Research Article

The Kirchhoff Index of Folded Hypercubes and Some Variant Networks

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The n -dimensional folded hypercube FQ_n is an important and attractive variant of the n -dimensional hypercube Q_n , which is obtained from Q_n by adding an edge between any pair of vertices complementary edges. FQ_n is superior to Q_n in many measurements, such as the diameter of FQ_n which is $\lceil n/2 \rceil$, about a half of the diameter in terms of Q_n . The Kirchhoff index $Kf(G)$ is the sum of resistance distances between all pairs of vertices in G . In this paper, we established the relationships between the folded hypercubes networks FQ_n and its three variant networks $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$ on their Kirchhoff index, by deducing the characteristic polynomial of the Laplacian matrix in spectral graph theory. Moreover, the explicit formulae for the Kirchhoff indexes of FQ_n , $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$ were proposed, respectively.

1. Introduction

It is well known that interconnection networks play an important role in parallel communication systems. An interconnection network is usually modelled by connected graphs $G = (V, E)$, where V denotes the set of processors and E denotes the set of communication links between processors in networks. In this work, we are concerned with finite undirected connected simple graphs (networks). For the graph theoretical definitions and notations, we follow [1].

The adjacency matrix $A(G)$ of G is an $n \times n$ matrix with the (i, j) -entry equal to 1 if vertices i and j are adjacent and 0 otherwise. Suppose that $D(G) = \text{diag}(d_1(G), d_2(G), \dots, d_n(G))$ is the degree diagonal matrix of G , where $d_i(G)$ is the degree of the vertex i , $i = 1, 2, \dots, n$. Let $L(G) = D(G) - A(G)$ be called the Laplacian matrix of G . Then, the eigenvalues of $A(G)$ and $L(G)$ are called eigenvalues and Laplacian eigenvalues of G , respectively. For more details, we refer to [1].

Let G be a graph with vertices labelled $1, 2, \dots, n$. The resistance distances between vertices i and j , denoted by r_{ij} ,

are defined to be the effective electrical resistance between them if each edge of G is replaced by a unit resistor [2]. A famous distance-based topological index as the Kirchhoff index $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices in G . Define

$$Kf(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n r_{ij}(G), \quad (1)$$

known as the Kirchhoff index of G [2].

The Kirchhoff index attracted extensive attention due to its wide applications in physics, chemistry, graph theory, and so forth [3–6]. For example, Zhu et al. [7] and Gutman and Mohar [8] proved that relations between Kirchhoff index of a graph and Laplacian eigenvalues of the graph. The Kirchhoff index also is a structure descriptor [9].

However, it is difficult to design some algorithms [7, 10, 11] to calculate resistance distances and the Kirchhoff indexes of graphs. Hence, it makes sense to find explicit closed form

for some special classes of graphs, for instance, the Kirchhoff index for cycles and complete graphs [12] which has been computed, geodetic graphs [13], some composite graphs [14], and composite networks [15]. Besides, many efforts were also made to obtain the Kirchhoff index bounds for some graphs [11, 16] and characterize extremal graphs as well, such as bicyclic graphs and cacti graphs [6, 17]. Details on its theory can be found in recent papers [11, 16, 17] and the references cited therein.

The hypercube Q_n is one of the most popular and efficient interconnection networks due to its excellent performance for some practical applications. There is a large amount of literature on the properties of hypercubes networks [18–20].

As an important variant of Q_n , the folded hypercube networks FQ_n , proposed by El-Amawy and Latifi [18], are the graph obtained from Q_n by adding an edge between any pair of vertices complementary addresses. The folded hypercube FQ_n , in which diameter of FQ_n is $\lceil n/2 \rceil$, about half the diameter of Q_n , has the same number of vertices as a hypercube Q_n and 2^{n-1} edges more than hypercube; at the same time, the folded hypercubes preserve the symmetric properties of the hypercubes. The folded hypercubes FQ_n obtained considerable attention due to its perfect properties, such as symmetry, regular structure, strong connectivity, and small diameter, and many of its properties have been explored [21–26].

But few works appear on the Kirchhoff index for the combinatorial networks, such as hypercubes Q_n and folded hypercubes FQ_n , except that Palacios and Renom [11] studied the bounds of the Kirchhoff index of hypercubes Q_n by using probabilistic tools in 2010. In present paper, we established the relationships between the folded hypercubes networks FQ_n and three variant networks $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$ on their Kirchhoff index, by deducing the characteristic polynomial of the Laplacian matrix in spectral graph theory.

