 into six steps.

- (I) We first show that $\mathfrak{F}$ is a closed and convex subset in K .

By Lemma 4, it is trivial to show that $F(S)$ and $F(T)$ are two closed and convex subsets of K . Therefore $\mathfrak{F}$ is closed and convex in K .

- (II) Next we prove that C_n is a closed and convex subset in K for all $n \geq 1$.

As a matter of fact, by hypothesis, $C_1 = K$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. From the definition of ϕ , we may know that

$$\begin{aligned} C_{k+1} &= \{z \in C_k : \phi(z, y_k) \leq \alpha_k \phi(z, x_0) \\ &\quad + (1 - \alpha_k) \phi(z, x_k) + (1 - \alpha_k) \xi_k\}, \\ &= \{z \in C_k : 2\langle z, Jx_k - Jy_k \rangle + 2\alpha_k \langle z, Jx_0 \rangle \\ &\quad \leq \alpha_k \|x_0\|^2 + (1 - \alpha_k) \|x_k\|^2 \\ &\quad - \|y_k\|^2 + (1 - \alpha_k) \xi_k\} \cap C_k \end{aligned} \quad (6)$$

and thus C_{k+1} is closed and convex. Therefore, by induction principle, C_n are closed and convex for all $n \geq 1$. This also shows that $\Pi_{\mathfrak{F}} x_0$ is well defined.

- (III) Now we show that $\mathfrak{F} \subset C_n, \forall n \geq 1$.

Indeed, it is obvious that $\mathfrak{F} \subset C_1 = K$. Suppose that $\mathfrak{F} \subset C_n$ for some $n \geq 1$. Hence for any $u \in \mathfrak{F} \subset C_n$, by Lyapunov functional (1) and Lemma 5, we have

$$\begin{aligned} \phi(u, z_n) &= \phi(u, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n)) \\ &= \|u\|^2 - 2\langle u, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n \rangle \\ &\quad + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n\|^2 \\ &\leq \|u\|^2 - 2\beta_n^{(1)} \langle u, Jx_n \rangle - 2\beta_n^{(2)} \langle u, JT^n x_n \rangle \\ &\quad - 2\beta_n^{(3)} \langle u, JS^n x_n \rangle + \beta_n^{(1)} \|x_n\|^2 + \beta_n^{(2)} \|T^n x_n\|^2 \\ &\quad + \beta_n^{(3)} \|S^n x_n\|^2 - \beta_n^{(2)} \beta_n^{(3)} g(\|T^n x_n - S^n x_n\|) \\ &\leq \beta_n^{(1)} \phi(u, x_n) + \beta_n^{(2)} \phi(u, T^n x_n) + \beta_n^{(3)} \phi(u, S^n x_n) \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|T^n x_n - S^n x_n\|) \\ &\leq \beta_n^{(1)} \phi(u, x_n) \\ &\quad + \beta_n^{(2)} [\phi(u, x_n) + \mu_n^{(2)} \zeta(\phi(u, x_n)) + \nu_n^{(2)}] \\ &\quad + \beta_n^{(3)} [\phi(u, x_n) + \mu_n^{(1)} \zeta(\phi(u, x_n)) + \nu_n^{(1)}] \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|JT^n x_n - JS^n x_n\|) \\ &\leq \phi(u, x_n) + \xi_n - \beta_n^{(2)} \beta_n^{(3)} g(\|JT^n x_n - JS^n x_n\|) \\ &\leq \phi(u, x_n) + \xi_n. \end{aligned} \quad (7)$$

Moreover, it follows from Property 1(ii) that we have

$$\begin{aligned} \phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) Jz_n)) \\ &\leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, z_n). \end{aligned} \quad (8)$$

Combining (7) with (8), we have

$$\phi(u, y_n) \leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) (\phi(u, x_n) + \xi_n). \quad (9)$$

This shows that $u \in C_{n+1}$. Therefore, by induction principle, we have $\mathfrak{F} \subset C_n$ for all $n \geq 1$.

- (IV) Next we prove that $\{x_n\}$ converges strongly to some point $p^* \in K$.