Recall the definitions of n -dimensional folded hypercubes networks FQ_n as follows [18].

Definition 1 (see [18]). The folded hypercubes FQ_n can be constructed from Q_n by adding an edge to every pair of

vertices with complementary addresses. Two vertices $X = x_1 x_2 \cdots x_n$ and $\bar{X} = \bar{x}_1 \bar{x}_2 \cdots \bar{x}_n$ are adjacent in the folded hypercubes FQ_n .

From above definition of FQ_n , it is easy to get that the folded hypercubes FQ_n have 2^n vertices and $(n+1)2^{n-1}$ edges, respectively.

The line graph of a graph G , denoted by $l(G)$, is the graph whose vertices correspond to the edges of G with two vertices of $l(G)$ being adjacent if and only if the corresponding edges in G share a common vertex. The subdivision graph of a graph G , denoted by $s(G)$, is the graph obtained by replacing every edge in G with a copy of P_2 (path of length two). The total graph of a graph G , denoted by $t(G)$, is the graph whose vertices correspond to the union of the set of vertices and edges of G , with two vertices of $t(G)$ being adjacent if and only if the corresponding elements are adjacent or incident in G .

Gao et al. [27] obtained special formulae for the Kirchhoff index of $l(G)$, $s(G)$, and $t(G)$, where G is a regular graph. Motivated by above results, we present the corresponding calculated formulae for the Kirchhoff index of the hypercubes networks Q_n and its three-variant networks $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$ in this paper.

The remainder of present paper is organized as follows. Section 2 gives some basic notations and some preliminaries in our discussion. The proofs of our main results are in Section 3 and some conclusions are given in Section 4, respectively.

2. Notations and Some Preliminaries

In this section, we introduced some basic properties which we need to use in the proofs of our main results.

M. Chen and B. X. Chen have studied the Laplacian spectra of folded hypercubes networks in 2011 [21].

Lemma 2 (see [21]). *For the folded hypercubes networks FQ_n with $n \geq 2$, the spectrum of Laplacian matrix in terms of hypercubes networks is as follows.*

(1) If $n \equiv 0 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-8 & 2n-4 & 2n \\ C_n^0 & C_n^{n-1} + C_n^{n-2} & C_n^{n-3} + C_n^{n-4} & \cdots & C_n^5 + C_n^4 & C_n^3 + C_n^2 & C_n^1 + C_n^0 \end{pmatrix}. \quad (2)$$

(2) If $n \equiv 1 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-6 & 2n-2 & 2n+2 \\ C_n^0 & C_n^1 + C_n^2 & C_n^3 + C_n^4 & \cdots & C_n^{n-4} + C_n^{n-3} & C_n^{n-2} + C_n^{n-1} & C_n^n \end{pmatrix}, \quad (3)$$

where C_n^i are the binomial coefficients; the elements in the first and second rows are the eigenvalues of the Laplacian matrix of folded hypercubes networks and the multiplicities of the corresponding eigenvalues.

Lemma 3 (see [7, 8]). *Let G be a connected graph with $n \geq 2$ vertices; then,*

$$\text{Kf}(G) = n \sum_{i=1}^{n-1} \frac{1}{\lambda_i}. \quad (4)$$

Let $P_{(G)}(x)$ be the characteristic polynomial of the Laplacian matrix of a graph G ; the following results were shown in [27].

Lemma 4 (see [27]). *Let G be a r -regular connected graph with n vertices and m edges; then,*

$$\begin{aligned} P_{l(G)}(x) &= (x - 2r)^{m-n} P_G(x), \\ P_{s(G)}(x) &= (-1)^m (2 - x)^{m-n} P_G(x(r + 2 - x)), \\ P_{t(G)}(x) &= (-1)^m (r + 1 - x)^n \\ &\quad \times (2r + 2 - x)^{m-n} P_G\left(\frac{x(r + 2 - x)}{r - x + 1}\right), \end{aligned} \quad (5)$$

where $P_{l(G)}(x)$, $P_{s(G)}(x)$, and $P_{t(G)}(x)$ are the characteristic polynomial for the Laplacian matrix of graphs $l(G)$, $s(G)$, and $t(G)$, respectively.

It is worthwhile to note that the conclusion of Lemma 4 is not completely correct; the authors [28] recently show the Laplacian characteristic polynomial of $t(G)$, where G is a regular graph, which corrects Lemma 3 in Gao et al. [27] as follows.

Lemma 5 (see [28]). *Let G be a r -regular connected graph with n vertices and m edges; then,*

$$\begin{aligned} P_{s(G)}(x) &= (-1)^n (2 - x)^{m-n} P_G(x(r + 2 - x)), \\ P_{t(G)}(x) &= x(x - r - 2)(x - 2r - 2)^{m-n} \\ &\quad \times \prod_{i=1}^{n-1} \left[(x^2 - 2x - rx) + (3 - 2x + r)\mu_i + \mu_i^2 \right], \end{aligned} \quad (6)$$

where $P_{s(G)}(x)$, $P_{t(G)}(x)$ are the characteristic polynomial for the Laplacian matrix of graphs $s(G)$ and $t(G)$, respectively.

The following lemma gives an expression on $Kf(t(G))$ and $Kf(G)$ of a regular graph G .

Lemma 6 (see [28]). *Let G be a r -regular connected graph with n vertices and m edges and $r \geq 2$; then,*

$$\begin{aligned} Kf(t(G)) &= \frac{(r+2)^2}{2(r+3)} Kf(G) + \frac{n^2(r^2-4)}{8(r+1)} + \frac{n}{2} \\ &\quad + \frac{n(r+2)(r+4)}{2(r+3)} \sum_{i=1}^{n-1} \frac{1}{\mu_i + 3 + r}. \end{aligned} \quad (7)$$

For proving the formula for the Kirchhoff index on the subdivision graph of hypercubes $s(FQ_n)$, we prove the following lemma utilizing Vieta's theorem; in our proof, some techniques in [27] are referred.

Lemma 7. *Let $P_{(FQ_n)}(x)$ be the characteristic polynomial of the Laplacian matrix of the folded hypercubes FQ_n with $n \geq 2$ and*

$$P_{(FQ_n)}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \cdots + a_{2^n-1} x; \quad (8)$$

then,

$$\frac{Kf(FQ_n)}{2^n} = -\frac{a_{2^n-2}}{a_{2^n-1}}, \quad (9)$$

where a_{2^n-1} , a_{2^n-2} are the coefficient of x and x^2 in the characteristic polynomial, respectively.

Proof. Let $\text{Spec}(FQ_n) = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{2^n-1}\}$. Then, λ_i , $i = 1, 2, \dots, 2^n - 1$, satisfy the following equation:

$$x^{2^n-1} + a_1 x^{2^n-2} + \cdots + a_{2^n-1} = 0; \quad (10)$$

it is not difficult to check that $1/\lambda_i$, $i = 1, 2, \dots, 2^n - 1$, are the roots of equation

$$a_{2^n-1} x^{2^n-1} + a_{2^n-2} x^{2^n-2} + \cdots + a_1 x + 1 = 0. \quad (11)$$

Note that G is connected graph and the multiplicity of 0 as an eigenvalue of $L(G)$ is equal to the number of the connected components in G . So $a_{2^n-1} \neq 0$, by Lemma 3 and Vieta's theorem,

$$\frac{Kf(FQ_n)}{2^n} = \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i} = -\frac{a_{2^n-2}}{a_{2^n-1}}, \quad (12)$$

where a_{2^n-1} , a_{2^n-2} are the coefficient of x and x^2 in the characteristic polynomial of the Laplacian matrix of the hypercubes FQ_n . $\square$

3. Main Results

3.1. The Kirchhoff Index in Folded Hypercubes Networks FQ_n . In this section, we firstly give formula for the Kirchhoff index of the folded hypercubes FQ_n with any positive integer n .

Theorem 8. *For the folded hypercubes networks FQ_n with any positive integer n ,*

$$\begin{aligned} (1) \quad Kf(FQ_n) &= 2^n \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i}, \\ i &= 1, 2, \dots, \frac{n}{2}, \quad \text{if } n \equiv 0 \pmod{2}; \\ (2) \quad Kf(FQ_n) &= 2^n \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{2^{n-1}}{n+1}, \\ i &= 1, 2, \dots, \frac{n-1}{2}, \quad \text{if } n \equiv 1 \pmod{2}, \end{aligned} \quad (13)$$

where the $4i$, $i = 1, 2, \dots, (n-1)/2$ or $n/2$ are the eigenvalues of the Laplacian matrix of hypercubes networks FQ_n and C_n^i are the binomial coefficients,

Proof. The hypercubes networks FQ_n have 2^n vertices and $(n+1)2^{n-1}$ edges.