In fact, since $x_n = \Pi_{C_n} x_0$, it follows from Lemma 1(ii) that we have

$$\langle x_n - y, Jx_0 - Jx_n \rangle \geq 0, \quad \forall y \in C_n. \quad (10)$$

Again since $\mathfrak{F} \subset C_n$ for all $n \geq 1$, we have

$$\langle x_n - u, Jx_0 - Jx_n \rangle \geq 0, \quad \forall u \in \mathfrak{F}. \quad (11)$$

It follows from Lemma 1(i) that for each $u \in \mathfrak{F}$ and for each $n \geq 1$ we have

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0).\end{aligned}\quad (12)$$

Therefore $\{\phi(x_n, x_0)\}$ is bounded, and hence $\{x_n\}$ is bounded.

On the other hand, since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 1$. This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing, and so $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists. By the construction of $\{\phi(x_n, x_0)\}$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m > n$. Therefore, using Lemma 1(i), we have

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+m}, x_0) - \phi(x_n, x_0)\end{aligned}\quad (13)$$

for all $n \geq 1$. Since $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$ exists, we obtain that

$$\phi(x_{n+m}, x_n) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty), \quad \forall m \geq 1. \quad (14)$$

Thus, by Lemma 3, we have

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0, \quad \forall m \geq 1. \quad (15)$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in K . Since K is a nonempty closed subset of Banach space E , this implies that it is complete. Hence there exists an $p^* \in K$ such that

$$\lim_{n \rightarrow \infty} x_n = p^*. \quad (16)$$

By the way, it is easy from (16) to see that

$$\begin{aligned}\lim_{n \rightarrow \infty} \xi_n &= \lim_{n \rightarrow \infty} \left[\left(\beta_n^{(2)} \mu_n^{(2)} + \beta_n^{(3)} \mu_n^{(1)} \right) \sup_{p \in \mathfrak{F}} \zeta(\phi(p, x_n)) \right. \\ &\quad \left. + \left(\beta_n^{(2)} \nu_n^{(2)} + \beta_n^{(3)} \nu_n^{(1)} \right) \right] = 0.\end{aligned}\quad (17)$$

(V) Now we prove that $p^* \in \mathfrak{F}$.

Since $x_{n+1} \in C_{n+1}$ and by the structure of C_{n+1} , we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n) (\phi(x_{n+1}, x_n) + \xi_n). \quad (18)$$

Hence, by means of limits (14), (17) and $\lim_{n \rightarrow \infty} \alpha_n = 0$ and using Lemma 3, we get that $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0$. But

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|. \quad (19)$$

Thus $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$. This implies that

$$\lim_{n \rightarrow \infty} y_n = p^*. \quad (20)$$

Meanwhile, it follows from (7) and (8) that we have

$$\begin{aligned}(1 - \alpha_n) \beta_n^{(2)} \beta_n^{(3)} g(\|JT^n x_n - JS^n x_n\|) \\ \leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) (\phi(u, x_n) + \xi_n) - \phi(u, y_n) \\ \leq \alpha_n (\phi(u, x_0) - \phi(u, y_n)) + (1 - \alpha_n) \xi_n \\ + (1 - \alpha_n) (\phi(u, x_n) - \phi(u, y_n)),\end{aligned}\quad (21)$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous strictly increasing convex function with $g(0) = 0$. Thus, by virtue of (16), (17), (20), (21), $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\liminf_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$, we have $\lim_{n \rightarrow \infty} g(\|JT^n x_n - JS^n x_n\|) = 0$, and hence $\lim_{n \rightarrow \infty} \|JT^n x_n - JS^n x_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we get

$$\lim_{n \rightarrow \infty} \|T^n x_n - S^n x_n\| = \lim_{n \rightarrow \infty} \|J^{-1}(JT^n x_n - JS^n x_n)\| = 0. \quad (22)$$