By Lemma 3, if $n \equiv 0 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-8 & 2n-4 & 2n \\ C_n^n & C_n^{n-1} + C_n^{n-2} & C_n^{n-3} + C_n^{n-4} & \cdots & C_n^5 + C_n^4 & C_n^3 + C_n^2 & C_n^1 + C_n^0 \end{pmatrix}, \quad (14)$$

then

$$\begin{aligned} \text{Kf}(FQ_n) &= 2^n \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i} \\ &= 2^n \left(\frac{C_n^{n-1} + C_n^{n-2}}{4 \times 1} + \frac{C_n^{n-3} + C_n^{n-4}}{4 \times 2} + \cdots \right. \\ &\quad \left. + \frac{C_n^{n-i} + C_n^{n-i-1}}{4 \times i} + \cdots + \frac{C_n^1 + C_n^0}{4 \times (n/2)} \right) \end{aligned}$$

$$= 2^n \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i},$$

$$i = 1, 2, \dots, \frac{n}{2}.$$

(15)

By Lemma 3, if $n \equiv 1 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-6 & 2n-2 & 2n+2 \\ C_n^0 & C_n^1 + C_n^2 & C_n^3 + C_n^4 & \cdots & C_n^{n-4} + C_n^{n-3} & C_n^{n-2} + C_n^{n-1} & C_n^n \end{pmatrix}, \quad (16)$$

then

$$\begin{aligned} \text{Kf}(FQ_n) &= 2^n \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i} \\ &= 2^n \left(\frac{C_n^1 + C_n^2}{4 \times 1} + \frac{C_n^3 + C_n^4}{4 \times 2} + \cdots + \frac{C_n^{2i} + C_n^{2i-1}}{4 \times i} + \cdots \right. \\ &\quad \left. + \frac{C_n^{n-2} + C_n^{n-1}}{4 \times ((n-1)/2)} + \frac{C_n^n}{2n+2} \right) \\ &= 2^n \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{2^{n-1}}{n+1}, \\ &\quad i = 1, 2, \dots, \frac{n-1}{2}. \end{aligned} \quad (17)$$

The proof of Theorem 8 is completed. $\square$

3.2. The Kirchhoff Index in the Line Graph of Folded Hypercubes Networks $l(FQ_n)$. In the following theorem, we proposed the formula for calculating the Kirchhoff index, denoted by $\text{Kf}(l(FQ_n))$, on the line graph of folded hypercubes $l(FQ_n)$.

Theorem 9. Let $l(FQ_n)$ be line graphs of hypercubes FQ_n with any positive integer n ; then

$$\begin{aligned} (1) \quad \text{Kf}(l(FQ_n)) &= (n+1) 2^{n-1} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\ &\quad + (n-1) 2^{2n-3}, \\ &\quad i = 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2}; \\ (2) \quad \text{Kf}(l(FQ_n)) &= (n+1) 2^{n-1} \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} \\ &\quad + (n-1) 2^{2n-3} + 2^{n-2}, \\ &\quad i = 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2}, \end{aligned} \quad (18)$$

where the $4i$, $i = 1, 2, \dots, (n-1)/2$ or $n/2$ are the eigenvalues of the Laplacian matrix of folded hypercubes networks FQ_n and C_n^i are the binomial coefficients.

Proof. Now for convenience, we denote the numbers of vertices and edges in the folded hypercubes networks FQ_n by $p = 2^n$ and $q = (n+1)2^{n-1}$, respectively.

By Lemma 4,

$$P_{l(FQ_n)}(x) = (x - 2r)^{q-p} P_{FQ_n}(x). \quad (19)$$

Notice that folded hypercubes networks FQ_n are regular graphs with the degree of any vertex being $r = n+1$; compare the spectrum of FQ_n as follows.

(1) If $n \equiv 0 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-8 & 2n-4 & 2n \\ C_n^n & C_n^{n-1} + C_n^{n-2} & C_n^{n-3} + C_n^{n-4} & \cdots & C_n^5 + C_n^4 & C_n^3 + C_n^2 & C_n^1 + C_n^0 \end{pmatrix}. \quad (20)$$

(2) If $n \equiv 1 \pmod{2}$,

$$\text{Spec}(FQ_n) = \begin{pmatrix} 0 & 4 & 8 & \cdots & 2n-6 & 2n-2 & 2n+2 \\ C_n^0 & C_n^1 + C_n^2 & C_n^3 + C_n^4 & \cdots & C_n^{n-4} + C_n^{n-3} & C_n^{n-2} + C_n^{n-1} & C_n^n \end{pmatrix}. \quad (21)$$

We can easily obtain the spectrum of $l(FQ_n)$ as follows.

(1) If $n \equiv 0 \pmod{2}$,

$$\text{Spec}(l(FQ_n)) = \begin{pmatrix} 2r & 0 & 4 & 8 & \cdots & 2n-8 & 2n-4 & 2n \\ q-p & C_n^n & C_n^{n-1} + C_n^{n-2} & C_n^{n-3} + C_n^{n-4} & \cdots & C_n^5 + C_n^4 & C_n^3 + C_n^2 & C_n^1 + C_n^0 \end{pmatrix}. \quad (22)$$

(2) If $n \equiv 1 \pmod{2}$,

$$\text{Spec}(l(FQ_n)) = \begin{pmatrix} 2r & 0 & 4 & 8 & \cdots & 2n-6 & 2n-2 & 2n+2 \\ q-p & C_n^0 & C_n^1 + C_n^2 & C_n^3 + C_n^4 & \cdots & C_n^{n-4} + C_n^{n-3} & C_n^{n-2} + C_n^{n-1} & C_n^n \end{pmatrix}, \quad (23)$$

where $\lambda_0, \lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{2^n-1}$ are the eigenvalues of FQ_n and $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \lambda_{n+1} \leq \dots \leq \lambda_{2^n-1}$.

By Lemma 4,

Substituting the results of Theorem 8 into (25), we can get the formula for the Kirchhoff index on the line graph of folded hypercubes $\text{Kf}(l(FQ_n))$:

$$\begin{aligned} \frac{\text{Kf}(l(FQ_n))}{(n+1)2^{n-1}} &= \sum_{i=1}^{n-1} \frac{1}{\lambda_i} + \frac{(n+1)2^{n-1} - 2^n}{2r} \\ &= \frac{\text{Kf}(FQ_n)}{2^n} + \frac{(n+1)2^{n-1} - 2^n}{2n+2} \\ &= \frac{\text{Kf}(FQ_n)}{2^n} - \frac{2^{n-1}}{n+1} + 2^{n-2}. \end{aligned} \quad (24)$$

$$\begin{aligned} (1) \text{Kf}(l(FQ_n)) &= (n+1)2^{n-1} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\ &\quad + (n-1)2^{2n-3}, \\ i &= 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2}; \end{aligned}$$

$$\begin{aligned} (2) \text{Kf}(l(FQ_n)) &= (n+1)2^{n-1} \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} \\ &\quad + (n-1)2^{2n-3} + 2^{n-2}, \\ i &= 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2}, \end{aligned} \quad (26)$$

Therefore, we clearly obtained

$$\begin{aligned} \text{Kf}(l(FQ_n)) &= (n+1)2^{n-1} \left(\frac{\text{Kf}(FQ_n)}{2^n} - \frac{2^{n-1}}{n+1} + 2^{n-2} \right) \\ &= \frac{n+1}{2} \text{Kf}(FQ_n) + n2^{2n-3} - 2^{2n-3} \\ &= \frac{n+1}{2} \text{Kf}(FQ_n) + (n-1)2^{2n-3}. \end{aligned} \quad (25)$$

which completes the proof. $\square$

3.3. The Kirchhoff Index in the Subdivision Graph of Folded Hypercubes Networks $s(FQ_n)$. In an almost identical way as Theorem 9, we derived the formula for the Kirchhoff index on the subdivision graph of folded hypercubes $s(FQ_n)$, denoted by $\text{Kf}(s(FQ_n))$.