Next, using the convexity of $\|\cdot\|^2$, Property 1(iv), and hypothesis (ii), we obtain that

$$\begin{aligned}\phi(T^n x_n, z_n) \\ = \phi(T^n x_n, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n)) \\ = \|T^n x_n\|^2 - 2 \langle T^n x_n, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n \rangle \\ + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n\|^2 \\ \leq \|T^n x_n\|^2 - 2\beta_n^{(1)} \langle T^n x_n, Jx_n \rangle - 2\beta_n^{(2)} \langle T^n x_n, JT^n x_n \rangle \\ - 2\beta_n^{(3)} \langle T^n x_n, JS^n x_n \rangle + \beta_n^{(1)} \|x_n\|^2 \\ + \beta_n^{(2)} \|T^n x_n\|^2 + \beta_n^{(3)} \|S^n x_n\|^2 \\ = \beta_n^{(1)} \phi(T^n x_n, x_n) \\ + \beta_n^{(3)} \phi(T^n x_n, S^n x_n) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\end{aligned}\quad (23)$$

Hence, from Property 1(ii), hypothesis (i), and (23), we have

$$\begin{aligned}\phi(T^n x_n, y_n) &= \phi(T^n x_n, J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) Jz_n)) \\ &\leq \alpha_n \phi(T^n x_n, x_0) + (1 - \alpha_n) \phi(T^n x_n, z_n) \longrightarrow 0 \\ &\quad (\text{as } n \longrightarrow \infty).\end{aligned}\quad (24)$$

Thus, by Lemma 3, we have $\lim_{n \rightarrow \infty} \|T^n x_n - y_n\| = 0$ and hence

$$\begin{aligned}\|T^n x_n - x_n\| &\leq \|T^n x_n - y_n\| + \|y_n - x_n\| \longrightarrow 0 \\ &\quad (\text{as } n \longrightarrow \infty).\end{aligned}\quad (25)$$

Moreover, we observe that

$$\begin{aligned}\|S^n x_n - x_n\| &\leq \|S^n x_n - T^n x_n\| + \|T^n x_n - x_n\| \longrightarrow 0 \\ &\quad (\text{as } n \longrightarrow \infty).\end{aligned}\quad (26)$$

Since $\lim_{n \rightarrow \infty} x_n = p^*$ and E is uniformly smooth, it follows from (25) and (26) that we have

$$\lim_{n \rightarrow \infty} T^n x_n = p^*, \quad (27)$$

$$\lim_{n \rightarrow \infty} S^n x_n = p^*. \quad (28)$$

Next, by the assumption that T is uniformly L -Lipschitz continuous, we have

$$\begin{aligned} \|T^{n+1} x_n - T^n x_n\| &\leq \|T^{n+1} x_n - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\quad + \|x_n - T^n x_n\| \leq (L+1) \|x_{n+1} - x_n\| \\ &\quad + \|T^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - T^n x_n\|. \end{aligned} \quad (29)$$

Combining (15) and (25) with (29), we have

$$\lim_{n \rightarrow \infty} \|T^{n+1} x_n - T^n x_n\| = 0. \quad (30)$$

Hence, it follows from (27) that we have

$$\lim_{n \rightarrow \infty} T^{n+1} x_n = p^*, \quad (31)$$

that is,

$$\lim_{n \rightarrow \infty} TT^n x_n = p^*. \quad (32)$$

Furthermore, in view of (27) and the closeness of T , it yields that $Tp^* = p^*$. Similarly, we have $Sp^* = p^*$. Therefore, this implies that $p^* \in \mathfrak{F}$.

(VI) Finally we prove that $p^* = \Pi_{\mathfrak{F}} x_0$ and so $x_n \rightarrow \Pi_{\mathfrak{F}} x_0$.

Let $w = \Pi_{\mathfrak{F}} x_0$. Since $w \in \mathfrak{F} \subset C_n$ and $x_n \rightarrow \Pi_{\mathfrak{F}} x_0$, we have $\phi(x_n, x_0) \leq \phi(w, x_0)$, $\forall n \geq 1$. This implies that

$$\phi(p^*, x_0) = \lim_{n \rightarrow \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \quad (33)$$

Since $w = \Pi_{\mathfrak{F}} x_0$, this implies that $p^* = w$. Therefore, $x_n \rightarrow \Pi_{\mathfrak{F}} x_0$. This completes the proof. $\square$

If $S = T$, $\beta_n^{(1)} = 0$ or $\beta_n^{(1)} = \beta_n^{(3)} = 0$ for all $n \geq 1$ in Theorem 6, then we have the following corollary.

Corollary 7. Let E, K be the same as Theorem 6. Let $T : K \rightarrow K$ be closed, uniformly L -Lipschitz continuous, and totally quasi- ϕ -asymptotically nonexpansive mappings with nonnegative real sequences $\{\mu_n\}$, $\{\nu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n = 0$ and $\zeta(0) = 0$. Let $\{x_n\}$ be a sequence defined by

$$\begin{aligned} x_0 &\in C_1 = K \text{ arbitrarily chosen,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) JT^n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_0) \\ &\quad + (1 - \alpha_n) \phi(z, x_n) + \xi_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \quad (34)$$

where $\xi_n = (1 - \alpha_n)(\mu_n \sup_{p \in \mathfrak{F}} \zeta(\phi(p, x_n)) + \nu_n)$ and $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1} . Assume that the sequence $\{\alpha_n\}$ satisfies hypothesis (i). If $\mathfrak{F} := F(T) \neq \emptyset$ is bounded and $\nu_1 = 0$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathfrak{F}} x_0$.

It is well known that the quasi- ϕ -asymptotically nonexpansive mappings are a special case of the total quasi- ϕ -asymptotically nonexpansive mappings. So we obtain the following theorem.

Theorem 8. Let E, K be the same as Theorem 6. Let $S, T : K \rightarrow K$ be two closed, uniformly L -Lipschitz continuous, and quasi- ϕ -asymptotically nonexpansive mappings with nonnegative real sequences $\{\mu_n^{(1)}\}$, $\{\mu_n^{(2)}\}$ such that $\lim_{n \rightarrow \infty} \mu_n^{(1)} = \lim_{n \rightarrow \infty} \mu_n^{(2)} = 0$. Let $\{x_n\}$ be a sequence defined by

$$\begin{aligned} x_0 &\in C_1 = K \text{ arbitrarily chosen,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) Jz_n), \\ z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT^n x_n + \beta_n^{(3)} JS^n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_0) \\ &\quad + (1 - \alpha_n) \phi(z, x_n) + (1 - \alpha_n) \xi_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \quad (35)$$

where $\xi_n = (\beta_n^{(2)} \mu_n^{(2)} + \beta_n^{(3)} \mu_n^{(1)}) \sup_{p \in \mathfrak{F}} \phi(p, x_n)$. Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n^{(j)}\}$ ($j = 1, 2, 3$) satisfy hypotheses (i) and (ii). If $\mathfrak{F} := F(S) \cap F(T) \neq \emptyset$ is bounded and $\nu_1 = 0$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathfrak{F}} x_0$.

Proof. The proof is similar to that of Theorem 6 and hence we omit it. This completes the proof. $\square$

If $S = T$, $\beta_n^{(1)} = 0$ or $\beta_n^{(1)} = \beta_n^{(3)} = 0$ for all $n \geq 1$ in Theorem 8, then we have the following corollary.

Corollary 9. Let E, K be the same as Theorem 6. Let $T : K \rightarrow K$ be closed, uniformly L -Lipschitz continuous, and quasi- ϕ -asymptotically nonexpansive mappings with nonnegative real sequences $\{\mu_n\}$ such that $\lim_{n \rightarrow \infty} \mu_n = 0$. Let $\{x_n\}$ be a sequence defined by

$$\begin{aligned} x_0 &\in C_1 = K \text{ arbitrarily chosen,} \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) JT^n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_0) \\ &\quad + (1 - \alpha_n) \phi(z, x_n) + \xi_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{aligned} \quad (36)$$

where $\xi_n = (1 - \alpha_n) \mu_n \sup_{p \in \mathfrak{F}} \phi(p, x_n)$ and $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1} . Assume that the sequence $\{\alpha_n\}$ satisfies hypothesis (i). If $\mathfrak{F} := F(T) \neq \emptyset$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathfrak{F}} x_0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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