Theorem 10. Let $s(FQ_n)$ be subdivision graphs of the folded hypercubes FQ_n with any positive integer n ; then,

$$\begin{aligned}
 (1) \quad \text{Kf}(s(FQ_n)) &= (n+3)^2 2^{n-1} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\
 &\quad + \frac{2^n + (n+1)2^{n-1}}{n+3} \\
 &\quad + \frac{(n+1)^2 4^{n-1} - 4^n}{2}, \\
 &\quad i = 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2}; \\
 (2) \quad \text{Kf}(s(FQ_n)) &= (n+3)^2 2^{n-1} \\
 &\quad \times \left(\sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{2^{n-1}}{n+1} \right) \\
 &\quad + \frac{2^n + (n+1)2^{n-1}}{n+3} \\
 &\quad + \frac{(n+1)^2 4^{n-1} - 4^n}{2}, \\
 &\quad i = 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2}.
 \end{aligned} \tag{27}$$

Proof. Now suppose that $P_{s(FQ_n)}(x)$ is the characteristic polynomial of the Laplacian matrix of the $s(FQ_n)$; let $P_{s(FQ_n)}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \dots + a_{2^n-1} x$ and

$$\text{Spec}(s(FQ_n)) = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{2^n-1}\}, \tag{28}$$

where $\lambda_i, i = 0, 1, 2, \dots, 2^n - 1$, are the Laplacian eigenvalues of $s(FQ_n)$.

Then, by Lemma 7,

$$\frac{\text{Kf}(s(FQ_n))}{2^n} = -\frac{a_{2^n-2}}{a_{2^n-1}}. \tag{29}$$

By Lemma 5,

$$\begin{aligned}
 P_{s(FQ_n)}(x) &= (-1)^{2^n} (2-x)^{(n+1)2^{n-1}-2^n} \\
 &\quad \times [x^{2^n}(r+2-x)^{2^n} + a_1 x^{2^n-1}(r+2-x)^{2^n-1} \\
 &\quad + \dots + a_{2^n-2} x^2 (r+2-x)^2 \\
 &\quad + a_{2^n-1} x (r+2-x)].
 \end{aligned} \tag{30}$$

Consequently, the coefficient of x^2 in $P_{s(FQ_n)}(x)$ is

$$\begin{aligned}
 (-1)^{2^n} \{ &2^{(n+1)2^{n-1}-2^n} a_{2^n-2} (r+2)^2 \\
 &- 2^{(n+1)2^{n-1}-2^n} a_{2^n-1} [(n+1)2^{n-1} - 2^n] \\
 &\times 2^{(n+1)2^{n-1}-2^n-1} (r+2) a_{2^n-1} \},
 \end{aligned} \tag{31}$$

and the coefficient of x in $P_{s(FQ_n)}(x)$ is

$$(-1)^{2^n} 2^{(n+1)2^{n-1}-2^n} (r+2) a_{2^n-1}. \tag{32}$$

Since $s(FQ_n)$ has $2^n + (n+1)2^{n-1}$ vertices. For convenience, denote $p = 2^n$, $q = (n+1)2^{n-1}$, and $r = n+1$, respectively. By Lemma 7 and substituting the coefficients into (12), we get

$$\frac{\text{Kf}(s(FQ_n))}{p+q} = -\frac{2^{q-p}(r+2)^2 - 2^{q-p} a_{2^n-1}}{2^{q-p} a_{2^n-1} (r+2)} \tag{33}$$

$$\begin{aligned}
 &- \frac{(q-p) 2^{q-p-1} a_{2^n-1} (r+2)}{2^{q-p} a_{2^n-1} (r+2)} \\
 &= -\frac{a_{2^n-2} (r+2)}{a_{2^n-1}} + \frac{1}{r+2} + \frac{q-p}{2}.
 \end{aligned} \tag{34}$$

Combining the results of Lemma 7 and (34),

$$\frac{\text{Kf}(s(FQ_n))}{p+q} = (r+2) \frac{\text{Kf}(FQ_n)}{2^n} + \frac{1}{r+2} + \frac{q-p}{2}. \tag{35}$$

Hence,

$$\begin{aligned}
 &\text{Kf}(s(FQ_n)) \\
 &= (p+q)(r+2) \frac{\text{Kf}(FQ_n)}{2^n} + \frac{p+q}{r+2} + \frac{q^2 - p^2}{2}.
 \end{aligned} \tag{36}$$

Simplifying (36) by substituting $p = 2^n$, $q = (n+1)2^{n-1}$, and $r = n+1$, we can get

$$\begin{aligned}
 &\text{Kf}(s(FQ_n)) \\
 &= [2^n + (n+1)2^{n-1}] (n+3) \frac{\text{Kf}(FQ_n)}{2^n} \\
 &\quad + \frac{2^n + (n+1)2^{n-1}}{n+3} + \frac{(n+1)^2 4^{n-1} - 4^n}{2} \\
 &= \frac{(n+3)^2}{2} \text{Kf}(FQ_n) + \frac{2^n + (n+1)2^{n-1}}{n+3} \\
 &\quad + \frac{(n+1)^2 4^{n-1} - 4^n}{2}.
 \end{aligned} \tag{37}$$

Note that the results of Theorem 8 and (37), we can get the formulae for the Kirchhoff index of the subdivision graph of the folded hypercubes $\text{Kf}(s(FQ_n))$:

$$\begin{aligned}
 (1) \text{Kf}(s(FQ_n)) &= (n+3)^2 2^{n-1} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\
 &\quad + \frac{2^n + (n+1)2^{n-1}}{n+3} + \frac{(n+1)^2 4^{n-1} - 4^n}{2}, \\
 &\quad i = 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2}; \\
 (2) \text{Kf}(s(FQ_n)) &= (n+3)^2 2^{n-1} \\
 &\quad \times \left(\sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{2^{n-1}}{n+1} \right) \\
 &\quad + \frac{2^n + (n+1)2^{n-1}}{n+3} + \frac{(n+1)^2 4^{n-1} - 4^n}{2}, \\
 &\quad i = 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2},
 \end{aligned} \tag{38}$$

which completes the proof. $\square$

3.4. The Kirchhoff Index in the Total Graph of Folded Hypercubes Networks $t(FQ_n)$. We now proved the formula for the Kirchhoff index in the total graph of the folded hypercubes $t(FQ_n)$, denoted by $\text{Kf}(t(FQ_n))$.

Theorem 11. Let $t(FQ_n)$ be the total graphs of the folded hypercubes networks FQ_n with any positive integer n ; then,

$$\begin{aligned}
 (1) \text{Kf}(t(FQ_n)) &= \frac{2^{n-1}(n+3)^2}{n+4} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\
 &\quad + \frac{2^{2n-3}(n-1)(n+3)}{n+2} + 2^{n-1} \\
 &\quad + \frac{2^{n-1}(n+3)(n+5)}{n+4} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i+n+4}, \\
 &\quad i = 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2}; \\
 (2) \text{Kf}(t(FQ_n)) &= \frac{2^{n-1}(n+3)^2}{n+4} \\
 &\quad \times \left(\sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{1}{2n+2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{2^{2n-3}(n-1)(n+3)}{n+2} + 2^{n-1} \\
 &+ \frac{2^{n-1}(n+3)(n+5)}{n+4} \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i+n+4}, \\
 &i = 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2},
 \end{aligned} \tag{39}$$

where the $4i$, $i = 1, 2, \dots, (n-1)/2$ or $n/2$, are the eigenvalues of the Laplacian matrix of folded hypercubes networks FQ_n and C_n^i are the binomial coefficients.

Proof. Let $P_{t(FQ_n)}(x) = x^{2^n} + a_1 x^{2^n-1} + a_2 x^{2^n-2} + \dots + a_{2^n-1} x$ and

$$\text{Spec}(t(FQ_n)) = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{2^n-1}\}, \tag{40}$$

where λ_i , $i = 0, 1, 2, \dots, 2^n - 1$, are the Laplacian eigenvalues of FQ_n .

Applying Lemma 5, the Laplacian characteristic polynomial of $t(FQ_n)$ is

$$\begin{aligned}
 P_{t(FQ_n)}(x) &= x(x-n-3)(x-2n-4)^{q-p} \\
 &\quad \times \prod_{i=1}^{2^n-1} \left\{ \left[x^2 - 2x - (n+1)x \right] \right. \\
 &\quad \left. + [3 - 2x + (n+1)] \lambda_i + \lambda_i^2 \right\} \\
 &= x(x-n-3)(x-2n-4)^{q-p} \\
 &\quad \times \prod_{i=1}^{2^n-1} \left[x^2 - (3+n+2\lambda_i)x + \lambda_i^2 + (n+4)\lambda_i \right].
 \end{aligned} \tag{41}$$

Notice that $t(FQ_n)$ has $2^n + (n+1)2^{n-1}$ vertices, that is, $p+q$ vertices. By Lemma 6, we can get the following:

$$\begin{aligned}
 \text{Kf}(t(FQ_n)) &= \frac{(r+2)^2}{2(r+3)} \text{Kf}(FQ_n) + \frac{n^2(r^2-4)}{8(r+1)} + \frac{n}{2} \\
 &\quad + \frac{n(r+2)(r+4)}{2(r+3)} \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i + 3 + r}.
 \end{aligned} \tag{42}$$

Consider that FQ_n has 2^n vertices and $r = n + 1$, we have

$$\begin{aligned}
 \text{Kf}(t(FQ_n)) &= \frac{(n+3)^2}{2(n+4)} \text{Kf}(FQ_n) \\
 &\quad + \frac{4^n [(n+1)^2 - 4]}{8(n+2)} + 2^{n-1} \\
 &\quad + \frac{2^n (n+3)(n+5)}{2(n+4)} \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i + n + 4} \\
 &= \frac{(n+3)^2}{2(n+4)} \text{Kf}(FQ_n) \\
 &\quad + \frac{2^{2n-3} (n-1)(n+3)}{n+2} + 2^{n-1} \\
 &\quad + \frac{2^{n-1} (n+3)(n+5)}{n+4} \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i + n + 4}.
 \end{aligned} \tag{43}$$

From (43), the relationships between the folded hypercubes networks FQ_n and its variant networks $t(FQ_n)$ for Kirchhoff index are as follows:

$$\begin{aligned}
 \text{Kf}(t(FQ_n)) &= \frac{(n+3)^2}{2(n+4)} \text{Kf}(FQ_n) \\
 &\quad + \frac{2^{2n-3} (n-1)(n+3)}{n+2} + 2^{n-1} \\
 &\quad + \frac{2^{n-1} (n+3)(n+5)}{n+4} \sum_{i=1}^{2^n-1} \frac{1}{\lambda_i + n + 4}.
 \end{aligned} \tag{44}$$

Substituting results of Theorem 8 into (44) and simplifying (44), we can get the formula for the Kirchhoff index on the total graph of folded hypercubes $\text{Kf}(t(FQ_n))$; we consider two cases in terms of if $n \equiv 0 \pmod{2}$ and if $n \equiv 1 \pmod{2}$,

$$\begin{aligned}
 (1) \quad \text{Kf}(t(FQ_n)) &= \frac{2^{n-1}(n+3)^2}{n+4} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i} \\
 &\quad + \frac{2^{2n-3} (n-1)(n+3)}{n+2} + 2^{n-1} \\
 &\quad + \frac{2^{n-1} (n+3)(n+5)}{n+4} \sum_{i=1}^{n/2} \frac{C_n^{n-i} + C_n^{n-i-1}}{4i + n + 4}, \\
 &\quad i = 1, 2, \dots, \frac{n}{2}, \text{ if } n \equiv 0 \pmod{2};
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{Kf}(t(FQ_n)) &= \frac{2^{n-1}(n+3)^2}{n+4} \\
 &\quad \times \left(\sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i} + \frac{1}{2n+2} \right) \\
 &\quad + \frac{2^{2n-3} (n-1)(n+3)}{n+2} + 2^{n-1} \\
 &\quad + \frac{2^{n-1} (n+3)(n+5)}{n+4} \sum_{i=1}^{(n-1)/2} \frac{C_n^{2i-1} + C_n^{2i}}{4i + n + 4}, \\
 &\quad i = 1, 2, \dots, \frac{n-1}{2}, \text{ if } n \equiv 1 \pmod{2},
 \end{aligned} \tag{45}$$

where the $4i$, $i = 1, 2, \dots, (n-1)/2$ or $n/2$, are the eigenvalues of the Laplacian matrix of folded hypercubes networks FQ_n and C_n^i are the binomial coefficients.

This completes the proof of Theorem 11. $\square$

Remark 12. The authors [27] gave the formulae for the Kirchhoff index of $l(G)$, $s(G)$, and $t(G)$, where G is a regular graph. In this paper, we proposed the special formulae for calculating the Kirchhoff indexes of the folded hypercubes networks FQ_n and its three variant networks $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$, respectively, which enhanced related results on the Kirchhoff index.

4. Conclusions

The main purpose of this paper is to investigate the Kirchhoff index of the folded hypercubes networks and related networks, which are important networks topology indexes for parallel processing computer systems. We established the relationships between the folded hypercubes networks FQ_n and its three variant networks $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$ for their Kirchhoff index, respectively, by deducing the characteristic polynomial of the Laplacian matrix related networks.

Moreover, the explicit formulae were proposed for calculating the Kirchhoff indexes of FQ_n , $l(FQ_n)$, $s(FQ_n)$, and $t(FQ_n)$, respectively, by making use of the characteristic polynomial of the Laplacian matrix in spectral graph theory and Vieta's theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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